A certain 2-coloring of the reals

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Abstract

There is a function $F : [c]^{<\omega} \to \{0, 1\}$ such that if $A \subseteq [c]^{<\omega}$ is uncountable, then $\{F(a \cup b) : a, b \in A, a \neq b\} = \{0, 1\}$. A corollary is that there is a function $f : \mathbb{R} \to \{0, 1\}$ such that if $A \subseteq \mathbb{R}$ is uncountable, $2 \leq k < \omega$, then both 0 and 1 occur as the value of $f$ at the sum of $k$ distinct elements of $A$. This was originally proved by Hindman, Leader, and Strauss under CH, and they asked if it holds in general.

Here we solve a problem left open in the paper [2]. We prove that there is a coloring with two colors of the finite subsets of $\mathbb{R}$ such that if $A$ is an uncountable subfamily of this set, then both colors occur as the color of $a \cup b$ for some $a, b \in A, a \neq b$. Consequently—and this is what Hindman, Leader, and Strauss were interested in—there is a 2-coloring of $\mathbb{R}$ such that if $A \subseteq \mathbb{R}$ is uncountable, then both colors occur as the color of $a + b$ for some $a, b \in A, a \neq b$. In fact, this holds for $k$-sums in place of 2-sums. In [2] this was proved under CH, and the authors raised the question if it holds without it. The statement is a generalization of Sierpiński’s theorem, by which there is a coloring of the pairs of $\mathbb{R}$ with two colors, with no monocolored uncountable set ([5], see also e.g., in [1], Lemma 9.4.). The proof combines the main idea of Sierpiński’s construction with some ideas in a current theory of Shelah, Todorcevic, and others producing very complicated colorings of pairs of sets (see e.g., [3], [4], [6]).

We just learned that the same result was independently proved by Dániel Soukup and William Weiss (Toronto).

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Notation. Definitions. We use the notation and definitions of axiomatic set theory. In particular, ordinals are von Neumann ordinals, and each cardinal is identified with the least ordinal of that cardinality. Specifically, 
\[2 = \{0, 1\}\] and \(c\) denotes the least ordinal of cardinality continuum.

If \(S\) is a set, \(\kappa\) a cardinal, we define \([S]^\kappa = \{x \subseteq S : |x| = \kappa\}\), \([S]^{<\kappa} = \{x \subseteq S : |x| < \kappa\}\). For \(n < \omega\), \(\omega^2\) denotes the set of all \(n \rightarrow 2\) functions. Similarly, \(\leq^\omega 2 = \bigcup\{\omega^2 : n < \omega\}\), \(\omega^2 = \{f : \omega \rightarrow 2\}\), and \(\leq^\omega 2 = \leq^\omega 2 \cup \omega^2\). If \(f, g \in \leq^\omega 2\), then \(f \triangleleft g\) denotes that \(f\) is a proper initial segment of \(g\), i.e., \(f = g|_{n \neq g}\) for some \(n < \omega\). If \(f \in \omega^2\), \(x < 2\), then \(\hat{f} x\) is that function \(g \in n^+ 2\), such that \(f \triangleleft g\) and \(g(n) = x\). If \(n \leq \omega\), then \(<_{\text{lex}}\) is the lexicographic ordering on \(\omega^2\), i.e., \(f <_{\text{lex}} g\) iff there is \(i < n\) with \(f|i = g|i\), \(f(i) < g(i)\).

Theorem. There is a function \(F : [c]^{<\omega} \rightarrow 2\) such that if \(\{a_\alpha : \alpha < \omega_1\}\) are distinct finite subsets of \(c\), \(i < 2\), then there are \(\alpha < \beta\) such that \(F(a_\alpha \cup a_\beta) = i\).

Proof. Let \(\{r_\alpha : \alpha < c\} \subseteq \omega^2\) be distinct functions. For \(\alpha \neq \beta\) set
\[\Delta(\alpha, \beta) = \min\{n : r_\alpha(n) \neq r_\beta(n)\} .\]

If \(a \in [c]^{<\omega}\), \(|a| \geq 2\), let
\[N = \max\{\Delta(\alpha, \beta) : \alpha \neq \beta \in a\}\, .\]

Let \(s \in \omega^2\) be lexicographically minimal such that there are \(\beta_0, \beta_1 \in a\) with \(r_{\beta_0}|_{N} = r_{\beta_1}|_{N} = s\), \(r_{\beta_1}(N) = i\) \((i < 2)\). Define
\[F(a) = \begin{cases} 0, & \text{if } \beta_0 < \beta_1, \\ 1, & \text{if } \beta_1 < \beta_0. \end{cases}\]

For the other sets \(a\), i.e., when \(|a| \leq 1\), we define \(F(a)\) arbitrarily.

Claim. If \(A, B \subseteq c\), \(|A| = |B| = \aleph_1\), then there are \(g \in \leq^\omega 2\) and \(\varepsilon < 2\), such that \(A' = \{\alpha \in A : g\hat{\varepsilon} < r_\alpha\}\) and \(B' = \{\beta \in B : g(1 - \varepsilon) < r_\beta\}\) are both uncountable.

Proof. For \(s \in \leq^\omega 2\) define \(M(A, s) = \{\alpha \in A : s < r_\alpha\}\) and similarly \(M(B, s) = \{\beta \in B : s < r_\beta\}\).

Set
\[A^* = \{\alpha \in A : \exists s < r_\alpha, |M(A, s)| \leq \aleph_0\}\]

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and define $B^*$ analogously for $B$. $A^*$ is countable as the appropriate $\alpha \to s$ mapping maps $A^*$ to the countable $\preceq \omega 2$ such that each preimage is countable. Similarly, $B^*$ is countable.

Pick $\alpha \in A - A^*, \beta \in B - B^*, \alpha \neq \beta$. If $N = \Delta(\alpha, \beta)$, $g = r_\alpha|N = r_\beta|N$, $g^\epsilon \triangleleft r_\alpha$, $g^\epsilon(1 - \epsilon) \triangleleft r_\beta$, then

$$A' = \{\gamma \in A : r_\gamma|(N + 1) = g^\epsilon\}$$

and

$$B' = \{\gamma \in B : r_\gamma|(N + 1) = g^\epsilon(1 - \epsilon)\}$$

are uncountable by the choice of $\alpha, \beta$.

In order to show that the function $F$ defined above is good, assume that $\{a_\xi : \xi < \omega 1\} \subseteq [\epsilon]^{< \omega}$ are different. Using the $\Delta$-system lemma we can assume that $a_\xi = a \cup b_\xi$ where $a \cap b_\xi = b_\xi \cap b_\eta = \emptyset$ ($\xi < \eta$), $|a| = \ell$, $|b_\xi| = k$.

Here $\ell$ can be zero, but $k > 0$. Let $a = \{\gamma_i : i < \ell\}$, $b_\xi = \{\gamma_\xi^j : j < k\}$ be the increasing enumerations. By shrinking, we can achieve that for each $j < k$, $\{\gamma_\xi^j : \xi < \omega 1\}$ is of order type $\omega_1$. With further shrinking, we can obtain that for each $j < k$, $\gamma_j^\xi < \gamma_j^\eta$ holds for $\xi < \eta$. (Another possibility is to use the Dushnik–Miller partition theorem $\omega_1 \to (\omega_1, (\omega_1)^2)$.) Still more shrinking and re-indexing gives that there is $M < \omega$, such that $r_\gamma|M = f_i$ ($i < \ell$), $r_{\gamma_j^\eta}|M = g_j$ ($j < k$) and the functions $f_i, g_j$ are different.

We construct by recursion the uncountable sets $U_j, V_j$ ($j \leq k$) as follows. $U_0 = V_0 = \omega_1$. Given $U_j, V_j$, we apply the Claim to $A = \{\gamma_\xi^j : \xi \in U_j\}$, $B = \{\gamma_\eta^j : \xi \in V_j\}$, and obtain the uncountable $U_{j+1} \subseteq U_j, V_{j+1} \subseteq V_j$, $N_j < \omega, g_j \in N_j, \epsilon_j < 2$ such that

$$r_{\gamma_j^\xi}|(N_j + 1) = g_j^\epsilon\epsilon_j \quad (\xi \in U_{j+1})$$

and

$$r_{\gamma_j^\eta}|(N_j + 1) = g_j^\epsilon(1 - \epsilon_j) \quad (\eta \in V_{j+1}).$$

Set $N = \max\{N_j : j < k\}$. Notice that $N > M$. Let $g_j$ be the $\preceq_{\text{lex}}$-minimal element of $\{g_j : N_j = N\}$.

We now have that if $\xi \in U_k, \eta \in V_k$, then $F(a_\xi \cup a_\eta) = \epsilon_j$ iff $\gamma_\xi^j < \gamma_\eta^i$ iff $\xi < \eta$. As we can choose $\xi \in U_k, \eta \in V_k$ such that either of $\xi < \eta$ or $\eta < \xi$ hold, both 0 and 1 are attained as $F(a_\xi \cup a_\eta)$ for some $\xi, \eta$. □
Corollary. There is a function \( f : \mathbb{R} \to \{0, 1\} \) such that if \( A \subseteq \mathbb{R}, |A| = \aleph_1, \)
\( 2 \leq k < \omega \), then both 0 and 1 occur as \( f(a_0 + a_1 + \cdots + a_{k-1}) \) for some distinct \( a_0, a_1, \ldots, a_{k-1} \in A \).

Proof. Fix a Hamel basis \( B = \{ b_\alpha : \alpha < c \} \) over \( \mathbb{Q} \) for \( \mathbb{R} \). Each \( x \in \mathbb{R} \), can uniquely be written as
\[ x = \sum_{\alpha < \omega_1} \lambda_\alpha b_\alpha \]
where each \( \lambda_\alpha \) is rational and \( \text{supp}(x) = \{ \alpha : \lambda_\alpha \neq 0 \} \) is finite.

We define \( f(x) = F(\text{supp}(x)). \) We show that \( f \) is as required.

Assume first that \( k = 2 \). Let \( \{ x_\xi : \xi < \omega_1 \} \) be distinct reals. Set \( a_\xi = \text{supp}(x_\xi) \in [c]^{<\omega}. \) By repeatedly shrinking the system, we can assume that every \( a_\xi \) has the same number of elements, \( k \), and the sets \( \{ a_\xi : \xi < \omega_1 \} \) form a \( \Delta \)-system, i.e., \( a_\xi \cap a_\eta = a \) (\( \xi \neq \eta \)). Let \( a_\xi = \{ \gamma_\xi^i : i < k \} \) be the increasing enumeration of \( a_\xi \) and \( \lambda_\xi^i \) be the corresponding coefficients, that is,
\[ x_\xi = \sum_{i<k} \lambda_\xi^i b_{\gamma_\xi^i}. \]

By further shrinking the system we can assume that \( \lambda_\xi^i = \lambda_i \) and that there is a set \( I \) such that \( a = \{ \gamma_i^k : i < I \} \), that is, the elements of \( a \) occupy the same positions in the \( a_\xi \)'s.

If now \( \xi < \eta \), then
\[ \text{supp}(x_\xi + x_\eta) = a_\xi \cup a_\eta \]
as
\[ x_\xi + x_\eta = \sum_{i \in I} 2 \lambda_i b_{\gamma_\xi^i} + \sum_{i \in I} \lambda_i b_{\gamma_\eta^i} + \sum_{i \not\in I} \lambda_i b_{\gamma_\eta^i}, \]
where the \( b_\cdot \)'s are different on the right hand side.

We can therefore apply the Theorem and obtain \( \xi_0 < \eta_0 \) and \( \xi_1 < \eta_1 \) such that \( f(x_{\xi_0} + x_{\eta_0}) = 0 \) and \( f(x_{\xi_1} + x_{\eta_1}) = 1 \).

We now consider the case \( k \geq 3 \). Assume that \( \{ x_\xi : \xi < \omega_1 \} \) are distinct reals and \( i < 2 \). Define
\[ y_\xi = \frac{1}{2}(x_0 + \cdots + x_{k-3}) + x_{k-2+i} \]
and apply the previous argument to \( \{ y_\xi : \xi < \omega_1 \} \). It gives \( \xi < \eta \) such that the value of \( f \) is \( i \) at
\[ y_\xi + y_\eta = x_0 + x_1 + \cdots + x_{k-3} + x_{k-2+i} + x_{k-2+i}, \]
and
the sum of \( k \) distinct elements of \( \{x_\xi : \xi < \omega_1\} \).

\[ \square \]

**References**


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