1. ORDERED SETS

Definitions. (Partially and totally ordered sets.)

(1) The ordered pair \((X, \rho)\) is called a partially ordered set if \(X\) is a set and \(\rho\) is a partial order relation in \(X\).

(2) The ordered pair \((X, \rho)\) is called a totally ordered set (or linear ordered set) if \(X\) is a set and \(\rho\) is a total (linear) order relation in \(X\).

Let \((X, \rho)\) be a (partially or totally) ordered set and \(S \subset X\).

Definitions. (Upper and lower bounds, bounded set.)

(1) \(u \in X\) is called an upper bound for \(S\) if \(\forall x \in S \ (x, u) \in \rho\).

(2) \(v \in X\) is called a lower bound for \(S\) if \(\forall x \in S \ (v, x) \in \rho\).

(3) \(S\) is called bounded above if \(\exists u \in X\) an upper bound for \(S\).

(4) \(S\) is called bounded below if \(\exists v \in X\) a lower bound for \(S\).

(5) \(S\) is called bounded if it is bounded both above and below.

Definitions. (Maximal and minimal elements.)

(1) \(M \in S\) is called a maximal element of \(S\) if there is no \(x \in S\) such that \(x \neq M\) and \((M, x) \in \rho\).

(2) \(m \in S\) is called a minimal element of \(S\) if there is no \(x \in S\) such that \(x \neq m\) and \((x, m) \in \rho\).

Definitions. (Greatest and least elements.)

(1) \(M \in S\) is called the greatest element of \(S\) if \(\forall x \in S \ (x, M) \in \rho\).

(2) \(m \in S\) is called the least element of \(S\) if \(\forall x \in S \ (m, x) \in \rho\).

Definitions. (Supremum and infimum.)

(1) If \(S\) is bounded above and the set of all upper bounds for \(S\) has a least element, we call it the least upper bound or the supremum for \(S\), and denote it by \(\text{sup} S\).

(I.e., \(\text{sup} S \in U := \{ u \in X : \forall x \in S \ (x, u) \in \rho \}\) and \(\forall u \in U \ (\text{sup} S, u) \in \rho\).)

(2) If \(S\) is bounded below and the set of all lower bounds for \(S\) has a greatest element, we call it the greatest lower bound or the infimum for \(S\), and denote it by \(\text{inf} S\).

(I.e., \(\text{inf} S \in V := \{ v \in X : \forall x \in S \ (v, x) \in \rho \}\) and \(\forall v \in V \ (v, \text{inf} V) \in \rho\).)

Example.

Let \(\rho\) be the divisibility in \(\mathbb{N}^+\). Then \((\mathbb{N}^+, \rho)\) is a partially ordered set.

1 is the only minimal (and least) element of \(\mathbb{N}^+\), 1 is also the infimum for \(\mathbb{N}^+\).

The primes are minimal elements of \(\mathbb{N}^+\{1\}\), there is no least element of \(\mathbb{N}^+\{1\}\), 1 is the infimum for \(\mathbb{N}^+\{1\}\).
2. REAL NUMBERS

The system of axioms for the ordered field \( \mathbb{R} \) of real numbers is given, as follows:

\( \mathbb{R} \) is a set whose elements are called real numbers. 
There exists distinguished elements \( 0 \in \mathbb{R} \) and \( 1 \in \mathbb{R} \).
Two operations, called addition and multiplication are defined in \( \mathbb{R} \), for which the notations \((x, y) \mapsto x + y\) and \((x, y) \mapsto x \cdot y\) are used, respectively.
A total order relation, denoted by \( \leq \), is defined in \( \mathbb{R} \), (i.e. \((\mathbb{R}, \leq)\) is a totally ordered set).
(We use the notation \( x \leq y \) instead of \((x, y) \in \leq\), and write \( x < y \) if \( x \leq y \) and \( x \neq y \).)

The following axioms are satisfied:

(1) \( \forall x, y, z \in \mathbb{R} \quad (x + y) + z = x + (y + z) \), (the addition is associative)

(2) \( \forall x \in \mathbb{R} \quad x + 0 = x \),

(3) \( \forall x \in \mathbb{R} \exists u \in \mathbb{R} \quad x + u = 0 \), \quad \(-x := u, \quad y - x := y + (-x)\)

(4) \( \forall x, y \in \mathbb{R} \quad x + y = y + x \), (the addition is commutative)

(5) \( \forall x, y, z \in \mathbb{R} \quad (x \cdot y) \cdot z = x \cdot (y \cdot z) \), (the multiplication is associative)

(6) \( \forall x \in \mathbb{R} \quad x \cdot 1 = x \),

(7) \( \forall x \in \mathbb{R}\setminus\{0\} \exists u \in \mathbb{R} \quad x \cdot u = 1 \), \quad \( 1/x := u, \quad y/x := y \cdot 1/x \)

(8) \( \forall x, y \in \mathbb{R} \quad x \cdot y = y \cdot x \), (the multiplication is commutative)

(9) \( \forall x, y, z \in \mathbb{R} \quad x \cdot (y + z) = x \cdot y + x \cdot z \), (the multiplication is distributive with respect to the addition)

(10) \( \forall x, y, z \in \mathbb{R} \quad x \leq y \Rightarrow x + z \leq y + z \),

(11) \( \forall x, y, z \in \mathbb{R} \quad (x \leq y \text{ and } 0 \leq z) \Rightarrow x \cdot z \leq y \cdot z \),

(12) If a nonempty set \( S \subset \mathbb{R} \) is bounded above, then 
the set of all upper bounds for \( S \) has a least element, \( \text{(least upper bound for } S) \).
I.e., \( \exists \sup S \in \mathbb{R} \), 
(i) \( \forall x \in S \quad x \leq \sup S \),
(ii) \( \forall u \in \mathbb{R} \quad (\forall x \in S \quad x \leq u) \Rightarrow \sup S \leq u \).
\((\forall y \in \mathbb{R} \quad y < \sup S \Rightarrow \exists x \in S \quad y < x.) \)

Remarks.

(i) Axioms (1)–(9) are the axioms for a field.
Axioms (1)–(11) are the axioms for an ordered field.
Axiom (12) is usually called as axiom of least upper bound.

(ii) We show how some elementary laws of algebra and inequalities follow 
from the axioms (1)–(11):

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Proposition 1. \( \forall x \in \mathbb{R} \quad x \cdot 0 = 0 \).

Proof.
\[
x \cdot 0 = x \cdot 0 + 0 = x \cdot 0 + (x \cdot 0 - x \cdot 0) = (x \cdot 0 + x \cdot 0) - x \cdot 0 = x \cdot (0 + 0) - x \cdot 0 = x \cdot 0 - x \cdot 0 = 0.
\]

Proposition 2. \( \forall x, y \in \mathbb{R} \quad x \cdot y = 0 \Rightarrow x = 0 \text{ or } y = 0 \).

Proof.
\[
x \neq 0 \Rightarrow y = 1 \cdot y = (1/x \cdot x) \cdot y = 1/x \cdot (x \cdot y) = 1/x \cdot 0 = 0.
\]

Proposition 3. \( \forall x \in \mathbb{R} \quad -(x) = x \).

Proof.
\[
-(x) = -(x) + 0 = -(x) + (x + x) = -(x) + (-x) + x = 0 + x = x.
\]

Proposition 4. \( \forall x, y \in \mathbb{R} \quad -(x + y) = -(x) + -(y) = -x - y \).

Proof.
\[
-(x + y) = -(x + y) + 0 = -(x + y) + (0 + 0) = -(x + y) + ((-x + x) + (-y + y)) =
= -(x + y) + ((x + y) + (-x - y)) = -(x + y) + (x + y) + (-x - y) = 0 + (-x - y) = -x - y.
\]

Proposition 5. \( \forall x \in \mathbb{R} \setminus \{0\} \quad 1/x \neq 0 \text{ and } 1/(1/x) = x \).

Proof.
(i) \( 1/x = 0 \Rightarrow 1 = x \cdot 1/x = x \cdot 0 = 0 \), a contradiction.
(ii) \( 1/(1/x) = 1/(1/x) \cdot 1 = 1/1 \cdot (1/x \cdot x) = (1/(1/x) \cdot 1/x) \cdot x = 1 \cdot x = x \).

Proposition 6. \( \forall x, y \in \mathbb{R} \setminus \{0\} \quad 1/(x \cdot y) = 1/x \cdot 1/y \).

Proof.
\[
1/(x \cdot y) = 1/(x \cdot y) \cdot 1 = 1/(x \cdot y) \cdot (1 \cdot 1) = 1/(x \cdot y) \cdot ((1/x \cdot x) \cdot (1/y \cdot y)) =
= 1/(x \cdot y) \cdot ((x \cdot y) \cdot (1/x \cdot 1/y)) = (1/(x \cdot y) \cdot (x \cdot y)) \cdot (1/x \cdot 1/y) = 1/(1/x \cdot 1/y) = 1/x \cdot 1/y.
\]

Proposition 7. \( \forall x, y \in \mathbb{R} \quad x \cdot (-y) = -(x \cdot y) \).

Proof.
\[
x \cdot (-y) = x \cdot (-y) + 0 = x \cdot (-y) + (x \cdot y - (x \cdot y)) = (x \cdot (-y) + x \cdot y) - (x \cdot y) =
x \cdot (-y + y) - (x \cdot y) = x \cdot 0 - (x \cdot y) = 0 - (x \cdot y) = -x \cdot y.
\]

Proposition 8. \( \forall x, y \in \mathbb{R} \quad (-x) \cdot (-y) = x \cdot y \).

Proof.
\[
(-x) \cdot (-y) = -((x) \cdot (-y)) = -(y \cdot (-x)) = -(-y \cdot x) = y \cdot x = x \cdot y.
\]

Proposition 9. \( \forall x \in \mathbb{R} \quad (i) \ 0 \leq x \Rightarrow -x \leq 0 \quad (ii) \ x \leq 0 \Rightarrow 0 \leq -x \).

Proof.
(i) \( 0 \leq x \Rightarrow 0 + (-x) \leq x + (-x) \Rightarrow -x \leq 0 \)
(ii) \( x \leq 0 \Rightarrow x + (-x) \leq 0 + (-x) \Rightarrow 0 \leq -x \).
Proposition 10. \( \forall x, y, u, v \in \mathbb{R} \ (x \leq y \text{ and } u \leq v) \Rightarrow x + u \leq y + v. \)
Proof. 
\( x \leq y \Rightarrow x + u \leq y + u, \ u \leq v \Rightarrow y + u \leq y + v. \) Therefore, \( x + u \leq y + v. \)

Proposition 11. \( \forall x, y, u, v \in \mathbb{R} \ (0 \leq x \leq y \text{ and } 0 \leq u \leq v) \Rightarrow x \cdot u \leq y \cdot v. \)
Proof. 
\( x \leq y \Rightarrow x \cdot u \leq y \cdot u, \ u \leq v \Rightarrow y \cdot u \leq y \cdot v. \) Therefore, it follows that \( x \cdot u \leq y \cdot v. \)

Proposition 12. \( 0 < 1. \)
Proof. 
\( 1 \leq 0 \Rightarrow 0 < -1 \Rightarrow 0 = (-1) \cdot 0 < (-1) \cdot (-1) = 1 \cdot 1 = 1, \) a contradiction.

Theorem (2.1).
The following propositions are equivalent:
(1) If a nonempty subset of \( \mathbb{R} \) is bounded above, then it has a least upper bound in \( \mathbb{R} \).
(2) If a nonempty subset of \( \mathbb{R} \) is bounded below, then it has a greatest lower bound in \( \mathbb{R} \).
(3) If \( A \) and \( B \) are nonempty subsets of \( \mathbb{R} \) such that \( \forall (a, b) \in A \times B \ a \leq b, \) then \( \exists c \in \mathbb{R} \ \forall (a, b) \in A \times B \ a \leq c \leq b. \)

Proof. 
(1) \( \Rightarrow \) (2) : Let \( \emptyset \neq S \subset \mathbb{R} \) be bounded below. Since the set of all lower bounds for \( S, \ V := \{v \in \mathbb{R} : \forall x \in S \ v \leq x\} \) is nonempty and bounded above, we know by (1) that \( \exists \sup V \in \mathbb{R}. \)
(i) \( \forall x \in S \ x \text{ is an upper bound for } V, \text{ therefore } \sup V \leq x, \) thus \( \sup V \) is a lower bound for \( S. \)
(ii) If \( y \in \mathbb{R} \) is a lower bound for \( S, \) then \( y \in V, \) thus \( y \leq \sup V. \)
Hence, \( \sup V \) is the greatest lower bound for \( S, \) i.e. \( \inf S = \sup V. \)

(2) \( \Rightarrow \) (3) :
Let \( A \) and \( B \) are nonempty subsets of \( \mathbb{R} \) such that \( \forall (a, b) \in A \times B \ a \leq b. \)
Since \( B \) is nonempty and bounded below, we know by (2) that \( \exists \inf B \in \mathbb{R}, \) and since \( \forall a \in A \ a \text{ is a lower bound for } B, \) we have \( a \leq \inf B. \)
Therefore, with \( c := \inf B \) we obtain \( \forall (a, b) \in A \times B \ a \leq c \leq b. \)

(3) \( \Rightarrow \) (1) :
Let \( \emptyset \neq S \subset \mathbb{R} \) be bounded above. Since the set of all upper bounds for \( S, \ U := \{u \in \mathbb{R} : \forall x \in S \ x \leq u\} \) is nonempty, we know by (3) that \( \exists c \in \mathbb{R}, \) such that \( \forall (x, u) \in S \times U \ x \leq c \leq u. \)
(i) \( c \) is an upper bound for \( S, \)
(ii) \( c \) is less or equal than any upper bound for \( S, \)
therefore, \( c \in U \) is the least upper bound for \( S, \) i.e. \( \exists \sup S = c. \)
Remark.
The propositions (2) and (3) of the theorem (2.1) are known as
the "axiom" of greatest lower bound and Dedekind’s "axiom", respectively.

Definition. (Inductive sets.)
A set \( S \subset \mathbb{R} \) is said to be inductive if
(i) \( 1 \in S \),
(ii) \( x \in S \) implies \( x + 1 \in S \).

Definition. (Positive integers.)
The set of all positive integers, denoted by the symbol \( \mathbb{N}^+ \), is defined by
\[ \mathbb{N}^+ := \bigcap \{ S \subset \mathbb{R} : S \text{ is inductive} \} \ (\subset \mathbb{R}). \]

Theorem (2.2). (Principle of mathematical induction.)
If \( S \subset \mathbb{N}^+ \) is an inductive set, then \( S = \mathbb{N}^+ \).

Proof.
Since \( S \) is an inductive set, we know from the definition of \( \mathbb{N}^+ \) that \( \mathbb{N}^+ \subset S \), thus we have \( S = \mathbb{N}^+ \).

Definition. (Modified inductive sets.)
A set \( S \subset \mathbb{R} \) has the modified inductive property (with respect to \( m \in \mathbb{R} \)) if
(i) \( m \in S \),
(ii) \( x \in S \) implies \( x + 1 \in S \).

Theorem (2.3). (Modified form of mathematical induction.)
If \( S \subset \mathbb{N}^+ \) is a modified inductive set with respect to \( m \in \mathbb{N}^+ \),
then \( \forall n \in \mathbb{N}^+ \ n \geq m \) implies \( n \in S \).

Proof.
Let \( S := \{ n \in \mathbb{N}^+ : n \leq m \} \cup S \).
Evidently \( \tilde{S} \subset \mathbb{N}^+ \) is an inductive set, thus \( \tilde{S} = \mathbb{N}^+ \).
Therefore, we have \( \{ n \in \mathbb{N}^+ : n \leq m \} \subset \tilde{S} \), and so \( \{ n \in \mathbb{N}^+ : n \geq m \} \subset S \) must be also satisfied.

Example. Prove that \( \forall n \in \mathbb{N}^+ \ n \geq 3 \Rightarrow n^{n+1} > (n+1)^n \).

Solution.
Let \( S := \{ k \in \mathbb{N}^+ : k \geq 3 \text{ and } k^{k+1} > (k+1)^k \} \).
(i) \( 3 \in S \), since \( 3^4 > 4^3 \).
(ii) If \( k - 1 \in S \), then \( (k-1)^k > k^{k-1} \), therefore
\[ k^{k+1} = k^{k+1} \cdot (k+1)^k \cdot (k-1)^k / ((k+1)^k \cdot (k-1)^k) > \]
\[ > (k+1)^k \cdot k^{k+1} \cdot k^{k-1} / ((k^2-1)^k) = (k+1)^k \cdot (k^2/(k^2-1))^k > (k+1)^k. \]
Hence \( S \) is an inductive set with respect to 3, thus \( \{ n \in \mathbb{N}^+ : n \geq 3 \} \subset S \).
Definitions. (Natural numbers, integers, rational numbers, irrational numbers.)

1. The set of all natural numbers, denoted by $\mathbb{N}$, is defined by $\mathbb{N} := \mathbb{N}^+ \cup \{0\}$.

2. The set of all integers, denoted by $\mathbb{Z}$, is defined by $\mathbb{Z} := \mathbb{N} \cup \{-n : n \in \mathbb{N}^+\}$.

3. The set of all rational numbers, denoted by $\mathbb{Q}$, is defined by $\mathbb{Q} := \{x \in \mathbb{R} : \exists (p, q) \in \mathbb{Z} \times \mathbb{N}^+ \ x = p/q\}$.

4. The set $\mathbb{R}\setminus\mathbb{Q}$ is called the set of all irrational numbers.

Theorem (2.4). (Archimedes’ theorem.)
The set $\mathbb{N}^+$ of all positive integers is not bounded above.

Proof.
Suppose that $\mathbb{N}^+$ is bounded above. Then $\exists \sup \mathbb{N}^+ \in \mathbb{R}$, $\forall n \in \mathbb{N}^+$, $n \leq \sup \mathbb{N}^+$.
Since $\sup \mathbb{N}^+ - 1$ is not an upper bound for $\mathbb{N}^+$, $\exists m \in \mathbb{N}^+$, $\sup \mathbb{N}^+ - 1 < m$, thus we have $\sup \mathbb{N}^+ < m + 1 \leq \sup \mathbb{N}^+$, a contradiction.

Remarks.
The following propositions are obvious consequences of Archimedes’ theorem:

(i) If $p \in \mathbb{R}^+$ and $x \in \mathbb{R}$, then $\exists n \in \mathbb{N}^+$ such that $x < n \cdot p$.

(ii) The set $\mathbb{Z}$ of all integers is not bounded below.

Theorem (2.5).
For all $x \in \mathbb{R}$ there exists a unique integer, denoted by $[x]$ and called the largest integer in $x$, such that $[x] \leq x < [x]+1$.

Proof.
Let $x \in \mathbb{R}$ and define the set $K := \{z \in \mathbb{Z} : z \leq x\}$.
$K \neq \emptyset$ is bounded above ($x$ is an upper bound for $K$), thus $\exists \sup K \leq x$.

We show that $\sup K \in \mathbb{Z}$:

(i) Suppose that $\sup K \notin \mathbb{Z}$, i.e. $\forall z \in K \ z < \sup K$.

(ii) Since $\sup K - 1$ is not an upper bound for $K$, $\exists w \in K \sup K - 1 < w < \sup K$.
We have $\sup K < w + 1$, thus $w + 1 \notin K$, i.e. $x < w + 1$. Therefore,
$\forall z \in \mathbb{Z} \ z > w$ implies $z \notin K$, thus we have $\forall z \in K \ z \leq w$, a contradiction.

Define $[x] := \sup K \in \mathbb{Z}$. Evidently, we have $[x] \leq x < [x]+1$.

Remarks.
The next two propositions follow immediately from the theorem (2.5).

(i) If $\emptyset \neq K \subset \mathbb{Z}$ is bounded above, then $\sup K \in K$ is the greatest element of $K$.

(ii) If $\emptyset \neq K \subset \mathbb{Z}$ is bounded below, then $\inf K \in K$ is the least element of $K$.

The following propositions are also consequences of the theorem (2.5).
Proposition (2.6).
If \( a, b \in \mathbb{R} \) and \( b-a > 1 \), then \( \exists z \in \mathbb{Z} \) such that \( a < z < b \).

**Proof.** \( a+1 < b \Rightarrow a < [a]+1 \leq a+1 < b \). ♠

Proposition (2.7).
If \( a, b \in \mathbb{R} \) and \( a < b \), then \( \exists r \in \mathbb{Q} \) such that \( a < r < b \).

**Proof.** \( b-a > 0 \Rightarrow \exists n \in \mathbb{N}^+ \ n > 1/(b-a) \Rightarrow n \cdot b - n \cdot a > 1 \Rightarrow \exists z \in \mathbb{Z} \ n \cdot a < z < n \cdot b \Rightarrow a < z/n < b \). ♠

Proposition (2.8).
If \( a, b \in \mathbb{R} \) and \( a < b \), then \( \exists x \in \mathbb{R} \setminus \mathbb{Q} \) such that \( a < x < b \).

**Proof.** \( a < b \Rightarrow a \cdot \sqrt{2} < b \cdot \sqrt{2} \Rightarrow \exists r \in \mathbb{Q} \ a \cdot \sqrt{2} < r < b \cdot \sqrt{2} \Rightarrow a < r / \sqrt{2} < b \). ♠

**Definition.** (The set \( \overline{\mathbb{R}} \) of extended real numbers.)
The set \( \overline{\mathbb{R}} \) is defined by \( \overline{\mathbb{R}} := \mathbb{R} \cup \{ -\infty, +\infty \} \), where the symbols \( -\infty \) and \( +\infty \) are introduced such that
(i) \( -\infty < +\infty \),
(ii) \( \forall x \in \mathbb{R} \ -\infty < x < +\infty \).
(Hence \( (\overline{\mathbb{R}}, \leq) \) is a totally ordered set, but we emphasize that \( \overline{\mathbb{R}} \) is not a field.)

**Definitions.** (Intervals in \( \mathbb{R} \).)
Let \( a, b \in \mathbb{R} \) and \( a \leq b \).
The following subsets of \( \mathbb{R} \) are called intervals in \( \mathbb{R} \) with endpoints \( a \) and \( b \):

1. \( (a,b) := \{ x \in \mathbb{R} : a < x < b \} \) (open interval)
2. \( [a,b] := \{ x \in \mathbb{R} : a \leq x \leq b \} \) (closed interval)
3. \( (a,b] := \{ x \in \mathbb{R} : a < x \leq b \} \) (semiclosed interval)
4. \( [a,b) := \{ x \in \mathbb{R} : a \leq x < b \} \) (semiclosed interval)

The following subsets of \( \mathbb{R} \) are unbounded intervals in \( \mathbb{R} \):

5. \( (-\infty, b) := \{ x \in \mathbb{R} : x < b \} \) (open interval)
6. \( (a, +\infty) := \{ x \in \mathbb{R} : a < x \} \) (open interval)
7. \( (-\infty, b) := \{ x \in \mathbb{R} : x \leq b \} \) (closed interval)
8. \( [a, +\infty) := \{ x \in \mathbb{R} : a \leq x \} \) (closed interval)
(We sometimes write \( (-\infty, +\infty) \) instead of \( \overline{\mathbb{R}} \).)

**Definition.** (Dense subsets of \( \mathbb{R} \).)
A set \( S \subset \mathbb{R} \) is said to be dense in \( \mathbb{R} \) if \( \forall x \in \mathbb{R} \) and \( \forall \varepsilon \in \mathbb{R}^+ \) the set \( (x-\varepsilon, x+\varepsilon) \cap S \) is not empty.

**Remarks.**
The propositions (2.7) and (2.8) show that \( \mathbb{Q} \) and \( \mathbb{R} \setminus \mathbb{Q} \) are dense in \( \mathbb{R} \).
The next three propositions follow immediately from the definitions of bounded sets, supremum and infimum in ordered sets.

**Proposition (2.9).**
Every set $S \subset \mathbb{R}$ is bounded and has supremum and infimum in $\mathbb{R}$.

**Proposition (2.10).**
Let $S \subset \mathbb{R} \subset \mathbb{R}$ be a nonempty set.

1. If $S$ is bounded above in $\mathbb{R}$, then $\sup S$ in $\mathbb{R}$ equals $\sup S$ in $\mathbb{R}$.
2. If $S$ is bounded below in $\mathbb{R}$, then $\inf S$ in $\mathbb{R}$ equals $\inf S$ in $\mathbb{R}$.
3. If $S$ is not bounded above in $\mathbb{R}$, then $\sup S$ in $\mathbb{R}$ equals $+\infty$.
4. If $S$ is not bounded below in $\mathbb{R}$, then $\inf S$ in $\mathbb{R}$ equals $-\infty$.

**Remark.**
If $S \subset \mathbb{R} \subset \mathbb{R}$ is any set, $\sup S$ in $\mathbb{R}$ and $\inf S$ in $\mathbb{R}$ will be denoted simply by $\sup S$ and $\inf S$, respectively. (The same notations as in $\mathbb{R}$, provided they exist in $\mathbb{R}$.)

**Proposition (2.11).**
Let $S \subset \mathbb{R} \subset \mathbb{R}$ be a set. Then

1. $\sup S = +\infty \iff S$ is not bounded above in $\mathbb{R}$.
2. $\inf S = -\infty \iff S$ is not bounded below in $\mathbb{R}$.
3. $\sup S = -\infty \iff \inf S = +\infty \iff S = \emptyset$.

**Definition.** *(Absolute value.)*
If $x \in \mathbb{R}$, the absolute value of $x$, denoted by $|x|$, is defined by

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

**Theorem (2.12).**
For all $x, y \in \mathbb{R}$ the following properties are satisfied:

1. $|x| \geq 0$ and $|x| = 0 \iff x = 0$.
2. $|x \cdot y| = |x| \cdot |y|$.
3. $|x + y| \leq |x| + |y|$.

**Proof.**
(1) and (2) are simple consequences of the definition of absolute value.

To prove (3) observe that $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$.

Adding these inequalities we get $-(|x| + |y|) \leq x + y \leq (|x| + |y|)$, which is equivalent to (3).

**Remark. (2.13).**
For all $x, y \in \mathbb{R}$ $| |x| - |y| | \leq |x - y|$.

**Proof.**
$|x| = |(x - y) + y| \leq |x - y| + |y| \Rightarrow |x| - |y| \leq |x - y|$.

Similarly, we have $|y| - |x| \leq |y - x| = |x - y|$.

Therefore, we obtain $|x| - |y| \leq |x - y|$ and $-(|x| - |y|) \leq |x - y|$, from which the conclusion follows.
3. INFINITE SEQUENCES

**Definition.** (Infinite sequences.)
A sequence in $X$ is defined to be a function $c : \mathbb{N} \rightarrow X$.
If the domain of the sequence is a finite subset of $\mathbb{N}$, we say the sequence is finite. Otherwise, the sequence is called infinite.

In the following, we study infinite sequences in $\mathbb{R}$ (simply said sequences in $\mathbb{R}$) with domains $\mathbb{N}_m := \{ n \in \mathbb{N} : n \geq m \}, \ m \in \mathbb{N}$, (usually $m = 0$ or $m = 1$).
The most common notation is to write $n \mapsto c_n$ instead of $n \mapsto c(n)$.
The elements of the range $\{ c_n : n \in \mathbb{N}_m \}$ are called the terms of the sequence.
In general, we write the sequence as $(c_n)_{n \in \mathbb{N}_m}$, or simply $(c_n)$ when the domain of the sequence is understood from the context, or is not relevant to the discussion.
Sometimes the notation $c_m, c_{m+1}, \ldots, c_n, \ldots$ is also used.

**Theorem (3.1).** ("Nested intervals theorem").
If $(a_n)$ and $(b_n)$ are sequences in $\mathbb{R}$, and
(i) $\forall n \in \mathbb{N}^+ \ a_n \leq a_{n+1} \leq b_{n+1} \leq b_n,$
(ii) $\forall \varepsilon \in \mathbb{R}^+ \ \exists n \in \mathbb{N}^+ \ b_n - a_n < \varepsilon,$
then $\exists$ unique $c \in \mathbb{R}$ such that $\forall n \in \mathbb{N}^+ \ a_n \leq c \leq b_n.$

**Proof.**
(1) The existence of $c$ follows from Dedekind’s "axiom", with the sets
$A := \{ a_n : n \in \mathbb{N}^+ \}$ and $B := \{ b_n : n \in \mathbb{N}^+ \}$, since $\forall m, n \in \mathbb{N}^+$
$(m \leq n) \Rightarrow a_m \leq a_n \leq b_n$ and $(n < m) \Rightarrow a_m \leq b_m \leq b_n$,
thus we have $\forall m, n \in \mathbb{N}^+ \ a_m \leq b_n$.

(2) Suppose $\exists c_1 < c_2$ such that $\forall n \in \mathbb{N}^+ \ a_n \leq c_1 < c_2 \leq b_n$.
Then from (ii) we know $\exists n \in \mathbb{N}^+$ such that $b_n - a_n < c_2 - c_1$,
thus we get $b_n - a_n < c_2 - c_1 \leq b_n - a_n$, a contradiction. ♠

**Remarks.**
(1) If we omit the condition (ii), the theorem can be formulate as follows:
If $(J_n)$ is a sequence of closed and bounded intervals such that
$J_{n+1} \subset J_n$ for all $n \in \mathbb{N}^+$, then $\bigcap_{n=1}^{\infty} J_n \neq \emptyset$.

(2) If the intervals are not closed or bounded, their intersection can be empty.
E.g. $J_n := (0, 1/n)$ or $J_n := [n, +\infty)$.

**Definition.** (Bounded sequences.)
A sequence $(c_n)$ in $\mathbb{R}$ is said to be bounded if the range of $(c_n)$ is a bounded set in $\mathbb{R}$.

**Definitions.** (Monotone sequences.)
Let $(c_n)$ be a sequence in $\mathbb{R}$.
(1) We say that $(c_n)$ is an increasing sequence if $\forall n \in \mathbb{N} \ c_n \leq c_{n+1}$;
we call it strictly increasing if $\forall n \in \mathbb{N} \ c_n < c_{n+1}$.
(2) We say that $(c_n)$ is a decreasing sequence if $\forall n \in \mathbb{N} \ c_n \geq c_{n+1}$;
we call it strictly decreasing if $\forall n \in \mathbb{N} \ c_n > c_{n+1}$.
$(c_n)$ is called monotone if it is either increasing or decreasing.
Definition. (Convergent and divergent sequences.)
Let \((c_n)\) be a sequence in \(\mathbb{R}\).

(1) We say that \((c_n)\) is **convergent** and **converges** to the limit \(C \in \mathbb{R}\)
if \(\forall \varepsilon \in \mathbb{R}^+ \exists M \in \mathbb{N}\) such that \(\forall n \in \mathbb{N}_M \ |c_n - C| < \varepsilon\).

(2) We say that \((c_n)\) is **divergent** if it **does not converge** to any limit in \(\mathbb{R}\).

(3) We say that \((c_n)\) tends to the limit \(+\infty\) (as \(n\) tends to \(+\infty\))
if \(\forall K \in \mathbb{R} \exists M \in \mathbb{N}\) such that \(\forall n \in \mathbb{N}_M \ K < c_n\).

(4) We say that \((c_n)\) tends to the limit \(-\infty\) (as \(n\) tends to \(+\infty\))
if \(\forall K \in \mathbb{R} \exists M \in \mathbb{N}\) such that \(\forall n \in \mathbb{N}_M \ c_n < K\).

If \(\alpha \in \mathbb{R}\) is the limit of \((c_n)\), we write \(\lim(c_n) = \alpha\), \(\lim_{n \to +\infty} c_n = \alpha\), or \(c_n \to \alpha\) as \(n \to +\infty\).

Remarks. (3.2).
(i) Evidently, the limit of a sequence is uniquely determined, provided it exists.
(ii) By (1) it follows that a sequence \((c_n)\) converges to the limit \(C \in \mathbb{R}\) if and only if
\(\forall \varepsilon \in \mathbb{R}^+\) the set \(\{n \in \mathbb{N} : |c_n - C| \geq \varepsilon\}\) is finite.
(iii) By (3)-(4) it follows that a sequence \((c_n)\) tends to \(+\infty\) \((-\infty)\) if and only if
\(\forall K \in \mathbb{R}\) the set \(\{n \in \mathbb{N} : c_n \leq K\ (K \leq c_n)\}\) is finite.

Theorem (3.3).
If \((c_n)\) is a convergent sequence in \(\mathbb{R}\), then it is a bounded sequence.

Proof.
Let \(\varepsilon := 1\), \(C := \lim(c_n)\). Then \(\{c_n: |c_n - C| \geq 1\}\) is bounded (since it is finite),
thus \(\{c_n: n \in \mathbb{N}\} = \{c_n: |c_n - C| < 1\} \cup \{c_n: |c_n - C| \geq 1\}\) is also bounded.

Theorem (3.4).
Let \((c_n)\) be a monotone sequence in \(\mathbb{R}\).

(1) If \((c_n)\) is bounded, then \((c_n)\) is convergent, and
\(\lim(c_n) = \sup\{c_n: n \in \mathbb{N}\}\), provided \((c_n)\) is increasing, or
\(\lim(c_n) = \inf\{c_n: n \in \mathbb{N}\}\), provided \((c_n)\) is decreasing.

(2) If \((c_n)\) is not bounded, then \((c_n)\) is divergent, and
\((c_n)\) tends to \(+\infty\), provided \((c_n)\) is increasing, or
\((c_n)\) tends to \(-\infty\), provided \((c_n)\) is decreasing.

Proof.
(1) Let \(\varepsilon \in \mathbb{R}^+\), \(S := \{c_n: n \in \mathbb{N}\}\), \(A := \sup S\), and \(B := \inf S\).
Then, since \(A - \varepsilon\) is not an upper bound for \(S\), \(\exists p \in \mathbb{N}\ A - \varepsilon < c_p\),
thus, if \((c_n)\) is increasing, then \(\forall n \in \mathbb{N}_p\ A - \varepsilon < c_n \leq A\).
Similarly, since \(B + \varepsilon\) is not a lower bound for \(S\), \(\exists p \in \mathbb{N}\ c_p < B + \varepsilon\),
thus, if \((c_n)\) is decreasing, then \(\forall n \in \mathbb{N}_p\ B \leq c_n \leq c_p < B + \varepsilon\).

(2) By the theorem (3.3) it follows that \((c_n)\) is divergent. Let \(K \in \mathbb{R}\).
If \((c_n)\) is increasing, then \(\exists p \in \mathbb{N}\ K < c_p\) and \(\forall n \in \mathbb{N}_p\ K < c_p \leq c_n\).
If \((c_n)\) is decreasing, then \(\exists p \in \mathbb{N}\ c_p < K\) and \(\forall n \in \mathbb{N}_p\ c_n \leq c_p < K\).
Remark.
Theorem (3.4) is equivalent to the statement that every monotone sequence in \( \mathbb{R} \) has a limit in \( \mathbb{R} \):
(1) \( \lim (c_n) = \sup \{c_n : n \in \mathbb{N}\} \), provided \((c_n)\) is increasing, and
(2) \( \lim (c_n) = \inf \{c_n : n \in \mathbb{N}\} \), provided \((c_n)\) is decreasing.

Definition. (Rearrangement of a sequence.)
Let \((c_n)\) be a sequence in \( \mathbb{R} \).
A sequence \((b_n)\) is said to be a rearrangement of \((c_n)\) if there exists a bijection \( \varphi : \mathbb{N} \to \mathbb{N} \) such that \( \forall k \in \mathbb{N} \ b_k = c_{\varphi(k)} \).

Definition. (Subsequences of a sequence.)
Let \((c_n)\) be a sequence in \( \mathbb{R} \).
A sequence \((b_n)\) is said to be a subsequence of \((c_n)\) if there exists a strictly increasing sequence \((n_k)\) in \( \mathbb{N} \) such that \( \forall k \in \mathbb{N} \ b_k = c_{n_k} \).

Theorem (3.5).
Let \((c_n)\) be a sequence in \( \mathbb{R} \) and suppose that \( \exists \lim (c_n) \in \mathbb{R} \).
(1) If \((a_n)\) is a rearrangement of \((c_n)\), then \( \exists \lim (a_n) = \lim (c_n) \).
(2) If \((b_n)\) is a subsequence of \((c_n)\), then \( \exists \lim (b_n) = \lim (c_n) \).

Proof.
Both (1) and (2) are immediate consequences of the propositions (ii) and (iii) of remarks (3.2).

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Theorem (3.6).
Every sequence in \( \mathbb{R} \) has a monotone subsequence.

Proof.
Let \((c_n)\) be a sequence in \( \mathbb{R} \) and \( P := \{ n \in \mathbb{N} : \forall m \in \mathbb{N}_n \setminus \{n\} \ c_m < c_n \} \).
(1) If \( P \) is an infinite set, we define a sequence \((b_n)\) recursively, as follows:
(i) Let \( n_0 \) be any element of \( P \) (e.g. the least one), and define \( b_0 := c_{n_0} \).
(ii) If \( n_k \in P \) is chosen for \( k \in \mathbb{N} \), and \( b_k := c_{n_k} \) is defined, then we choose for \( k+1 \) an element \( n_{k+1} \in \{n \in P : n > n_k\} \) (e.g. the least one), and define \( b_{k+1} := c_{n_{k+1}} \).
Since \( n_{k+1} > n_k \) and \( n_k \in P \), we have \( c_{n_{k+1}} < c_{n_k} \), thus \( b_{k+1} < b_k \).
Therefore, \((b_n)\) is a strictly decreasing subsequence of \((c_n)\).

(2) If \( P \) is a finite set, then \( \exists M \in \mathbb{N} \ \forall n \in \mathbb{N}_M \ n \notin P \), thus \( \forall n \in \mathbb{N}_M \ \exists m \in \mathbb{N}_M \) such that \( c_n \leq c_m \).
(i) Let \( n_0 \) be any element of \( \mathbb{N}_M \) (e.g. \( n_0 := M \)), and define \( b_0 := c_{n_0} \).
(ii) If \( n_k \in \mathbb{N}_M \) is chosen for \( k \in \mathbb{N} \), and \( b_k := c_{n_k} \) is defined, then we choose for \( k+1 \) an element \( n_{k+1} \in \{n \in \mathbb{N}_M : n > n_k\} \) such that \( c_{n_{k+1}} \leq c_{n_{k+1}} \), and define \( b_{k+1} := c_{n_{k+1}} \).
Hence, \((b_n)\) is an increasing subsequence of \((c_n)\).

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**Theorem (3.7).** (Bolzano-Weierstrass theorem.)

Every bounded sequence in \( \mathbb{R} \) has a convergent subsequence.

**Proof.**

This is an immediate consequence of the theorems (3.6) and (3.4):

Let \((c_n)\) be a bounded sequence in \( \mathbb{R} \) and \((b_n)\) be a monoton subsequence of \((c_n)\).

Since \((b_n)\) is also bounded in \( \mathbb{R} \), \((b_n)\) is convergent. \( \star \)

**Definition.** (Cauchy sequences.)

Let \((c_n)\) be a sequence in \( \mathbb{R} \). We say that \((c_n)\) is a Cauchy sequence if \( \forall \varepsilon \in \mathbb{R}^+ \exists M \in \mathbb{N} \) such that \( \forall n, m \in \mathbb{N} \) \( |c_n - c_m| < \varepsilon \).

**Theorem (3.8).** (Cauchy criterion for convergence.)

A sequence in \( \mathbb{R} \) is convergent if and only if it is a Cauchy sequence.

**Proof.**

(1) Let \((c_n)\) be a convergent sequence in \( \mathbb{R} \), \( C := \lim (c_n) \) and \( \varepsilon \in \mathbb{R}^+ \).

Then \( \exists M \in \mathbb{N} \ \forall n \in \mathbb{N}_M \ |c_n - C| < \varepsilon /2 \). Thus, \( \forall n, m \in \mathbb{N}_M \) we have \( |c_n - c_m| = |(c_n - C) + (C - c_m)| \leq |(c_n - C)| + |(C - c_m)| < \varepsilon /2 + \varepsilon /2 = \varepsilon \).

(2) Let \((c_n)\) be a Cauchy sequence in \( \mathbb{R} \) and \( \varepsilon \in \mathbb{R}^+ \).

Then \( \exists M_1 \in \mathbb{N} \ \forall n, m \in \mathbb{N}_{M_1} \ |c_n - c_m| < \varepsilon /2 \).

(i) Since \( \exists L \in \mathbb{N} \ \forall n \in \mathbb{N}_L \ |c_n - c_L| < 1 \) and \( \{c_n: n < L\} \) is bounded, \((c_n)\) is a bounded sequence.

(ii) Therefore, \( \exists (c_{n_k}) \) a convergent subsequence of \((c_n)\), and since \((n_k)\) is a strictly increasing sequence in \( \mathbb{N} \), \( \forall k \in \mathbb{N} \ n_k \geq k \) is satisfied.

Let \( C := \lim (c_{n_k}) \). Then \( \exists M \in \mathbb{N}_{M_1} \ \forall k \in \mathbb{N}_M \ |c_{n_k} - C| < \varepsilon /2 \).

Hence, we obtain that \( \forall k \in \mathbb{N}_M \ |c_k - C| = |(c_k - c_{n_k}) + (c_{n_k} - C)| \leq |(c_k - c_{n_k})| + |(c_{n_k} - C)| < \varepsilon /2 + \varepsilon /2 = \varepsilon \). \( \star \)

**Theorem (3.9).** (Limits and order.)

Let \((a_n)\) and \((b_n)\) be sequences in \( \mathbb{R} \) having limits in \( \mathbb{R} \), \( A := \lim (a_n), \ B := \lim (b_n) \).

(1) If \( A < B \), then \( \exists M \in \mathbb{N} \) such that \( \forall n \in \mathbb{N}_M \ a_n < b_n \).

(2) If \( \exists M \in \mathbb{N} \) such that \( \forall n \in \mathbb{N}_M \ a_n \leq b_n \), then \( A \leq B \).

**Proof.**

(1) Let \( C \in \mathbb{R} \) such that \( A < C < B \) holds.

Then by the definition of the limits \( \exists M \in \mathbb{N} \ \forall n \in \mathbb{N}_M \ a_n < C < b_n \).

(2) Suppose \( B < A \). Then it follows from (1) that \( \exists L \in \mathbb{N} \) such that \( \forall n \in \mathbb{N}_L \ b_n < a_n \), a contradiction. \( \star \)

**Theorem (3.10).** (”Sandwiching” theorem.)

Let \((a_n), (b_n), (x_n)\) be sequences in \( \mathbb{R} \) such that \( \exists \lim (a_n) = \lim (b_n) =: A \in \mathbb{R} \), and \( \exists M \in \mathbb{N} \ \forall n \in \mathbb{N}_M \ a_n \leq x_n \leq b_n \). Then \( \exists \lim (x_n) = A \).

**Proof.** Let \( \varepsilon \in \mathbb{R}^+ \).

Then \( \exists M^* \in \mathbb{N}_M \) such that \( \forall n \in \mathbb{N}_{M^*} \ A - \varepsilon < a_n \leq x_n \leq b_n < A + \varepsilon \). \( \star \)
Definition. (Null sequences.)
Let \((c_n)\) be a sequence in \(\mathbb{R}\). We say that \((c_n)\) is a null sequence if \(\exists \lim (c_n) = 0\).

Proposition (3.11). (Convergence and null sequences.)
Let \((c_n)\) be a sequence in \(\mathbb{R}\) and \(C \in \mathbb{R}\).
Then \(\lim (c_n) = C\) if and only if \((c_n - C)\) is a null sequence.

Proof. Let \(\varepsilon \in \mathbb{R}^+\).
\[
\lim (c_n - C) = 0 \iff \exists M \in \mathbb{N} \ \forall n \in \mathbb{N}_M \ | (c_n - C) - 0 | < \varepsilon \iff \exists M \in \mathbb{N} \ \forall n \in \mathbb{N}_M \ | c_n - C | < \varepsilon \iff \lim (c_n) = C.
\]

Proposition (3.12). (Null sequences and algebraic operations.)
Let \((a_n)\) and \((b_n)\) be sequences in \(\mathbb{R}\).
(1) If \((a_n)\) and \((b_n)\) are null sequences, then \((a_n + b_n)\) is a null sequence.
(2) If \((a_n)\) is bounded in \(\mathbb{R}\), and \((b_n)\) is a null sequence, then \((a_n \cdot b_n)\) is a null sequence.

Proof. Let \(\varepsilon \in \mathbb{R}^+\).
(1) \(\exists M_1 \in \mathbb{N} \ \forall n \in \mathbb{N}_{M_1} \ | a_n | < \varepsilon / 2\), and \(\exists M_2 \in \mathbb{N} \ \forall n \in \mathbb{N}_{M_2} \ | b_n | < \varepsilon / 2\) \(\Rightarrow \ \forall n \geq \max \{M_1, M_2\} \ | a_n + b_n | \leq | a_n | + | b_n | < \varepsilon / 2 + \varepsilon / 2 = \varepsilon\).
(2) \(\exists K \in \mathbb{R}^+ \ \forall n \in \mathbb{N} \ | a_n | \leq K\), and \(\exists M \in \mathbb{N} \ \forall n \in \mathbb{N}_M \ | b_n | < \varepsilon / K\) \(\Rightarrow \ \forall n \in \mathbb{N}_M \ | a_n \cdot b_n | = | a_n | \cdot | b_n | < K \cdot \varepsilon / K = \varepsilon\).

Theorem (3.13). (Convergence and algebraic operations.)
If \((a_n)\) and \((b_n)\) are convergent sequences in \(\mathbb{R}\), \(A := \lim (a_n), B := \lim (b_n)\), then
(1) \((a_n + b_n)\) and \((a_n \cdot b_n)\) are convergent sequences,
\n\[
\lim (a_n + b_n) = A + B \text{ and } \lim (a_n \cdot b_n) = A \cdot B.
\]
(2) If \(B \neq 0\) and \(\forall n \in \mathbb{N} \ | b_n | \neq 0\), then
\n\[
(1 / b_n) \text{ is convergent, and } \lim (1 / b_n) = 1 / B.
\]

Proof. (1) \((a_n - A)\) and \((b_n - B)\) are null sequences \(\Rightarrow \)
\[
\Rightarrow ((a_n - A) + (b_n - B)) = ((a_n + b_n) - (A + B)) \text{ is a null sequence } \Rightarrow \lim (a_n + b_n) = A + B.
\]
\[
\forall n \in \mathbb{N} \ (a_n \cdot b_n) - (A \cdot B) = (a_n - A) \cdot b_n + A \cdot (b_n - B); \ (b_n) \text{ and } (A) \text{ are bounded, } (a_n - A) \text{ and } (b_n - B) \text{ are null sequences } \Rightarrow \Rightarrow ((a_n \cdot b_n) - (A \cdot B)) \text{ is a null sequence } \Rightarrow \lim (a_n \cdot b_n) = A \cdot B.
\]
(2) \(\forall n \in \mathbb{N} \ 1 / b_n - 1 / B = (B - b_n) / (b_n \cdot B); \ \lim (b_n \cdot B) = B^2 \Rightarrow \Rightarrow \exists M \in \mathbb{N} \ \forall n \in \mathbb{N}_M \ B^2 / 2 < b_n \cdot B \Rightarrow 0 < 1 / (b_n \cdot B) < 2 / B^2 \Rightarrow \Rightarrow (1 / (b_n \cdot B)) \text{ is bounded } (and \ (B - b_n) \text{ is a null sequence}) \Rightarrow \Rightarrow ((B - b_n) \cdot (1 / (b_n \cdot B))) \text{ is a null sequence } \Rightarrow (1 / b_n - 1 / B) \text{ is a null sequence } \Rightarrow \lim (1 / b_n) = 1 / B.

Proposition (3.14).
Let \((a_n)\) be a convergent sequence in \(\mathbb{R}\), and \(A := \lim (a_n)\). Then \(\exists \lim(\ | a_n |) = | A |\).

Proof. Let \(\varepsilon \in \mathbb{R}^+\). \(\exists M \in \mathbb{N} \ \forall n \in \mathbb{N}_M \ | | a_n | - | A | | \leq | a_n - A | < \varepsilon\).
Definition. \textit{(Limes superior, limes inferior.)}\newline
Let $(c_n)$ be a sequence in $\mathbb{R}$.

(1) If $(c_n)$ is bounded above in $\mathbb{R}$, let $a_n := \sup \{c_k : k \in \mathbb{N}_n\}$ for all $n \in \mathbb{N}$, if $(c_n)$ is bounded below in $\mathbb{R}$, let $b_n := \inf \{c_k : k \in \mathbb{N}_n\}$ for all $n \in \mathbb{N}$.

Clearly, $(a_n)$ is a decreasing sequence in $\mathbb{R}$, $(b_n)$ is an increasing sequence in $\mathbb{R}$.

By the theorem (3.15) $(a_n)$ and $(b_n)$ have limits in $\mathbb{R}$.

The limit of $(a_n)$ is called the \textit{limes superior} or \textit{upper limit} of the sequence $(c_n)$, and is denoted by the symbol $\limsup(c_n)$.

The limit of $(b_n)$ is called the \textit{limes inferior} or \textit{lower limit} of the sequence $(c_n)$, and is denoted by the symbol $\liminf(c_n)$.

It follows from the theorem (3.4) that if $(c_n)$ is bounded above in $\mathbb{R}$, then $\limsup(c_n) = \inf \{a_n : n \in \mathbb{N}\} = \inf \{\sup \{c_k : k \in \mathbb{N}_n\} : n \in \mathbb{N}\}$, if $(c_n)$ is bounded below in $\mathbb{R}$, then $\liminf(c_n) = \sup \{a_n : n \in \mathbb{N}\} = \sup \{\inf \{c_k : k \in \mathbb{N}_n\} : n \in \mathbb{N}\}$.

(2) If $(c_n)$ is not bounded above in $\mathbb{R}$, we define $\limsup(c_n) := +\infty$, and if $(c_n)$ is not bounded below in $\mathbb{R}$, we define $\liminf(c_n) := -\infty$.

\textbf{Theorem (3.15).}\newline
Let $(c_n)$ be a sequence in $\mathbb{R}$ and $A \in \mathbb{R}$. The following propositions are equivalent:

(1) $\limsup(c_n) = A$.

(2) For all $G \in \mathbb{R}$, if $G > A$ then $\{n \in \mathbb{N} : c_n \geq G\}$ is a finite set, and for all $L \in \mathbb{R}$, if $L < A$ then $\{n \in \mathbb{N} : c_n > L\}$ is an infinite set.

\textbf{Proof.} \\
(1) $\Rightarrow$ (2) \\
Let $G$, $L \in \mathbb{R}$ such that $G > A$ (if $A \neq +\infty$) and $L < A$ (if $A \neq -\infty$).

Then $\exists M \in \mathbb{N}$, $a_M = \sup \{c_k : k \in \mathbb{N}_M\} < G$, thus $\forall k \in \mathbb{N}_M$, $c_k < G$, and $\forall m \in \mathbb{N}$, $a_m = \sup \{c_k : k \in \mathbb{N}_m\} \geq A > L$, hence $\forall m \in \mathbb{N}$, $\exists n \in \mathbb{N}_m$, $c_n > L$.

(2) $\Rightarrow$ (1) \\
If $G \in \mathbb{R}$ and $G > A$, then $\{n \in \mathbb{N} : c_n \geq G\}$ is a finite set, thus $\exists M \in \mathbb{N}$, $\forall k \in \mathbb{N}_M$, $c_k < G$, and hence $a_M = \sup \{c_k : k \in \mathbb{N}_M\} \leq G$.

Therefore, $\limsup(c_n) \leq G$ for all $G > A$, which implies that $\limsup(c_n) \leq A$.

If $L \in \mathbb{R}$ and $L < A$, then $\{n \in \mathbb{N} : c_n > L\}$ is an infinite set, thus $\forall m \in \mathbb{N}$, $\exists n \in \mathbb{N}_m$, $c_n > L$, and hence $\forall m \in \mathbb{N}$, $a_m = \sup \{c_k : k \in \mathbb{N}_m\} > L$.

Therefore, $\limsup(c_n) \geq L$ for all $L < A$, which implies that $\limsup(c_n) \geq A$.\hfill\blackdiamond

\textbf{Remark. (3.16).}\newline
The corresponding characterization of the \textit{lower limit} can be formulated as follows:

$\liminf(c_n) = B \in \mathbb{R}$ \textit{if and only if}

for all $L \in \mathbb{R}$, $L < B$ implies that $\{n \in \mathbb{N} : c_n \leq L\}$ is a finite set, and for all $G \in \mathbb{R}$, $G > B$ implies that $\{n \in \mathbb{N} : c_n < G\}$ is an infinite set.
Proposition (3.17).
Let \((c_n)\) be a sequence in \(\mathbb{R}\) and \((d_n)\) be a subsequence of \((c_n)\).
If \(\exists \lim (d_n) \in \mathbb{R}\), then \(\liminf (c_n) \leq \lim (d_n) \leq \limsup (c_n)\).

Proof.
Suppose that \(\limsup (c_n) < \lim (d_n)\). Then \(\exists D \in \mathbb{R}\) \(\limsup (c_n) < D < \lim (d_n)\).
By the theorem (3.15), \(\{n \in \mathbb{N} : D \leq c_n\} \supset \{n \in \mathbb{N} : D \leq d_n\}\) is a finite set,
which is impossible, since \(D < \lim (d_n)\) implies that \(\exists M \in \mathbb{N} \ \forall n \in \mathbb{N}_M\) \(D < d_n\).
On the other hand, if \(\lim (d_n) < \liminf (c_n)\), then \(\exists K \in \mathbb{R}\) \(\lim (d_n) < K < \liminf (c_n)\).
By the remark (3.16), \(\{n \in \mathbb{N} : c_n \leq K\} \supset \{n \in \mathbb{N} : d_n \leq K\}\) is a finite set,
which is impossible, since \(\lim (d_n) < K\) implies that \(\exists M \in \mathbb{N} \ \forall n \in \mathbb{N}_M\) \(d_n < K\).

Proposition (3.18).
Let \((c_n)\) be a sequence in \(\mathbb{R}\). There exist \((d_n)\) and \((s_n)\), subsequences of \((c_n)\),
such that \(\exists \lim (d_n) = \limsup (c_n)\) and \(\exists \lim (s_n) = \liminf (c_n)\).

Proof.
(1) Let \(A := \limsup (c_n)\).
We define a subsequence \((d_n)\) of \((c_n)\), using the theorem (3.15), as follows:
If \(A \in \mathbb{R}\):
\[\exists n_0 \in \mathbb{N} \quad A - 1 < c_{n_0} < A + 1.\]
Define \(d_0 := c_{n_0}\).
Suppose that \(n_k \in \mathbb{N}\) has been already chosen for \(k \in \mathbb{N}\), and \(d_k := c_{n_k}\) is defined.
\[\exists n_{k+1} \in \mathbb{N} \quad n_{k+1} > n_k \quad \text{and} \quad A - 1/(k+1) < c_{n_{k+1}} < A + 1/(k+1).\]
Define \(b_{k+1} := c_{n_{k+1}}\).
Evidently, \((d_n)\) is a convergent subsequence of \((c_n)\) and \(\lim (d_n) = A\).
If \(A = +\infty\):
\[\exists n_0 \in \mathbb{N} \quad 1 < c_{n_0}.\]
Define \(d_0 := c_{n_0}\).
Suppose that \(n_k \in \mathbb{N}\) has been already chosen for \(k \in \mathbb{N}\), and \(d_k := c_{n_k}\) is defined.
\[\exists n_{k+1} \in \mathbb{N} \quad n_{k+1} > n_k \quad \text{and} \quad (k+1) < c_{n_{k+1}}.\]
Define \(b_{k+1} := c_{n_{k+1}}\).
Evidently, \((d_n)\) is a subsequence of \((c_n)\) and \(\lim (d_n) = +\infty\).
If \(A = -\infty\):
\[\exists n_0 \in \mathbb{N} \quad c_{n_0} < -1.\]
Define \(d_0 := c_{n_0}\).
Suppose that \(n_k \in \mathbb{N}\) has been already chosen for \(k \in \mathbb{N}\), and \(d_k := c_{n_k}\) is defined.
\[\exists n_{k+1} \in \mathbb{N} \quad n_{k+1} > n_k \quad \text{and} \quad c_{n_{k+1}} < -(k+1).\]
Define \(b_{k+1} := c_{n_{k+1}}\).
Evidently, \((d_n)\) is a subsequence of \((c_n)\) and \(\lim (d_n) = -\infty\).

(2) Let \(B := \liminf (c_n)\). In an analogous way, using the remark (3.16),
we can define a subsequence \((s_n)\) of \((c_n)\), so that \(\exists \lim (s_n) = B\).

Proposition (3.19).
Let \((c_n)\) be a sequence in \(\mathbb{R}\).
\(\exists \lim (c_n) \in \mathbb{R}\) \(\iff \liminf (c_n) = \limsup (c_n)\),
and then \(\lim (c_n) = \limsup (c_n)\).

Proof.
This is an immediate consequence of the theorems (3.18), (3.15) and remark (3.16).