On the number of representations of integers as the sum of $k$ terms

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November 25, 2009

Abstract

Let $\mathcal{A} = \{a_1, a_2, \ldots \}$ ($a_1 < a_2 < \ldots$) be a fixed infinite sequence of positive integers, and let $R_k(n)$ denote the number of solutions of $a_{i_1} + a_{i_2} + \cdots + a_{i_k} = n$, $a_{i_1} \in \mathcal{A}, \ldots, a_{i_k} \in \mathcal{A}$. For $k = 2$, P. Erdős and A. Sárközy proved if $F(n)$ is a “nice” arithmetic function then there exists a sequence $\mathcal{A}$ such that $|R_2(n) - F(n)| \ll (F(n) \log n)^{1/2}$.

The aim of this paper is to extend their result to $k > 2$ by using probabilistic methods.

2000 AMS Mathematics subject classification number: 11B34

Key words and phrases: additive number theory, general sequences, additive representation function.

1 Introduction

Let $\mathbb{N}$ denote the set of positive integers. Let $k > 2$ be a fixed integer and let $\mathcal{A} = \{a_1, a_2, \ldots \}$ ($a_1 < a_2 < \ldots$) be an infinite sequence of positive integers. For $n = 1, 2, \ldots$ let $R_k(n)$ denote the number of solutions of $a_{i_1} + a_{i_2} + \cdots + a_{i_k} = n$, $a_{i_1} \in \mathcal{A}, \ldots, a_{i_k} \in \mathcal{A}$. For $k = 2$, P. Erdős and A. Sárközy studied how regular the behaviour of the function $R_2(n)$ can be. In [2] they proved the following theorem:

**Theorem 1** If $F(n)$ is an arithmetic function such that

$$F(n) \to +\infty,$$

$$F(n + 1) \geq F(n) \text{ for } n \geq n_0,$$

$$F(n) = o\left(\frac{n}{(\log n)^2}\right),$$
and we write
\[ \Delta(N) = \sum_{n=1}^{N} (R_2(n) - F(n))^2, \]
then
\[ \Delta(N) = o(NF(N)) \]
cannot hold.

In [3] they showed that the above result is nearly best possible:

**Theorem 2** If \( F(n) \) is an arithmetic function satisfying
\[ F(n) > 36 \log n \quad \text{for} \quad n > n_0, \]
and there exist a real function \( g(x) \), defined for \( 0 < x < +\infty \), and real numbers \( x_0, n_1 \) such that

(i) \( g'(x) \) exists and it is continuous for \( 0 < x < +\infty \),
(ii) \( g'(x) \leq 0 \) for \( x \geq x_0 \),
(iii) \( 0 < g(x) < 1 \) for \( x \geq x_0 \),
(iv) \( |F(n) - 2 \int_0^{n/2} g(x)g(n-x)dx| < (F(n) \log n)^{1/2} \)
for \( n > n_1 \),
then there exists a sequence \( A \) such that
\[ |R_2(n) - F(n)| < 8(F(n) \log n)^{1/2} \quad \text{for} \quad n > n_2. \]

In [6] G. Horváth extended Theorem 1 to any \( k > 2 \):  

**Theorem 3** If \( F(n) \) is an arithmetic function such that
\[ F(n) \to +\infty, \]
\[ F(n + 1) \geq F(n) \quad \text{for} \quad n \geq n_0, \]
\[ F(n) = o\left(\frac{n}{(\log n)^2}\right), \]
and we write
\[ \Delta(N) = \sum_{n=1}^{N} (R_k(n) - F(n))^2, \]
then
\[ \Delta(N) = o(NF(N)) \]
cannot hold.
A. Sárközy proposed to prove the analogue of Theorem 2 for \( k > 2 \) [8, Problem 3]. In this paper my goal is to extend Theorem 2 to any \( k > 2 \), i.e., to show that Theorem 3 is nearly best possible. In fact I will prove the following theorem:

**Theorem 4** If \( k > 2 \) is a positive integer, \( c_8 \) is a constant large enough in terms \( k \), \( F(n) \) is an arithmetic function satisfying

\[
F(n) > c_8 \log n \quad \text{for} \quad n > n_0,
\]

and there exists a real function \( g(x) \), defined for \( 0 < x < +\infty \), and real numbers \( x_0, n_1 \) and \( c_7, c_9 \) constants such that

(i) \( 0 < g(x) \leq \frac{(\log x)^k}{x^{1+1/k}} < 1 \) for \( x \geq x_0 \),

(ii) \[
\left| F(n) - k! \sum_{1 \leq x_1 < x_2 < \ldots < x_k < n} g(x_1)g(x_2)\ldots g(x_k) \right| < c_7(F(n) \log n)^{1/2}
\]

for \( n > n_1 \),

then there exists a sequence \( A \) such that

\[
|R_k(n) - F(n)| < c_9(F(n) \log n)^{1/2} \quad \text{for} \quad n > n_2.
\]

It is easy to see that the following functions satisfy the conditions of Theorem 4: \( g(x) = c_{10}\left(\frac{(\log x)^\alpha}{x^\beta}\right) \), where \( c_{10} \) is a positive constant, \( \alpha > 1 - \frac{k+1}{k^2} \), or \( \alpha = 1 - \frac{k+1}{k^2} \) and \( \beta \leq 1/k \). It follows that for \( F(n) = n^\delta(\log n)^\gamma \) with \( 0 < \delta \leq 1/k \), or \( 0 \leq \gamma < 1 \) there is a sequence \( A \) for which \( R_k(n) \) satisfies the inequality at the end of the theorem. For \( k = 2 \) in [3] P. Erdős and A. Sárközy used probabilistic method to construct a sequence \( A \). In the case \( k = 2 \), in their paper certain events were mutually independent. For \( k > 2 \) the independency fails, thus in order to prove Theorem 4 we need deeper probabilistic tools.

### 2 Probabilistic tools

The proof of Theorem 4 is based on the probabilistic method due to Erdős and Rényi. There is an excellent summary of this method in the Halberstam - Roth book [5]. We use the notation and terminology of this book. First we give a survey of the probabilistic tools and notations which we use in the proof of Theorem 4. Let \( \Omega \) denote the set of strictly increasing sequences of positive integers. In this paper we denote the probability of an event \( E \) by \( P(E) \).
Lemma 1 Let
\[ \alpha_1, \alpha_2, \alpha_3 \ldots \]
be real numbers satisfying
\[ 0 \leq \alpha_n \leq 1 \quad (n = 1, 2, \ldots). \]
Then there exists a probability space \((\Omega, S, P)\) with the following two properties:

(i) For every natural number \(n\), the event \(E^{(n)} = \{ A \in \Omega : n \in A \} \) is measurable, and \(P(E^{(n)}) = \alpha_n\).

(ii) The events \(E^{(1)}, E^{(2)}, \ldots\) are independent.

See Theorem 13. in [5], p. 142. We denote the characteristic function of the event \(E^{(n)}\) by \(\varrho(A, n)\):

\[ \varrho(A, n) = \begin{cases} 
1, & \text{if } n \in A \\
0, & \text{if } n \notin A.
\end{cases} \]

Furthermore, we denote the number of solutions of \(a_{i_1} + a_{i_2} + \ldots + a_{i_k} = n\) by \(r_k(n)\), where \(a_{i_1} \in A, a_{i_2} \in A, \ldots, a_{i_k} \in A, 1 \leq a_{i_1} < a_{i_2} \ldots < a_{i_k} < n\). Thus

\[ r_k(n) = \sum_{\substack{(a_1, a_2, \ldots, a_k) \in \mathbb{N}^k \\
1 \leq a_1 < a_2 < \ldots < a_k < n \\
a_1 + a_2 + \ldots + a_k = n}} \varrho(A, a_1) \varrho(A, a_2) \ldots \varrho(A, a_k). \]

Let \(r_k^*(n)\) denote the number of those representations of \(n\) in the form \(a_{i_1} + a_{i_2} + \ldots + a_{i_k} = n\) in which there are at least two equal terms. Thus we have

\[ R_k(n) = k! r_k(n) + r_k^*(n). \]

It is easy to see from (3) that \(r_k(n)\) is the sum of random variables. However for \(k > 2\) these variables are not independent because the same \(\varrho(A, a_i)\) may appear in many terms, thus we need deeper probabilistic tools.

Our proof is based on a method of J. H. Kim and V. H. Vu. In the next section we give a short survey of this method. Interested reader can find more details in [7], [9], [10]. Assume that \(t_1, t_2, \ldots, t_n\) are independent binary (i.e., all \(t_i\)'s are in \(\{0, 1\}\)) random variables. Consider a polynomial \(Y\) in \(t_1, t_2, \ldots, t_n\) with degree \(k\). We say a polynomial \(Y\) is positive if it can be written in the form \(Y = \sum e_i \Gamma_i\), where the \(e_i\)'s are positive and \(\Gamma_i\) is a product of some \(t_j\)'s. Given a (multi-) set \(A\), \(\partial_A(Y)\) denotes the partial derivative of \(Y\) with respect to the variables with indices in \(A\). For
instance, if $Y = t_1 t_2$ and $A_1 = \{1, 2\}$ and $A_2 = \{2, 2\}$ then $\partial_{A_1}(Y) = 2t_2$ and $\partial_A Y = 2t_1$. If $A$ is empty then $\partial_A(Y) = Y$. $E_A(Y)$ denotes the expectation of $\partial_A(Y)$. Furthermore, set $E_j(Y) = \max_{|A| \geq j} E_A(Y)$, for all $j = 0, 1, \ldots, k$, thus $E_0(Y) = E(Y)$.

**Theorem 5** (J. H. Kim - V. H. Vu) For every positive integer $k$ there are positive constants $d_k$ and $b_k$ depending only on $k$ such that the following holds. For any positive polynomial $Y = Y(t_1, t_2, \ldots, t_n)$ of degree $k$, where the $t_i$’s are independent binary random variables,

$$P\left(|Y - E(Y)| \geq d_k \lambda^k \sqrt{E_0(Y) E_1(Y)}\right) \leq b_k e^{-\lambda/4 + (k-1) \log n}.$$  

See [7] for the proof. Finally we need the Borel - Cantelli lemma (see in [5]):

**Lemma 2** Let $\{B_i\}$ be a sequence of events in a probability space. If

$$\sum_{j=1}^{+\infty} P(B_j) < \infty,$$

then with probability 1, at most a finite number of the events $B_j$ can occur.

### 3 Proof of Theorem 4

Fix a number $n$ and write

$$S_n = \{(a_1, a_2, \ldots, a_k) \in \mathbb{N}^k : 0 < a_1 < a_2 \ldots < a_k < n, a_1 + a_2 + \ldots + a_k = n\}.$$  

Define the sequence (1) of real numbers by

$$\alpha_n = \begin{cases} 
  g(n), & \text{if } n \geq x_0 \\
  0, & \text{otherwise}
\end{cases}$$

and the probability space $(\Omega, S, P)$ as described in Lemma 1. Clearly the sequence $\alpha_n$ satisfies (2). Thus

$$r_k(n) = \sum_{(a_1, a_2, \ldots, a_k) \in S_n} t_{a_1} t_{a_2} \ldots t_{a_k},$$

where

$$t_{a_i} = \begin{cases} 
  1, & \text{if } a_i \in A \\
  0, & \text{if } a_i \notin A
\end{cases}.$$
Then we have
\[ \lambda_n = E(r_k(n)) = \sum_{(a_1, a_2, \ldots, a_k) \in S_n} P(a_1 \in \mathcal{A})P(a_2 \in \mathcal{A}) \ldots P(a_k \in \mathcal{A}), \]
where \( E(\zeta) \) denotes the expectation of the random variable \( \zeta \). To prove Theorem 4 we will give an upper estimation for \( |R_k(n) - k!\lambda_n| \). As Vu in [10] we split \( r_k(n) \) into two parts, as follows. Let \( a \) be a small positive constant say \( a < \frac{1}{2(k+1)} \) and let \( S_n^{[1]} \) be the subset of \( S_n \) consisting of all \( k \)-tuples whose smallest element is at least \( na \), i.e., \( S_n^{[1]} = \{ (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k : na \leq a_1 < a_2 \ldots < a_k < n, a_1 + a_2 + \ldots + a_k = n \} \) and \( S_n^{[2]} = S_n \setminus S_n^{[1]} \). We split \( r_k(n) \) into the sum of two terms corresponding to \( S_n^{[1]} \) and \( S_n^{[2]} \), respectively:
\[ r_k(n) = r_k^{[1]}(n) + r_k^{[2]}(n), \]
where
\[ r_k^{[j]}(n) = \sum_{(a_1, a_2, \ldots, a_k) \in S_n^{[j]}} t_{a_1}t_{a_2} \ldots t_{a_k}, \quad (5) \]
and
\[ \lambda_n^{[j]} = E(r_k^{[j]}(n)). \]
Clearly
\[ |R_k(n) - k!\lambda_n| \leq |R_k(n) - k!r_k(n)| + k!|r_k(n) - \lambda_n| \]
\[ = r_k^*(n) + k!|r_k^{[1]}(n) + r_k^{[2]}(n) - \lambda_n^{[1]} - \lambda_n^{[2]}| \]
\[ \leq r_k^*(n) + k!|r_k^{[1]}(n) - \lambda_n^{[1]}| + k!|r_k^{[2]}(n) - \lambda_n^{[2]}| \]
\[ = r_k^*(n) + I_1 + I_2. \]

The remaining part of the proof of Theorem 4 has four parts. In the first part we give an upper estimation for \( I_1 \), in the second part we give an upper estimation for \( I_2 \), in the third part we give an upper estimation for \( r_k^*(n) \), and in the last part we complete the proof of Theorem 4.

To estimate \( I_1 \) we will apply Theorem 5 so we need an upper bound for \( E_1(r_k^{[1]}(n)) \). To do this, it is clear from the definition of \( E_1 \) that we need the following lemma, which guarantees that every partial derivative of \( r_k^{[1]}(n) \) has small expectation.

**Lemma 3** For all non-empty multi-sets \( A \) of size at most \( k - 1 \),
\[ E(\partial_A(r_k^{[1]}(n))) = O(n^{-a/2k^2}). \]
**Proof.** This can be proved similarly to Lemma 5.3 in [10]. For the sake of completeness I will present the proof. Consider a multi-set \( A \) of \( k-l \) elements and \( \sum_{x \in A} x = n - m \). There exists a constant \( c(k) \) such that

\[
\partial_A(r_k^{[l]}(n)) \leq c(k) \sum_{n^a_1<n^a_2<\ldots<n^a_l \atop a_1+\ldots+a_l=m} t_{a_1} t_{a_2} \cdots t_{a_l}.
\]

As \( a_l \geq m/l \), and using the fact that \( \sum_{x=1}^m x^{1/k-1} \approx \int_1^m z^{1/k-1} dz \approx m^{1/k} \), and (i) in Theorem 4, we have

\[
E(\partial_A(r_k^{[l]})) = O \left( \sum_{n^a_1<n^a_2<\ldots<n^a_l \atop a_1+\ldots+a_l=m} P(a_1 \in A) \cdots P(a_l \in A) \right)
\]

\[
= O \left( \sum_{n^a_1<n^a_2<\ldots<n^a_l \atop a_1+\ldots+a_l=m} g(a_1) g(a_2) \cdots g(a_l) \right)
\]

\[
= O(\log n) \sum_{n^a_1<n^a_2<\ldots<n^a_l \atop a_1+\ldots+a_l=m} a_1^{k+1} - 1 \cdot a_2^{k+1} - 1 \cdot \cdots \cdot a_l^{k+1} - 1
\]

\[
= O(\log n) O \left( \sum_{x=1}^m x^{1/k-1} \right)^{i-1} (m/l)^{k+1-1}
\]

\[
= O(\log n) O(m^{(l-1)(k+1)} (m/l)^{k+1/k^2-1}) = O(\log n) O(m^{(l(k-1)-k^2)/k^2}) = O(n^{-a/2k^2}),
\]

since \( k-1 \geq l \) and \( m \geq n^a \). The proof of Lemma 3 is completed.

By the definition of \( E_1(r_k^{[l]}(n)) \), and from Lemma 3 it is clear that \( E_1(r_k^{[l]}(n)) = \max_{|A| \geq 1} E_A(r_k^{[l]}(n)) \leq cn^{-a/2k^2} \), where \( c \) is a constant. It is clear from (5) that \( r_k^{[l]}(n) \) is a positive polynomial of degree \( k \). Now we apply Theorem 5 with \( \lambda = \left( \frac{\log n}{E_1(r_k^{[l]}(n))} \right)^{1/k} \). If \( n \) is large enough we have

\[
P(\|r_k^{[l]}(n) - \lambda_n^{[l]}\| \geq d_k \left\lfloor \frac{\log n}{E_1(r_k^{[l]}(n))} \right\rfloor \sqrt{\lambda_n^{[l]} E_1(r_k^{[l]}(n))} \leq
\]

\[
\leq b_k \exp \left( \frac{-1}{4} \sqrt\frac{\log n}{E_1(r_k^{[l]}(n))} \right) + (k-1) \log n \leq b_k \exp \left( \frac{-1}{4} \sqrt\frac{\log n}{n^{-a/2k^2}} \right) + (k-1) \log n
\]

\[
< \exp(-2 \log n) < \frac{1}{n^2}.
\]
Applying the above result we obtain
\[ \sum_{n=1}^{+\infty} P\left( |r_k^{[1]}(n) - \lambda_n^{[1]}| \geq d_k \sqrt{\lambda_n^{[1]} \log n} \right) < \sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty. \]

By the Borel - Cantelli lemma with probability 1, there exists a number \( n_0 \) such that
\[ |r_k^{[1]}(n) - \lambda_n^{[1]}| < d_k \sqrt{\lambda_n^{[1]} \log n} \text{ for } n > n_0. \]  

(7)

In the next section we will give an upper estimation for \( I_2 \). We prove similarly to the proof in [10] that for almost every sequence \( A \), there is a finite number \( c_{11}(A) \) such that \( r_k^{[2]}(n) \leq c_{11}(A) \) for all sufficiently large \( n \).

Let \( r_l(n) \) denote the number of representations of \( n \) as the sum of \( l \) distinct numbers from \( A \). First we give an upper estimation for \( E(r_l(n)) \) similarly to the estimate in [4]. Let \( 2 \leq l \leq (k-1) \) be fixed. By \( n/l < a_l \), and (i) in Theorem 4, we have
\[ E(r_l(n)) \leq \sum_{a_1 + a_2 + \ldots + a_l = n, 1 \leq a_1 < a_2 < \ldots < a_l < n} P(a_1 \in A) \ldots P(a_l \in A) \]
(8)

\[ < \sum_{a_1 + a_2 + \ldots + a_l = n, 1 \leq a_1 < a_2 < \ldots < a_l < n} g(a_1)g(a_2)\ldots g(a_l) \]

\[ \leq \sum_{a_1 + a_2 + \ldots + a_l = n, 1 \leq a_1 < a_2 < \ldots < a_l < n} \frac{(\log a_1)^{\frac{1}{2}}}{a_1^{1 - \frac{k+1}{k^2}}} \ldots \frac{(\log a_l)^{\frac{1}{2}}}{a_l^{1 - \frac{k+1}{k^2}}} = n^{o(1)} \sum_{a_1 + a_2 + \ldots + a_l = n, 1 \leq a_1 < a_2 < \ldots < a_l < n} \frac{1}{(a_1 \ldots a_l)^{1 - \frac{k+1}{k^2}}} \]

\[ \leq n^{o(1)} \left( n^{\frac{k+1}{k^2} - 1 + o(1)} \sum_{a_1 + a_2 + \ldots + a_l = n, 1 \leq a_i < a_{i+1} < \ldots < a_l < n} \frac{1}{(a_1 \ldots a_l)^{1 - \frac{k+1}{k^2}}} \right) \]

\[ \leq n^{\frac{k+1}{k^2} - 1 + o(1)} \left( \sum_{1 \leq a_i < a_{i+1} < \ldots < a_l < n} \frac{1}{a_1^{1 - \frac{k+1}{k^2}}} \right)^{l-1} \]

\[ = n^{\frac{k+1}{k^2} - 1 + o(1)} \left( n^{\frac{k+1}{k^2} + o(1)} \right)^{l-1} = n^{-1 + \frac{k+1}{k^2} + o(1)}. \]

Let \( T_1 = \{a_1, a_2, \ldots a_k\}, T_2 = \{b_1, b_2, \ldots b_k\}, T_1 \neq T_2, T_1, T_2 \subset A \) and
\[ a_1 + a_2 + \ldots + a_k = b_1 + b_2 + \ldots + b_k = n. \]
We say these representations are disjoint if they share no element in common. Let \( f_l(n) \) denote the maximum number of pairwise disjoint representations of \( n \) as the sum of \( l \) distinct numbers from \( A \). We show that with probability 1, \( f_l(n) \) is bounded. We will apply the following result due to Erdős and Tetali which is called disjointness lemma. We say events \( G_1, \ldots, G_n \) are independent if for all subsets \( I \subseteq \{1, \ldots, n\} \), \( P(\cap_{i \in I} G_i) = \prod_{i \in I} P(G_i) \).

**Lemma 4** If \( \sum_i P(B_i) \leq \mu \), then

\[
\sum_{(B_1, \ldots, B_l) \text{ independent}} P(B_1 \cap \ldots \cap B_l) \leq \frac{\mu^l}{l!}.
\]

**Proof.** This is Lemma 1 in [4]. Let

\( B = \{(a_1, \ldots, a_l) : a_1 + \ldots + a_l = n, a_i \in A, 1 \leq a_1 < \ldots < a_l < n\} \).

Let \( H(B) = \{T \subset B : \text{all the } K \in T \text{ are pairwise disjoint}\} \) and \( c_1 \) be a constant. It is clear that the pairwise disjointness of the sets implies the independence of the associated events, i.e., if \( K_1 \) and \( K_2 \) are pairwise disjoint representations, the events \( K_1 \subset A, K_2 \subset A \) are independent. Thus by (8) and Lemma 4 we have

\[
P(f_l(n) > c_1) \leq P\left( \bigcup_{T \in H(B)} \bigcap_{K \in T} K \right) \leq \sum_{T \in H(B)} P\left( \bigcap_{K \in T} K \right) \leq \sum_{(K_1, \ldots, K_{c_1+1}) \text{ pairwise disjoint}} P(K_1 \cap \ldots \cap K_{c_1+1})
\]

\[
\leq \frac{1}{(c_1 + 1)!} \left( E(f_l(n)) \right)^{c_1+1} \leq \frac{1}{(c_1 + 1)!} \left( E(r_l(n)) \right)^{c_1+1}
\]

if \( c_1 \) large enough. By the Borel - Cantelli lemma, with probability 1 for almost every random sequence \( A \) there is a finite number \( c_1(A) \) such that for any \( l < k \) and all \( n \), the maximal number of disjoint \( l \)-representations of \( n \) from \( A \) is at most \( c_1(A) \). In the next step we give an upper estimation for \( E(r_k^{[2]}(n)) \) similarly as in Lemma 3. Using also the fact that \( \sum_{x=1}^n x^{1/k-1} \approx \int_1^n x^{1/k-1} \, dx \approx n^{1/k} \), and \( a_k \geq n/k, a < \frac{1}{2(k+1)} \), and (i) in Theorem 4, we have

\[
E(r_k^{[2]}(n)) = E\left( \sum_{(a_1, a_2, \ldots, a_k) \in S_n^{[2]}} t_{a_1} \ldots t_{a_k} \right)
\]
\[
O\left(\sum_{(a_1, a_2, \ldots, a_k) \in S_n^k} p(a_1 \in A) \ldots p(a_k \in A)\right)
\]

\[
= O(\log n) \sum_{a_1 + a_2 + \ldots + a_k = n, a_1 \leq n^a} \left(\sum_{x = 1}^{n^a} x^{k+1-1} \left(\sum_{x = 1}^{n^a} x^{k+1-1}\right)^{k-2} \left(n/k\right)^{k+1-1}\right)
\]

\[
= O(\log n) O\left(\sum_{x = 1}^{n^a} x^{k+1-1} \left(\sum_{x = 1}^{n^a} x^{k+1-1}\right)^{k-2} \left(n/k\right)^{k+1-1}\right)
\]

\[
= O\left(n^{a(k+1)-1} \log n\right) = O(n^{-1/2k^2}).
\]

Thus by Lemma 4 and the Borel-Cantelli lemma, with probability 1, there is a constant \(c_2\) such that almost surely the maximum number of disjoint representations of \(n\) in \(r_k^{[2]}(n)\) is at most \(c_2\) for all large \(n\). To finish the proof it suffices to show that \(r_k^{[2]}(n)\) is bounded by a constant. The proof is purely combinatorial. We need the following well-known result due to Erdős and Rado [1]. Let \(r\) be a positive integer, \(r \geq 3\). A collection of sets \(D_1, D_2, \ldots, D_r\) forms a \(\Delta\)-system if the sets have pairwise the same intersection.

**Lemma 5** If \(H\) is a collection of sets of size at most \(k\) and \(|H| > (r - 1)^{k!}\) then \(H\) contains \(r\) sets forming a \(\Delta\)-system.

Set \(C(A) = \left(max(c_1(A), c_2)\right)^k\) and assume that \(n\) is sufficiently large. To each representation of \(n\) counted in \(r_k^{[2]}(n)\) we assign the set formed by the \(k\) terms occurring in this representation. We will apply Lemma 5 with the collection of these sets in place of \(H\). It is clear that if \(r_k^{[2]}(n) > C(A)\), then by Lemma 5, \(r_k^{[2]}(n)\) contains a \(\Delta\) - system with \(c_3 = max(c_1(A), c_2) + 1\) sets. If the intersection of these sets is empty, then they form a family of \(c_3\) disjoint \(k\)-representations of \(n\), which contradicts the definition of \(c_3\). Otherwise, assume that the intersection of these sets is \(\{y_1, y_2, \ldots, y_j\}\), where \(1 \leq j \leq k - 1\), and \(\sum_{i=1}^{j} y_i = m\). Removing the common intersection of these sets we can find \(c_1(A) + 1\) \((k - j)\) representations of \(n - m = n - \sum_{i=1}^{j} y_i\). These \(c_1(A) + 1\) sets are disjoint due to the definition of the \(\Delta\) - system. Therefore in both cases we obtain a contradiction.

In the next section we will give an upper estimation for \(r_k^*(n)\). If we collect the equal terms, we have

\[
u_1 a_1 + u_2 a_2 + \ldots + u_h a_h = n,
\]  

(10)

where the \(u_i\)'s are positive integers, and

\[
u_1 + u_2 + \ldots + u_h = k.
\]

(11)
Thus $r^*_k(n)$ denotes the number of representations of $n$ in the form (10), where the $a_i$'s are different. It can be proved similarly to the estimate of $r^*_k(n)$, that $r^*_k(n)$ is also bounded by a constant. For the sake of completeness we sketch the proof and we leave the details to the reader. Let $2 \leq h \leq k - 1$ be fixed. For a fixed $u_1, \ldots, u_h$ let $s_h(n)$ denote the number of representations of $n$ in the form (10). We show that $s_h(n)$ is bounded by a constant. (Note that in the previous section we proved this in the case when all $u_i$'s are equal to one, and $h = k$). First we will give an upper estimation for $E(s_h(n))$, with a calculation similar to (8). Using the definition of $s_h(n)$, and $n/h < a_h$, we have

$$E(s_h(n)) \leq \sum_{a_1 < a_2 < \ldots < a_h < n} P(a_1 \in \mathcal{A})P(a_2 \in \mathcal{A}) \ldots P(a_h \in \mathcal{A}) (12)$$

$$= \sum_{a_1 < a_2 < \ldots < a_h < n} g(a_1)g(a_2) \ldots g(a_h)$$

$$\leq \sum_{a_1 < a_2 < \ldots < a_h < n} \frac{(\log a_1)^{\frac{k}{2}}}{a_1^{1 - \frac{k+1}{2}}} \ldots \frac{(\log a_h)^{\frac{k}{2}}}{a_h^{1 - \frac{k+1}{2}}}$$

$$= n^{-1 + \frac{k+1}{2} + o(1)}.$$ 

Let $s^*_h(n)$ denote the size of a maximal collection of pairwise disjoint representations in the form (10). The same argument as in (9) shows that there exists a constant $v_h$ such that for $n$ large enough $s^*_h(n) < v_h$. In view of (12), and applying Lemma 4 we have

$$P(s^*_h(n) > v_h) < n^{-2 + o(1)},$$

if $v_h$ is large enough. Thus by the Borel - Cantelli lemma we get $s^*_h(n) < v_h$ for $n$ large enough. We say that an $m$ - tuple $(a_1, \ldots, a_m)$ $(m \leq h)$ is an $m$ - representation of $n$ in the form (10) if there is a permutation $\pi$ of the numbers \{1, 2, $\ldots$, $h$$\}$ such that $\sum_{i=1}^{m} a_{\pi(i)} = n$. For all $m < h$, let $s^*_m(n)$ denote the size of a maximal collection of pairwise disjoint such representations of $n$. The same argument as above shows that there exists a constant $p_m$ such that for every $n$, $s^*_m(n) < p_m$. In the last step we apply Lemma 5 to prove that $s_h(n)$ is bounded by a constant. Let $C = \left(\max(p_m h!, v_h)\right)^h$. Let $H$ in Lemma 5 is the collection of representations of $n$ in the form (10). Clearly $|H| = s_h(n)$. If $s_h(n) > C$, and $n$ is sufficiently large then by Lemma 5, $H$ contains a $\Delta$ - system with $C + 1$ sets. If the intersection of these sets is
empty, then they form a family of disjoint $h$-representations in the form 
(10). Otherwise let the common intersection of the sets be \{\(y_1, \ldots, y_s\)\}, where \(1 \leq s \leq h - 1\). By the pigeon hole principle there exists a permutation \(\pi\) of the numbers \(\{1, 2, \ldots, h\}\) such that we can find \(p_m + 1\) \((k - s)\) representations of \(n'' = n - \sum_{i=1}^{s} u_{\pi(i)} y_s\). These \(p_m + 1\) sets are disjoint, thus in both cases we obtain a contradiction. Since there are only finite number of partitions of \(k\) in the form (11), we get that \(r_k^*(n)\) is bounded by a constant, i.e., there exists a constant \(C_3\) such that \(r_k^*(n) < C_3\). Let \(c_4, c_5, c_6\) be constants. Thus by (6) and (7) we have

\[
|R_k(n) - k!\lambda_n| \leq |R_k(n) - k!r_k(n)| + k!|r_k(n) - \lambda_n| < C_3 + k!|r^{[1]}_n + r^{[2]}_n - \lambda^{[1]}_n - \lambda^{[2]}_n| \\
\leq C_3 + k!|r^{[1]}_n - \lambda^{[1]}_n| + k!|r^{[2]}_n - \lambda^{[2]}_n| \leq C_3 + d_k k! \sqrt{\lambda^{[1]}_n \log n + 2k!c_4} \\
\leq c_5 + d_k k! \sqrt{\lambda_n \log n}.
\]

In the last section we complete the proof of Theorem 4, similarly as in [3]. In view of the estimate above and (ii) in Theorem 4, for large \(n\) we have

\[
|R_k(n) - F(n)| \leq |R_k(n) - k!\lambda_n| + |k!\lambda_n - F(n)| \\
< c_5 + d_k k! (\lambda_n \log n)^{1/2} + |k!\lambda_n - F(n)| \\
\leq c_5 + c_6 \left( \frac{1}{k!} F(n) + \frac{1}{k!} |k!\lambda_n - F(n)| \right) \log n^{1/2} + |k!\lambda_n - F(n)| \\
< c_5 + c_6 \left( \frac{1}{k!} F(n) + \frac{c_7}{k!} (F(n) \log n)^{1/2} \right) \log n^{1/2} + c_7 (F(n) \log n)^{1/2} \\
< c_5 + c_6 \left( \frac{1}{k!} F(n) + \frac{c_7}{k!} \left(F(n) \frac{F(n)}{c_8}\right)^{1/2} \right) \log n^{1/2} + c_7 (F(n) \log n)^{1/2} \\
= c_5 + c_6 \left( \frac{1}{k!} + \frac{c_7}{\sqrt{c_8 k!}} \right) F(n) \log n^{1/2} + c_7 (F(n) \log n)^{1/2} < c_9 (F(n) \log n)^{1/2}.
\]

The proof of Theorem 4 is completed.

ACKNOWLEDGEMENT: The author would like to thank Professor András Sárközy for the valuable discussions.
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