ON REGULAR GRAPHS OF GIRTH SIX ARISING FROM PROJECTIVE PLANES

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Abstract. In 1967, Brown constructed small k-regular graphs of girth six as induced subgraphs of the incidence graph of a projective plane of order q, $q \geq k$. Examining the construction method, we prove that starting from $\mathrm{PG}(2,q)$, $q = p^h$, p prime, there are no other constructions using this idea resulting in a $(q+1-t)$-regular graph of girth six than the known ones, if t is not too large ($t \leq p$ and roughly $t < q^{1/6}/8$). Both algebraic and combinatorial tools are used.

1. Introduction

A (k,g)-*graph* is a simple k-regular graph of girth g. We denote by $c(k,g)$ the smallest number of vertices a (k,g)-graph may have. Determining $c(k,g)$ is very difficult for most values of k and g and has been studied by several authors. A (k,g)-graph with $c(k,g)$ vertices is called a (k,g)-*cage*. An easy combinatorial lower bound for $c(k,g)$ is given by the Moore bound, and graphs attaining equality in this bound are called Moore graphs. Since the present paper is about the $g = 6$ case, we only formulate the bound for this: $c(k,6) \geq 2((k-1)^2 + (k-1) + 1)$. It was shown by Kárteszi [13] that a Moore graph for the $g = 6$ case is the incidence graph of a projective plane of order $k-1$. Hence a Moore graph exists for $g = 6$ if and only if there exists a projective plane of order $k-1$. This is true in a more general setting: when $g \geq 6$ is even, then a Moore graph is the incidence graph of a so-called generalized $g/2$-gon of order $k-1$. These exist for $g = 6, 8, 12$ whenever $k-1$ is a prime power. We refer to the survey of Wong [17], to the dynamic survey of Exoo and Jajcay [9] and the web page of Royle [14] for further introduction and results on (k,g)-graphs, cages and Moore graphs.

Considering the cases $g = 6, 8, 12$, many papers have focused on constructing small (k,g)-graphs as induced regular subgraphs of the incidence graphs of generalized polygons. Some authors use $0/1$ matrices to construct the adjacency matrix of the (k,g)-graphs, but in many cases these turn out to give rise to subgraphs of generalized polygons ([1], [2], [3], [8], [11]). When $g = 6$, the generalized polygon is a projective plane. For these, essentially two constructions are known, which we will describe in the next section. Our main result (to be stated in Section 2) is that under certain conditions, no other constructions can be obtained from this technique. However, it should be mentioned already at this point, that we only consider induced subgraphs of the incidence graph. We will make some comments on this in the last section.

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András Gács died while we were working on this article; he made a major contribution in the result. By his early death we lost a brilliant colleague, an excellent Ph.D. supervisor and a good friend.

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The paper is organized as follows. In Section 2 we describe the construction method, list the known constructions by this method and state the result. The proof will be given in Section 3. The proof uses deep and technical results on (weighted) t-fold blocking sets. Referring to these can be avoided in a special case to be discussed in Section 4. Finally, in Section 5 we make some comments and discuss some open problems.

We end this introduction with some facts about projective planes. For proofs and more information, we refer to [12].

A *projective plane* is an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ consisting of a point-set $\mathcal{P}$, a line-set $\mathcal{L}$ and an incidence relation $\mathcal{I}$ between points and lines satisfying that any two lines have a unique point in common and any two points are incident with a unique line. One can prove that besides two degenerate examples, any finite projective plane has an order q with the property that any line is incident with $q+1$ points, any point is incident with $q+1$ lines and both the number of points and the number of lines are q^2+q+1. The two degenerate projective planes are the following.

π_1: there is an incident point-line pair (P, l) such that all points are incident with l and all lines are incident with P;

π_2: there is a non-incident point-line pair (P, l) such that every point except P is incident with l and every line except l is incident with P.

A *subplane* of a projective plane $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a subset $\mathcal{P}'$ of points and $\mathcal{L}'$ of lines of the original plane such that $(\mathcal{P}', \mathcal{L}', \mathcal{I}')$ is a projective plane on its own right, where $\mathcal{I}'$ is the restriction of $\mathcal{I}$ on the pair $(\mathcal{P}', \mathcal{L}')$. The subplane is *degenerate*, if it is π_1 or π_2 (that is, it is degenerate as a projective plane). The order of a non-degenerate subplane is always at most the square root of the order of the big plane.

It is easy to see that any non-degenerate projective plane has degenerate subplanes of both types. A degenerate subplane of type π_1 is an incident point-line pair (P, l) together with some other points on l and some other lines through P. A degenerate subplane of type π_2 is a non-incident point-line pair (P, l) together with some more points on l and the lines joining P to these points. In a finite projective plane the numbers of points and lines are equal unless the plane is degenerate of type π_1.

We denote by $\mathrm{PG}(2, q)$ and $\mathrm{AG}(2, q)$ the desarguesian projective and affine plane of order q, respectively, i.e., the projective or affine plane over the Galois field of q elements, $\mathrm{GF}(q)$, where q is a power of the prime p. It is well known that all non-degenerate subplanes of $\mathrm{PG}(2, p^h)$ have order p^l where l divides h. Thus $\mathrm{PG}(2, p)$, p prime, does not have non-degenerate subplanes. On the other hand, for any square prime power q, $\mathrm{PG}(2, q)$ contains non-degenerate subplanes of order $\sqrt{q}$, which are called *Baer subplanes*. Moreover, one can partition the point-set of $\mathrm{PG}(2, q)$ into the point-sets of $q - \sqrt{q} + 1$ disjoint Baer subplanes. A non-degenerate subplane in $\mathrm{PG}(2, q)$, $q = p^h$ has at least $p^2 + p + 1$ points. Two distinct Baer subplanes in a finite projective plane of order q may have at most $\sqrt{q} + 2$ points in common [7].

If we consider $\mathrm{AG}(2, q)$ embedded into $\mathrm{PG}(2, q)$, then we call the line in $\mathrm{PG}(2, q)$ outside $\mathrm{AG}(2, q)$ *the line at infinity*, and we call its points *ideal points*. The common (ideal) point of vertical lines or lines of slope m will be denoted by (∞) and (m), respectively. Note that the axioms of a projective plane are symmetric, so there is a duality between points and lines; the *dual* of the plane $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is simply $(\mathcal{L}, \mathcal{P}, \mathcal{I})$. The dual of $\mathrm{PG}(2, q)$ is isomorphic to $\mathrm{PG}(2, q)$.

Furthermore we will use results on (weighted) multiple blocking sets of $\mathrm{PG}(2, q)$. A *t-fold blocking set* in a projective plane is a set of points that intersects every line in at least t points. A *t-fold weighted blocking set* is a set of points with positive integer weights (i.e., multiplicities) such that the sum of the weights of the points on an arbitrary line is at least t. If B is a weighted set with weight w we write $|B| = \sum_{b \in B} w(b)$. One-fold blocking sets are called *blocking sets*. The dual of a t-fold blocking set is a set of lines covering every point at least t times, i.e., a *t-fold covering set*. Thus if we have a result on t-fold (weighted) blocking sets in $\mathrm{PG}(2, q)$, then we have the dual result for t-fold covering sets as well.

2. t-good structures and the main theorem

For a non-degenerate projective plane of order q, the incidence graph (between lines and points) is easily seen to be a $(q + 1, 6)$ graph. As was mentioned in the introduction, this is the smallest possible construction for a $(q + 1, 6)$ graph. When there is no projective plane of order $k - 1$, one possibility to construct a $(k, 6)$ graph is to take the first $q \geq k$ for which a projective plane of order q exists, consider the incidence graph of the plane and try to delete vertices of this graph to make it k-regular. It is easy to see that for this we need the following structure within the plane.

Definition 2.1. *Let $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ be a pair of a point-set $\mathcal{P}_0$ and a line-set $\mathcal{L}_0$ such that*

- *$\forall P \notin \mathcal{P}_0$ there are exactly t lines in $\mathcal{L}_0$ through P,*
- *$\forall l \notin \mathcal{L}_0$ there are exactly t points in $\mathcal{P}_0$ on l.*

Then $\mathcal{T}$ is called a t-good structure*.*

Notation. If $\mathcal{T}$ is a fixed t-good structure, a line $l \in \mathcal{L}_0$ (a point $P \in \mathcal{P}_0$) will be called also a $\mathcal{T}$-line (a $\mathcal{T}$-point).

It is straightforward to check that to get a $(q + 1 - t)$-regular induced subgraph of the incidence graph, we have to delete vertices corresponding to points and lines of a t-good structure. The larger the point- and line-set of a t-good structure is, the smaller $(q + 1 - t, 6)$-graph we get (which is the goal usually). When deleting a t-good structure $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ from the incidence graph of the projective plane, we start from and result in a regular bipartite graph, whence $|\mathcal{P}_0| = |\mathcal{L}_0|$ follows. The *size* of $\mathcal{T}$ is defined as $|\mathcal{T}| = |\mathcal{P}_0| = |\mathcal{L}_0|$.

To review the known constructions of t-good structures, we need a definition first.

Definition 2.2. *A point P (or a line l) is $\mathcal{T}$-*complete *if P and all the lines through P (l and all the points on l) are in $\mathcal{T}$. We may also say that P (l) is completely in $\mathcal{T}$.*

A remark on the usage of expressions. In this paper a line is usually distinguished from the set of points it is incident with. Thus the phrase "l is in $\mathcal{T}$" does not mean that the points of l are in $\mathcal{P}_0$. To indicate the latter phenomenon, we will say "put the line l into $\mathcal{T}$ completely".

Essentially two types of t-good structures are known when $t < \sqrt{q}$ (see also [11]).

- **Construction 1: complete subplanes.** Take a (possibly degenerate) subplane which has t points and t lines, and put all its points and lines into $\mathcal{T}$ completely.
- **Construction 2: disjoint Baer subplanes.** Let $\mathcal{P}_0$ and $\mathcal{L}_0$ be the union of the points or the lines of t disjoint Baer subplanes, respectively.

Considering Construction 1, lines that intersect the point-set of the subplane in at least two points are $\mathcal{T}$-complete, hence through a point $P \notin \mathcal{P}_0$ there are precisely t lines intersecting the point-set of the subplane, which are precisely the $\mathcal{T}$-lines through P. Dually, the same can be said from the viewpoint of the lines, thus Construction 1 gives a t-good structure. Construction 2 can be seen to be t-good using the fact that a line may intersect only one of two disjoint Baer subplanes in more than one points. We refer to the graphs obtained by deleting a t-good structure according to Construction 1 or 2 as a *completely deleted subplane (CDS)* (also as a completely deleted degenerate subplane (CDDS) if the underlying subplane is degenerate) or *deleted Baer subplanes (DBS)*, respectively.

Historically, Brown [8] (1967) constructed $(k, 6)$-graphs as a CDDS with an underlying subplane of type π_1. Abreu, Funk, Labbate and Napolitano [1] (2006) used the Cayley tables of $\mathrm{GF}(q)$ to construct the adjacency matrices of $(k, 6)$-graphs. They gave two infinite families that can be rephrased as CDDSs of type π_1 and π_2 starting from $\mathrm{PG}(2, q)$. Moreover, for $q = 4, 9, 16$ they used the partition of $\mathrm{PG}(2, q)$ into Baer subplanes to construct graphs of the same size as a DBS. The notion of t-good structures was introduced in [11] (2008), where the translation of matrix techniques into geometry is also illustrated. Construction 2 was also given there for general square prime power q.

A little calculation shows that the size of the t-good structure we get from Construction 1 using a degenerate subplane of type π_1 or π_2, or a non-degenerate subplane of order t_1, where $t = t_1^2 + t_1 + 1$, is $tq + 1$, $tq - t + 3$ and $tq - (t_1 - 1)t$, respectively. From Construction 2 we get $|\mathcal{T}| = t(q + \sqrt{q} + 1)$.

Hence the second construction is much better than any type of the first one, but needs the existence of Baer subplanes in the projective plane of order q. By the remarks at the end of the introduction, whenever q is square, we can use $\mathrm{PG}(2, q)$, where the second construction is possible.

When q is not a square, we have to use the first construction, which is always better with a degenerate subplane than with a non-degenerate one, and for $t = 1$ the better one is to start from a subplane of type π_2; for $t \geq 3$ the better one is to start from a subplane of type π_1; finally, for $t = 2$ the two types are the same.

We remark that for $t < (q + 1)/2$, a t-good structure constructed by putting t points and t lines completely into it (and nothing else) is always a complete subplane. To see this, consider such a t-good structure $\mathcal{T}$, and let $\mathcal{P}^* = \{P_1, \ldots, P_t\}$ and $\mathcal{L}^* = \{l_1, \ldots, l_t\}$ be the set of points and lines that are to be put into $\mathcal{T}$ completely. Take a line connecting two (or more) of the P_is, and suppose that there is a point P on it that is not in $\mathcal{P}_0$. Then P is not incident with any $\ell \in \mathcal{L}^*$, hence there are less than t $\mathcal{T}$-lines through P, a contradiction. Hence lines connecting two points of $\mathcal{P}^*$ must be $\mathcal{T}$-complete. A line $l \notin \mathcal{L}^*$ may intersect $\mathcal{P}^*$ in at most t points and may contain t further $\mathcal{T}$-points on the l_is, hence $2t < q + 1$ implies that the set of $\mathcal{T}$-complete lines is $\mathcal{L}^*$. Together with the dual of this argument, we get that $(\mathcal{P}^*, \mathcal{L}^*)$ is a subplane.

In [11] the following was proved.

THEOREM 2.3 ([11]). *Let $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ be a t-good structure in a projective plane of order q.*

(i) *If $t \le 2\sqrt{q}$, then $|\mathcal{P}_0| = |\mathcal{L}_0| \le t(q + \sqrt{q} + 1)$, and in case of equality $\mathcal{P}_0$ meets every line in either t or $\sqrt{q} + t$ points. Moreover, if the plane is* $\mathrm{PG}(2, q)$ *and $t < \sqrt[4]{q}/2$, then equality holds only for Construction 2;*
(ii) *If $t = 1$, then every 1-good structure is one of the constructions above;*
(iii) *If $t = 2$, the plane is* $\mathrm{PG}(2, q)$, *and $q > 256$, then every 2-good structure is one of the constructions above;*
(iv) *If $t < \sqrt{q}$, then $\mathcal{P}_0$ is a blocking set.*

By the above result, one cannot get larger t-good structures than Construction 2 if t is not too large. The purpose of this paper is to describe all t-good structures in $\mathrm{PG}(2, q)$ (i.e., the $(q + 1 - t)$-regular induced subgraphs of the incidence graph of $\mathrm{PG}(2, q)$), provided that t is not too large. The following theorem will be proved.

THEOREM 2.4. *Let p be a prime and let $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ be a t-good structure in* $\mathrm{PG}(2, q)$, *$q = p^h$; furthermore,*

- *for $h = 1$ and $h = 2$, let $t < p^{1/2}/2$;*
- *for $h \ge 3$, let $t < \min\{p + 1, c_p q^{1/6} - 1, q^{1/4}/2\}$, where $c_2 = c_3 = 1/8$ and $c_p = 1$ for $p > 3$.*

Then $\mathcal{T}$ is a complete (degenerate) subplane or the union of t disjoint Baer subplanes.

One may ask whether the bounds of the above theorem are tight. As the authors of the present paper do not know of t-good structures for $t < \sqrt{q}$ other than the listed ones, we can not give a definite answer to this question. Further comments are made in Section 5.

3. PROOF OF THEOREM 2.4

Throughout this section $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ will denote a t-good structure in $\mathrm{PG}(2, q)$. We will suppose $t \ge 2$ (as Theorem 2.3 (ii) proves the $t = 1$ case).

DEFINITION 3.1. *We call a line* bad *if it does not intersect $\mathcal{P}_0$ in $t \mod p$ points. Dually, we call a point* bad *if it does not have $t \mod p$ lines from $\mathcal{L}_0$ through it. A point (line) is* good *if it is not bad.*

Clearly, every line not in $\mathcal{L}_0$ is good (as it intersects $\mathcal{P}_0$ in exactly t points); in other words, the bad lines are in $\mathcal{L}_0$. However, lines of $\mathcal{L}_0$ are not necessarily bad (see Figure 1). The dual observations stand for points as well.

Note that if we supposed that $q = p$ prime, then in the above definition $t \mod p$ and exactly t would be the same (assuming $2 \le t < p$).

DEFINITION 3.2. *Let the* index *of a point P be the number of bad lines going through it. Dually, the* index *of a line l is the number of bad points on it. We denote the index of a point P or a line l by* $\mathrm{ind}(P)$ *and* $\mathrm{ind}(l)$, *respectively. For the sake of simplicity, the index of the ideal point (m) will be denoted by* $\mathrm{ind}(m)$ *instead of* $\mathrm{ind}((m))$.

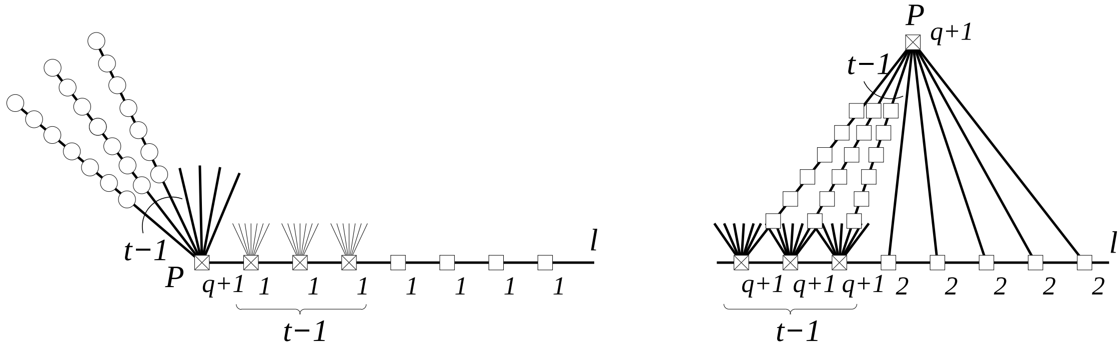

Figure 1. *We see the schematic pictures of two t-good structures, a complete degenerate subplane of type π_1 and π_2, respectively. Every depicted object is in $\mathcal{T}$. Good lines are thin, bad lines are thick; good points are round, bad points are square, $\mathcal{T}$-complete points are crossed. The numbers next to the points are their indices. Note that in case of the left construction, being bad and being in $\mathcal{T}$ are not equivalent; moreover, not every $\mathcal{T}$-complete point has large index.*

Next our aim is to show that the indices of the points are either quite small (at most t), or relatively large (at least $q+1-t$). First we will introduce a polynomial that encodes the intersection multiplicity of $\mathcal{P}_0$ with lines.

Let ℓ_∞ denote the line at infinity in an affine coordinate system. Let $\{(a_v, b_v)\}_v$ be the set of affine points of $\mathcal{T}$ (in this coordinate system), and let $\{(y_i)\}_i$ be the set of points of $\mathcal{T}$ on ℓ_∞. Consider the following polynomial in $\mathrm{GF}(q)[X,Y]$:

$$g(X,Y) = \sum_{v=1}^{|\mathcal{P}_0 \setminus \ell_\infty|} \left(1-(X+a_vY-b_v)^{q-1}\right) + \sum_{y_i \in \mathcal{P}_0 \cap \ell_\infty} \left(1-(Y-y_i)^{q-1}\right) - t = \sum_{i=0}^{q-1} r_i(Y) X^{q-1-i}.$$

Let $m, b \in \mathrm{GF}(q)$ and let l be the line defined by $Y = mX + b$. Then $g(b,m) = |l \cap \mathcal{P}_0| - t \pmod p$, as a term of the first or the second sum equals one iff the corresponding affine point (a_v, b_v) or the ideal point (y_i) is on l, respectively. Note that $\deg r_i \le i\ \forall 0 \le i \le q-1$.

Lemma 3.3. *Assume that ℓ_∞ (the line at infinity) is good. Then for any point $(y) \in \ell_\infty$, $\mathrm{ind}(y) = q - \deg \gcd\left(g(X,y), X^q - X\right)$.*

Proof. Since x is a root of $g(X,y)$ iff the line $Y = yX + x$ intersects $\mathcal{P}_0$ in t mod p points, then the number of affine good lines is the degree of the greatest common divisor. □

To be able to say something about the possible indices, we will need the following algebraic result, for more details see [16] or [15].

Notation. For $u \in \mathbb{R}$, let $u^+ = \max\{0, u\}$.

Result 3.4 (Szőnyi-Weiner). *Suppose that the polynomials $u(X,Y) = \sum_{i=0}^{n} u_i(Y) X^{n-i}$ and $v(X,Y) = \sum_{i=0}^{n-m} v_i(Y) X^{n-m-i}$, $m > 0$, satisfy $\deg u_i(Y) \le i$ for all $0 \le i \le n$, $\deg v_i(Y) \le i$ for all $0 \le i \le n-m$, and $u_0 \ne 0$.*

For $y \in \mathrm{GF}(q)$*, let* $k_y = n - \deg(\gcd(u(X,y), v(X,y)))$*. Then for any* $y \in \mathrm{GF}(q)$*,*

$$\sum_{y' \in \mathrm{GF}(q)} (k_y - k_{y'})^+ \leq k_y(k_y - m).$$

□

As a corollary of the above result and lemma, we can estimate the possible values of indices.

Proposition 3.5. *Let* k *be the index of a point or a line and let* $t \leq \sqrt{q}/2$*. Then either* $k \leq t$ *or* $k \geq q+1-t$*.*

Proof. By duality it is enough to prove the statement for the index of points only. Let δ denote the number of bad lines. Note that $\delta \leq |\mathcal{L}_0| \leq t(q+\sqrt{q}+1)$ (Theorem 2.3 (i)). Pick an arbitrary point P. If there is no good line through P, then $\mathrm{ind}(P) = q+1$ and there is nothing to prove. Otherwise choose our coordinate system so that ℓ_∞ is a good line through P and that $P = (y_0)$ is an ideal point different from (∞). Then $\delta \geq \sum_{y \in \mathrm{GF}(q)} \mathrm{ind}(y)$, as on the right-hand side we do not count the bad lines through (∞). Let $u(X,Y) = X^q - X$, $v(X,Y) = g(X,Y)$. Then Lemma 3.3 shows that $\mathrm{ind}(y) = \deg u - \deg\gcd(u(X,y), v(X,y))$, hence by Result 3.4 we get

$$q \cdot \mathrm{ind}(y_0) - \delta \leq \sum_{y \in \mathrm{GF}(q)} (\mathrm{ind}(y_0) - \mathrm{ind}(y)) \leq$$
$$\sum_{y \in \mathrm{GF}(q)} (\mathrm{ind}(y_0) - \mathrm{ind}(y))^+ \leq \mathrm{ind}(y_0)(\mathrm{ind}(y_0) - 1).$$

This implies that

$$\mathrm{ind}(P)(q+1-\mathrm{ind}(P)) \leq \delta. \tag{3.1}$$

As $\delta \leq t(q+\sqrt{q}+1)$, one can deduce that $\mathrm{ind}(P) \leq t$ or $\mathrm{ind}(P) \geq q+1-t$ (using $2 \leq t \leq \sqrt{q}/2$ and $t \in \mathbb{N}$). □

Now we see that the points (and the lines) can be split into two groups: ones with small and others with large index.

Definition 3.6. *The index of a point or a line is* large *(*small*) if it is at least* $q+1-t$ *(at most* t*).*

We shall examine points and lines with large index now.

Proposition 3.7. *If* $t \leq \sqrt{q}/2$*, then points and lines with large index are* $\mathcal{T}$*-complete.*

Proof. By duality it is enough to prove the proposition for points. Suppose to the contrary that there is a point P with large index and a line $l \notin \mathcal{L}_0$ passing through P. Thus $|l \cap \mathcal{P}_0| = t$. We count the number of $\mathcal{T}$-lines through the points of l. On each of the $q+1-t$ points of $l \setminus \mathcal{P}_0$ we see exactly t $\mathcal{T}$-lines and at least $q+1-t$ more through P (since the bad lines are in $\mathcal{L}_0$). Thus $|\mathcal{L}_0| \geq (q+1-t)t + q + 1 - t$, but this contradicts the upper bound in Theorem 2.3 (i) if $t \leq \sqrt{q}/2$. □

Proposition 3.8. *Suppose $t \leq \sqrt{q}/2$. Then the number of $\mathcal{T}$-complete points is at most t. Dually, the number of $\mathcal{T}$-complete lines is at most t.*

Proof. By duality it is enough to prove the proposition for points. Suppose to the contrary that there exist $t+1$ $\mathcal{T}$-complete points. Then the number of $\mathcal{T}$-lines through these is at least $(t+1)(q+1) - \binom{t+1}{2}$, thus by Theorem 2.3 (i) $(t+1)(q+1) - \binom{t+1}{2} \leq t(q+\sqrt{q}+1)$. This gives $2(q+1) \leq t(t+2\sqrt{q}+1)$, which contradicts $t \leq \sqrt{q}/2$. □

Proposition 3.9. *Suppose $t < \sqrt{q}$. Then the points with large index block the bad lines (that is, every bad line is incident with at least one point with large index). Dually, lines with large index cover the bad points.*

Proof. Suppose to the contrary that there exists a bad line l on which every point has index at most c, where $c \leq t$, and suppose that there exists a point P on l with $\mathrm{ind}(P) = c$. Then the total number δ of bad lines is at most $(c-1)(q+1)+1$. Then using inequality (3.1) we get

$$0 \leq c^2 - c(q+1) + (c-1)(q+1) + 1 = c^2 - q,$$

a contradiction since $c \leq t < \sqrt{q}$. □

Note that if $t \not\equiv 1 \pmod{p}$ then the existence of a point with large index is equivalent to the existence of a line with large index (if t is small enough to use the propositions). For instance a point P with large index is $\mathcal{T}$-complete (Proposition 3.7), that is, all lines through it are in $\mathcal{T}$, hence the number of $\mathcal{T}$-lines through it is $1 \mod p$, thus P is bad. On this bad point there should exist a line with large index (Proposition 3.9).

Proposition 3.10. *Suppose that $t \leq \sqrt{q}/2$ and also $t \leq p$. Then the line joining two $\mathcal{T}$-complete points has large index. Dually, the intersection point of two $\mathcal{T}$-complete lines has large index.*

Proof. Let P_1 and P_2 be two $\mathcal{T}$-complete points and denote by l the line joining them. As $2 \leq t \leq p$, P_1 and P_2 are bad. Suppose to the contrary that the index of l is at most t. Then there are at least $q+1-t$ good points on l, each having $t \mod p$, thus at least t $\mathcal{T}$-lines through them (here we use $t \leq p$ and l being a $\mathcal{T}$-line). Since l is a common $\mathcal{T}$-line on each of these points, we can deduce that $|\mathcal{L}_0| \geq (q+1-t)(t-1)+2q+1$, but this contradicts the upper bound $|\mathcal{L}_0| \leq t(q+\sqrt{q}+1)$ (Theorem 2.3 (i)) if $t \leq \sqrt{q}/2$. □

Corollary 3.11. *Suppose that $t \leq \sqrt{q}/2$ and also $t \leq p$. Let $\mathcal{P}'$ be either the set of $\mathcal{T}$-complete points or the set of points with large index. Let $\mathcal{L}'$ be either the set of $\mathcal{T}$-complete lines or the set of lines with large index. Then $(\mathcal{P}', \mathcal{L}')$ is a (possibly degenerate) subplane.*

Proof. We only need to check whether the intersection of two lines of $\mathcal{L}'$ is in $\mathcal{P}'$, and if the line joining two points of $\mathcal{P}'$ is in $\mathcal{L}'$. As points and lines with large index are $\mathcal{T}$-complete (Proposition 3.7), this follows from Proposition 3.10 in all the four cases. □

In Construction 1, the subplane formed by the $\mathcal{T}$-complete points and lines has t points and t lines, while in Construction 2 it is empty. In the proof of the main theorem, we will verify this property with the help of weighted t-fold blocking sets.

PROPOSITION 3.12. *Suppose that* $(\mathcal{P}_0, \mathcal{L}_0)$ *is a t-good structure in* $\mathrm{PG}(2,q)$, $t \le \sqrt{q}/2$, *and* $t \le p$. *Giving weight t to points with large index and weight one to the other points of* $\mathcal{P}_0$, *we obtain a weighted t-fold blocking set.*

Proof. By Proposition 3.9, there is at least one point with large index (thus with weight t) on each bad line. On the other hand, every good line is a $t \pmod p$ secant to $\mathcal{P}_0$, thus by $t \le p$ and $\mathcal{P}_0$ being a blocking set (Theorem 2.3 (iv)), a good line intersects $\mathcal{P}_0$ in a positive number of, hence in at least t points. □

REMARK 3.13. *If there are no points (and thus lines) with large index, then the above proposition yields that* $\mathcal{P}_0$ *is a t-fold blocking set (without weights).*

Weighted blocking sets were studied, e.g., in [10] and [5]. In our notion, the sum of some point-sets is a weighted set in which the weight of a point P is the number of sets in the sum in which P is contained. We will need the following results.

RESULT 3.14. ([10], Theorems 2.5, 2.13 and Proposition 2.15) *Let B be a weighted k-fold blocking set in* $\mathrm{PG}(2,p)$, *p prime, with* $|B| = kp + k + r$, $k + r < (p+2)/2$. *Then B contains the sum of k (not necessarily different) lines (considered as point-sets).*

RESULT 3.15. ([5], Theorem 3.10) *Let B be a weighted k-fold blocking set in* $\mathrm{PG}(2,q)$, $q = p^h$, *p prime,* $h > 1$. *Let* $c_2 = c_3 = 2^{-1/3}$ *and* $c_p = 1$ *for* $p > 3$. *Assume that* $|B| = kq + k + c - (k-1)(k-2)/2$, *where*

(1) $c < c_p q^{2/3}$ *and* $k < \min\{c_p q^{1/6}, q^{1/4}/2\}$;
(2) $q = p^2$, $k < q^{1/4}/2$ *and* $c < q^{3/4}/2$.

If the number of simple points (i.e., points with weight one) is at least $(k-2)(q+\sqrt{q}+1) + 16\sqrt{q} + 8q^{1/4}$ *in (1) and at least* $(k-2)(q+\sqrt{q}+1) + 16\sqrt{q} + 16q^{1/6}$ *in (2), then B contains the sum of the point-sets of k (not necessarily different) Baer subplanes and/or lines (considered as point-sets).*

We will need one more simple observation.

PROPOSITION 3.16. *Let* $1 \le t \le q/2$, $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ *and* $\mathcal{T}' = (\mathcal{P}'_0, \mathcal{L}'_0)$ *be two t-good structures. Then* $\mathcal{P}_0 \subset \mathcal{P}'_0$ *and* $\mathcal{L}_0 \subset \mathcal{L}'_0$ *implies* $\mathcal{T} = \mathcal{T}'$.

Proof. Suppose to the contrary that $\mathcal{T} \ne \mathcal{T}'$. By duality we may assume that $\mathcal{L}_0$ is a proper subset of $\mathcal{L}'_0$. Take a line $l \in \mathcal{L}'_0 \setminus \mathcal{L}_0$. Then l contains exactly $q+1-t$ points, $P_1, \ldots, P_{q+1-t}$, that are not in $\mathcal{P}_0$. Thus these are covered exactly t times by the lines of $\mathcal{L}_0$, that is, there are $q+1-t$ lines $\notin \mathcal{L}_0$ through each of the P_is ($i = 1, \ldots, q+1-t$) intersecting $\mathcal{P}_0$ in exactly t points. The set S of these lines has $(q+1-t)^2$ pairwise distinct elements, and $S \cap \mathcal{L}_0 = \emptyset$. As $l \in \mathcal{L}'_0$ and $\mathcal{L}_0 \subset \mathcal{L}'_0$, the P_is are covered at least $t+1$ times by $\mathcal{L}'_0$, hence they are in $\mathcal{P}'_0$. Thus the lines of S intersect $\mathcal{P}'_0$ in at least $t+1$ points, hence $S \subset \mathcal{L}'_0$, therefore the P_is ($1 \le i \le q+1-t$) are completely in $\mathcal{T}'$. Thus any point $P \notin l$ is covered at least $(q+1-t)$-times by $\mathcal{L}'_0$, which is more than t as $t < (q+1)/2$. This means that every point of the plane is in $\mathcal{P}'_0$, a contradiction. □

Proof of Theorem 2.4

Recall that the assumptions on $(2 \le)t \in \mathbb{N}$ are the following: if $q = p^h$, then assume $t < p^{1/2}/2$ for $h = 1, 2$, and $t < \min\{p + 1, c_p q^{1/6} - 1, q^{1/4}/2\}$ for $h \ge 3$, where $c_2 = c_3 = 1/8$ and $c_p = 1$ for $p > 3$. At this point we have to assume a somewhat stronger bound on t in case of $h = 1$, say, $t < \sqrt{q}/3$, mainly to satisfy the conditions of Result 3.14. In Section 4, we will refine the proof so that the original bound will be enough.

Step 1: $\mathcal{T}$ contains k $\mathcal{T}$-complete points and lines and $t-k$ Baer subplanes for some k ($0 \le k \le t$), which will be seen to be well defined in Step 2.

Counting the $\mathcal{T}$-lines through the points of a non $\mathcal{T}$-line we get that $|\mathcal{P}_0| = |\mathcal{L}_0| \ge t(q + 1 - t) = tq + t - t^2$. The number of $\mathcal{T}$-complete points is at most t (Proposition 3.8), thus giving weight t to the points with large index (which are $\mathcal{T}$-complete by Proposition 3.7) we obtain a weighted t-fold blocking set $\mathcal{P}_0^w$ (Proposition 3.12) with $|\mathcal{P}_0^w| \le |\mathcal{P}_0| + t(t-1) \le tq + t + t(\sqrt{q} + t - 1)$ (Theorem 2.3 (i)), in which at least $|\mathcal{P}_0| - t \ge tq - t^2$ points are simple. Dually, giving weights analogously to the lines of $\mathcal{L}_0$ we obtain a weighted t-fold covering set $\mathcal{L}_0^w$. The assumptions on t and q yield that we may use Result 3.14 and 3.15 together with their duals to see that $\mathcal{P}_0^w$ contains the sum of the point-sets of k lines $l_1, \dots, l_k$, and $t - k$ Baer subplanes $B_1^P, \dots, B_{t-k}^P$, while $\mathcal{L}_0^w$ contains the sum of the line-sets of k' points $P_1, \dots, P_{k'}$ (by the line-set of a point P we mean the set of lines incident with P) and $t - k'$ Baer subplanes $B_1^L, \dots, B_{t-k'}^L$. (Here a subplane is considered as a pair of a point-set and a line-set.)

Note that by the definitions of $\mathcal{P}_0^w$ and $\mathcal{L}_0^w$, the only points and lines in $\mathcal{P}_0^w$ and $\mathcal{L}_0^w$ with weight more than one are those with large index.

Let $\mathcal{P}^* = \{P_1, \dots, P_{k'}\}$, $\mathcal{B}_L^* = \{B_1^L, \dots, B_{t-k'}^L\}$, $\mathcal{L}^* = \{l_1, \dots, l_k\}$, and $\mathcal{B}_P^* = \{B_1^P, \dots, B_{t-k}^P\}$. Note that the elements of $\mathcal{P}^*$ and $\mathcal{L}^*$ are $\mathcal{T}$-complete. Moreover, as the line-set of a Baer subplane $B \in \mathcal{B}_L^*$ is in $\mathcal{L}_0$ and it covers all the points of B $\sqrt{q} + 1 > t$ times, the point-set of B is contained in $\mathcal{P}_0$ (and dually as well). However, in principle it could happen that $B \notin \mathcal{B}_P^*$. Next we show that this is not the case.

Step 2: There are no other $\mathcal{T}$-complete points, $\mathcal{T}$-complete lines, or Baer subplanes contained in $\mathcal{T}$ than the above found ones.

Let S be a line or a Baer subplane whose point-set is contained in $\mathcal{P}_0$. We show that $S \in \mathcal{L}^*$ or $S \in \mathcal{B}_P^*$. Suppose to the contrary. Then the union of the point-sets of the elements of $\mathcal{L}^*$ and $\mathcal{B}_P^*$ contains at least $t(q+1) - t(t-1)$ points (in $\mathcal{P}_0$, without multiplicities), as any one of them has at least $q + 1$ points, and at most t points may be in more than one of them (as $\mathcal{P}_0^w$ contains their sum, and there are at most t points with weight more than one in $\mathcal{P}_0^w$). Recall that the intersection of two lines, a line and a Baer subplane, or two Baer subplanes contains at most 1, $\sqrt{q} + 1$, or $\sqrt{q} + 2$ points, respectively (for the last fact see [7]). Thus, as $|S| \ge q + 1$, S adds at least $(q + 1) - t(\sqrt{q} + 2)$ new points to the union, whence $|\mathcal{P}_0| \ge (t + 1)(q + 1) - t(t - 1) - t(\sqrt{q} + 2)$. Compared with the upper bound $|\mathcal{P}_0| \le t(q + \sqrt{q} + 1)$ (Theorem 2.3 (iv)) and considering the assumed upper bounds on t, we get a contradiction. Together with the dual of this argument, we obtain the stated result, which yields $\mathcal{B}_L^* = \mathcal{B}_P^*$ and $k = k'$.

Step 3: $k = 0$ or $k = t$.

Suppose to the contrary that there is a $\mathcal{T}$-complete line l and a Baer subplane B as well in $\mathcal{T}$. As (the point-set of) a Baer subplane is a blocking set, there exists a point P in

$l \cap B$. As $l \in \mathcal{L}^*$ and $B \in \mathcal{B}_P^*$, P has weight at least two in $\mathcal{P}_0^w$. Thus P has large index, hence it is $\mathcal{T}$-complete (Proposition 3.7), consequently $P \in \mathcal{P}^*$. Therefore, as $B \in \mathcal{B}_L^*$ also holds, $\mathcal{L}_0^w$ contains the sum of the line-sets of P and B, thus the $\sqrt{q}+1 > t$ lines through P belonging to B have weight at least two in $\mathcal{L}_0^w$, hence they are $\mathcal{T}$-complete. However, the number of $\mathcal{T}$-complete lines is at most t (Proposition 3.8), a contradiction.

Step 4: $\mathcal{T}$ is a complete subplane or the union of t disjoint Baer subplanes.

Recall that $\mathcal{T}$-complete points and lines form a subplane (Corollary 3.11). As $k = 0$ or $k = t$, then $\mathcal{T}$ either contains the union of t disjoint Baer subplanes (as $\mathcal{P}_0^w$ contains the sum of the Baer subplanes, a point in the intersection would have weight at least two, hence it would be $\mathcal{T}$-complete) or a $\mathcal{T}$-complete subplane $(\mathcal{P}^*, \mathcal{L}^*)$ on t points and lines, which is degenerate as $t \leq p$. Both of these are t-good structures, thus $\mathcal{T}$ contains a t-good structure $\mathcal{T}'$. By Proposition 3.16 this may happen only if $\mathcal{T} = \mathcal{T}'$. □

4. The case $q = p$

If $q = p$ prime, then we can avoid referring to the cited results on weighted t-fold blocking sets. Supposing $q = p$ prime and $t \geq 2$, every good line (a $t \mod p$ secant) must intersect $\mathcal{P}_0$ in exactly t points, which is quite a strong property, yet we will need a lemma that is interesting in itself as well.

Lemma 4.1. *Let $\mathcal{L}$ be a set of non-vertical lines of* $\mathrm{AG}(2,q)$*, q a power of the prime p, which cover every point of* $\mathrm{AG}(2,q)$ *exactly k times ($k \geq 1$) except possibly the points of ν fixed vertical lines, where $\nu(k+1) \leq p$. Then $\mathcal{L}$ consists of the union of k parallel classes or $\mathcal{L}$ consists of the kq non-vertical lines passing through k fixed points on a fixed vertical line.*

Proof. Counting the lines of $\mathcal{L}$ through the points of a non-exceptional vertical line we see that $|\mathcal{L}| = kq$. Suppose $\nu = 0$. Fix an arbitrary line $l \in \mathcal{L}$, and consider $\mathrm{AG}(2,q)$ embedded into $\mathrm{PG}(2,q)$ (and also extend the lines by their ideal points). As the affine points of l are covered exactly k times by $\mathcal{L}$, there are $kq - q(k-1) = q$ lines of $\mathcal{L}$ incident with the ideal point of l, whence the statement follows.

Suppose $\nu \geq 1$. Since the lines of $\mathcal{L}$ are non-vertical, they are given by the equations $Y + m_iX + b_i = 0$, where $i \in \{1, \ldots, kq\}$. Consider the following polynomial (over $\mathrm{GF}(q)$):

$$f(X,Y) = \prod_{i=1}^{kq}(Y + m_iX + b_i).$$

Then $\deg_{X,Y} f = kq$. Let $S \subset \mathrm{GF}(q)$ be the subset of $q - \nu$ elements x for which on the vertical line $X = x$ every point is covered exactly k-times. Consider the polynomial

$$g(X) = \prod_{x \in S}(X - x).$$

Then $\deg_X g = q-\nu$. The elements of $S \times \mathrm{GF}(q)$ are zeros of the algebraic curve $f(X,Y) = 0$ with multiplicity k, thus by the Combinatorial Nullstellensatz with multiplicities (see [4]) we get

$$f(X,Y) = (Y^q - Y)^k + \ldots + f_i(X,Y)(Y^q - Y)^{k-i}g(X)^i + \ldots + f_k(X,Y)g(X)^k,$$

where $\deg_{X,Y} f_i \le kq - q(k-i) - (q-\nu)i = i\nu$ for all $1 \le i \le k$. Fix some arbitrary $x \notin S$. Then $g(x) \ne 0$, and including the arising constants into the f_is we get

$$f(x,Y) = (Y^q - Y)^k + f_1(x,Y)(Y^q - Y)^{k-1} + \ldots + f_k(x,Y).$$

Note that since the multiplicity of the root y is the number of lines of $\mathcal{L}$ passing through $(x,y) \in \mathrm{AG}(2,q)$, no root of $f(x,Y)$ can have multiplicity larger than q.

Now let l be the largest integer for which $f_l(x,Y) \not\equiv 0$ $(0 \le l \le k)$. Thus

$$f(x,Y) = (Y^q - Y)^k + f_1(x,Y)(Y^q - Y)^{k-1} + \ldots + f_l(x,Y)(Y^q - Y)^{k-l}.$$

Consider

$$f^*(x,Y) = \frac{f(x,Y)}{(Y^q - Y)^{k-l}} = (Y^q - Y)^l + (Y^q - Y)^{l-1} f_1(x,Y) + \ldots + f_l(x,Y).$$

Let R denote the set of distinct roots of $f^*(x,Y)$. Then $r(y) = \prod_{y \in R}(Y - y)$ divides $Y^q - Y$, and hence $f_l(x,Y)$ as well. Thus $|R| \le \deg f_l(x,Y) \le l\nu$. Let $'$ denote the derivation in the variable Y. As every root of $f^*(x,Y)$ is root of $f^{*\prime}(x,Y)$ with one less multiplicity, $\deg_Y f^{*\prime}(x,Y) \ge \deg_Y f^*(x,Y) - |R| \ge lq - l\nu \ge lq - k\nu$ holds except if $f^{*\prime}(x,Y) \equiv 0$. On the other hand,

$$f^{*\prime}(x,Y) = -l(Y^q - Y)^{l-1} + f_1'(x,Y)(Y^q - Y)^{l-1} - (l-1)f_1(x,Y)(Y^q - Y)^{l-2} + \ldots$$
$$\ldots - f_{l-1}(x,Y) + f_l'(x,Y),$$

and since $\deg f_i' < i\nu$, then $\deg f^{*\prime} < \max\{i\nu + (l-i)q \colon 0 \le i \le l\} \le (l-1)q + \nu$. As $\nu(k+1) \le q$, $lq - k\nu < (l-1)q + \nu$ can not hold. Thus $f^{*\prime} \equiv 0$, which means that $f^*(x,Y) \in \mathrm{GF}(q)[Y^p]$. As $k\nu < p$, we see that the terms of $f^*(x,Y)$ are of form Y^{qi+j} where $0 \le i \le k-l$ and $0 \le j < p$, thus only $j = 0$ occurs. This means that

$$f^*(x,Y) = Y^{lq} + a_1 Y^{(l-1)q} + \ldots + a_l$$

with proper $a_i \in \mathrm{GF}(q)$ (i.e., $f^*(x,Y) \in \mathrm{GF}(q)[Y^q]$ and it has degree l). Since $y^q = y$ for every $y \in \mathrm{GF}(q)$, then $f^*(x,Y)$ may have only l different zeros. Nevertheless, $f^*(x,Y)$ has lq zeros altogether (summing up the multiplicities), but each distinct zero has multiplicity at most $q - k + l$. Therefore $lq \le l(q - k + l)$. This can only happen if $l = 0$ or $l = k$. In the first case $f(x,Y) = (Y^q - Y)^k$, thus every point on the line $X = x$ is covered exactly k-times, while in the latter case $f(x,Y) = f^*(x,Y)$ and we find k points on the vertical line $X = x$ that are covered q-times. Thus the lemma is proven. □

Remark 4.2. *Note that the two possibilities on the structure of the set of lines in the above lemma are essentially the same: if we view* $\mathrm{AG}(2,q)$ *inside* $\mathrm{PG}(2,q)$*, then we see that our line-set consists of the lines intersecting a fixed line* l *in one of* k *fixed points. This line* l *may be the line at infinity or an affine line as well.*

Remark 4.3. *A similar result can be obtained using the results on weighted multiple blocking sets. E.g., in* $\mathrm{PG}(2,p)$ *one can do the following: give weight* k *to the* ν *exceptional vertical lines and the line at infinity. Then our line-set is a weighted* k*-fold covering set, hence we may apply Result 3.14 to its dual in* $\mathrm{PG}(2,p)$ *to deduce the same result, provided* *that $(\nu+1)k < (p+2)/2$. Note that in the actual application of Lemma 4.1 this bound is also satisfied.*

Proof of Theorem 2.4 in case of $q = p$.

We assume $2 \leq t < \sqrt{p}/2$. Recall that the points and lines with large index (which are $\mathcal{T}$-complete) form a subplane, which must be degenerate as we are in $\mathrm{PG}(2,p)$, and that the number of $\mathcal{T}$-complete points (and lines) is at most t; moreover, bad lines are blocked by points with large index (see Propositions 3.7, 3.8, 3.9, 3.11).

Case 1: there are no points or lines with large index. Then every line is good, hence intersects $\mathcal{P}_0$ in exactly t points. But then counting the points of $\mathcal{P}_0$ through the lines incident with a point inside or outside $\mathcal{P}_0$ we get that $|\mathcal{P}_0| = 1 + (t-1)(q+1)$ and $|\mathcal{P}_0| = (q+1)t$, a contradiction.

Case 2: the points and lines with large index form a subplane of type π_1. Then there exists an incident point-line pair (P,l), both having large index, such that every line with large index goes through P and dually, every point with large index lies on l. Take a line l through P with small index. Then every point Q on l except P is good (since the bad ones are blocked by the lines with large index, each of which intersects l in P), hence there are exactly t $\mathcal{T}$-lines through it. Thus there are $t-1$ $\mathcal{T}$-lines through Q different from PQ. We may choose the coordinate system in such a way that the line at infinity has large index and P is the common point of vertical lines. Then the non-vertical $\mathcal{T}$-lines cover all the points of the affine plane exactly $t-1$ times, except possibly the points of $t-1$ vertical lines with large index.

Applying Lemma 4.1 for the above situation, we get that there is a unique line with large index that contains $t-1$ $\mathcal{T}$-complete points (besides P) and that every point out of this line is good. Thus this line is the only line that has more than one bad point on it, i.e., this line is l. It follows from duality that P is the only point with large index and it has $t-1$ $\mathcal{T}$-complete lines through it (besides l). It is straightforward to see that this construction is what we get from Construction 1 starting with a degenerate subplane of type π_1.

Case 3: the points and lines with large index form a subplane of type π_2. Denote the points with large index by $P_1,\ldots,P_k$ ($k \leq t$) in such a way that $P_2,\ldots,P_k$ all lie on a line l, but $P_1 \notin l$. Here $k \neq 2$ may be assumed, as otherwise the degenerate subplane in question is of type π_1 as well.

Pick a point P on l with small index and denote by c the number of $\mathcal{T}$-points on the line PP_1 besides P and P_1. Counting the elements of $\mathcal{P}_0$ from P we get that $|\mathcal{P}_0| = q+1+(t-1)(q-1)+c+1$ $(\star)$, as there are $q+1$ $\mathcal{T}$-points on l, $t-1$ further $\mathcal{T}$-points on the $q-1$ good lines through P not incident with P_1 and $c+1$ more points on the line PP_1 (note that by Proposition 3.9 the only possibly bad lines through P are l and PP_1). This implies that c must be independent from the choice of P. Counting the $\mathcal{T}$-points via the lines passing through P_1 we get that $|\mathcal{P}_0| = 1+(k-1)(q-1)+(q+1)+c(q+1-(k-1))$ $(\star\star)$. Rearranging the equation obtained from $(\star)$ and $(\star\star)$ we get $c(k-2) = (c+k-t)(q-1)$.

If $k=1$, then $-c = (c+1-t)(q-1)$, hence by $c \leq q-1$ we get either $c=0$ and $t=1$, or $c = q-1$ and $t = q+1$. The latter case is out of interest, the first case corresponds to a 1-good complete subplane of type π_2.

If $k \geq 3$, then by $c \leq q-1$, we need to have $k-2 \geq c+k-t$, hence $c \leq t-2$. Using this and $k \leq t$, we get that $(c+k-t)(q-1) = c(k-2) \leq (t-2)^2 < q-1$ by the assumptions. Hence the only possibility is that $c+k-t=0$, whence $c=0$ and $k=t$ follows. It is easy to see that this implies that the pair $(\mathcal{P}_0, \mathcal{L}_0)$ is exactly the complete degenerate subplane of type π_2 spanned by the t points with large index. □

5. Concluding remarks

Theorem 2.4 describes all $(q+1-t, 6)$-graphs that can be obtained from $\mathrm{PG}(2,q)$ using the method in question if t is small enough. It should be emphasized however that we were looking for induced subgraphs of the incidence graph of the projective plane. Recently Araujo-Pardo and Balbuena [3] considered non induced subgraphs as well (that is, one is allowed to delete incidences from the plane, not only points and lines). They found a structure of size $tq+3$, that, if deleted, results in a construction that is better by 2 than Construction 1. However, when q is square, still Construction 2 seems to be the best one. Investigating the non induced case would be interesting.

Analyzing this method in generalized polygons is also of interest and seems to be difficult; partial results for generalized quadrangles were obtained by Beukemann and Metsch in [6].

Regarding the bounds on t in Theorem 2.4, we may consider the following two constructions. A unital is a set U of $q\sqrt{q}+1$ points such that every line intersects U in one or $\sqrt{q}+1$ points. Unitals exist in $\mathrm{PG}(2,q)$ for all square prime power q. Through every point $P \notin U$ there are exactly $\sqrt{q}+1$ tangent lines to U. Thus the points and the tangent lines of U form a $(\sqrt{q}+1)$-good structure. An $(n, \sqrt{q})$-arc is a set of n points such that every line intersects it in at most $\sqrt{q}$ points. For $n = \sqrt{q}\left(q - \sqrt{q}+1\right)$, $(n, \sqrt{q})$-arcs exist in $\mathrm{PG}(2,q)$ iff q is an even square prime power. Every line intersects such an arc K in zero or $\sqrt{q}$ points, and through every point $P \notin K$ there are exactly $\sqrt{q}$ lines skew to K. Hence the points and the skew lines of K form a $\sqrt{q}$-good structure.

The above two t-good structures are different from Constructions 1 and 2, which shows that some upper bound on t in Theorem 2.4 is needed. Note that if $q=p^2$, then the above two structures are $p+1$ and p-good, respectively, whereas the bound $t \leq \sqrt{p}/2$ is violated. However, the latter case is not at all general, as it works only in $\mathrm{PG}(2,4)$. It might be true though, that the condition $t \leq p$ is necessary for larger powers of p.

Question: Does there exist a $p+1$-good structure in $\mathrm{PG}(2,p^h)$ for some prime p and arbitrarily large h that is not a complete subplane nor the union of $p+1$ disjoint Baer subplanes?

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