

THE DEGREE OF THE DISCRIMINANT OF IRREDUCIBLE REPRESENTATIONS

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ABSTRACT. We present a formula for the degree of the discriminant of irreducible representations of a Lie group, in terms of the roots of the group and the highest weight of the representation. The proof uses equivariant cohomology techniques, namely, the theory of Thom polynomials, and a new method for their computation. We study the combinatorics of our formulas in various special cases.

1. INTRODUCTION

Let G be a complex connected reductive algebraic group, and let $\rho : G \rightarrow GL(V)$ be an irreducible algebraic representation. Then ρ induces an action of G on the projective space $\mathbb{P}V$. This action has a single closed orbit, the orbit of the weight vector of the highest weight λ . E.g., for $GL(n)$ acting on $\Lambda^k \mathbb{C}^n$, we get the Grassmannian $Gr_k(\mathbb{C}^n)$. The dual $\mathbb{P}D_\lambda$ of this orbit (or the affine cone D_λ over it) is called the *discriminant* of ρ since it generalizes the classical discriminant. The goal of the present paper is to give a formula for the degree of the discriminant in terms of the highest weight of the representation ρ and the roots of the Lie group G (Theorems 5.2 and 5.6).

The classical approach to find the degree of dual varieties is due to Kleiman [?] and Katz [?]. Their method, however, does not produce a formula in the general setting. Special cases were worked out by Holme [?], Lascoux [?], Boole, Tevelev [?], Gelfand-Kapranov-Zelevinsky [?, Ch.13,14], see a summary in [?, Ch.7]. De Concini and Weyman [?] showed that, if G is fix, then for regular highest weights the formula for the degree of the discriminant is a polynomial with positive coefficients, and they calculated the constant term of this polynomial. A corollary of our result is an explicit form for this polynomial (Cor. 5.8) with the additional fact that the same polynomial calculates the corresponding degrees for non-regular highest weights as well (modulo an explicit factor).

In special cases our formula can be expressed in terms of some basic concepts in the combinatorics of polynomials [?, ?], such as the Jacobi symmetrizer, divided difference operators, or the scalar product on the space of polynomials (Section 6).

For the group $GL_n(\mathbb{C})$ we further simplify the formula in many special cases in Section 7.

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2. DEGREE AND THOM POLYNOMIALS

In this paper we will use cohomology with rational coefficients.

2.1. Degree and cohomology. Suppose that Y is a smooth complex algebraic variety and $X \subset Y$ is a closed subvariety of complex codimension d . Then X represents a cohomology class $[X]$ in the cohomology group $H^{2d}(Y)$. This class is called the *Poincaré dual* of X . The existence of this class and its basic properties are explained e.g. in [?]. If Y is the projective space \mathbb{P}^n then $H^*(Y) \cong \mathbb{Q}[x]/(x^{n+1})$ and $[X] = \deg(X)x^d$, where x is the class represented by a hyperplane. By definition, the cone $CX \subset \mathbb{C}^{n+1}$ of X has the same degree.

2.2. Degree and equivariant cohomology. We would also like to express the degree in terms of *equivariant cohomology*. Let G be a complex connected reductive algebraic group (though some definitions and claims hold for more general groups as well). Let G act on a topological space Y . Then the equivariant cohomology ring $H_G^*(Y)$ is defined as the ordinary cohomology of the Borel construction $B_G Y = EG \times_G Y$. Here EG denotes the universal principal G -bundle over the classifying space BG . If Y is a smooth complex algebraic variety, G is a complex Lie group acting on Y and $X \subset Y$ is a G -invariant subvariety of complex codimension d then X represents an *equivariant cohomology class* (sometimes called equivariant Poincaré dual or Thom polynomial) $[X] \in H_G^{2d}(Y)$. See [?] or [?] for details.

Let $Y = V$ be a complex vector space and let $G = GL(1)$ act as scalars. If $X \subset V$ is a G -invariant subvariety of complex codimension d then X is the cone of a projective variety $\mathbb{P}X \subset \mathbb{P}V$. Then $H_G^*(V) \cong \mathbb{Q}[x]$ and $[X] = \deg(\mathbb{P}X)x^d$, where x is the class of a hyperplane. For more general complex Lie groups we can expect to extract the degree if G ‘contains the scalars’. As we will see even this condition is not necessary: we can simply replace G with $G \times GL(1)$.

Suppose that $\rho : G \rightarrow GL(V)$ ‘contains the scalars’ i.e. there is a homomorphism $h : GL(1) \rightarrow G$, such that $\rho \circ h$ is the scalar representation on V : for $v \in V$ and $z \in GL(1)$ we have $\rho \circ h(z)v = zv$. By basic properties of the equivariant Poincaré dual we have a ‘change of action’ formula:

$$(1) \quad [X]_{\rho \circ h} = h^*[X]_{\rho},$$

where

$$h^* : H_G^*(V) \cong H^*(BG) \rightarrow H_{GL(1)}^*(V) \cong H^*(BGL(1))$$

is induced by the map $Bh : BGL(1) \rightarrow BG$ classifying the principal G -bundle $EGL(1) \times_h G$. The ρ and $\rho \circ h$ in the lower index indicates whether we take the G - or $GL(1)$ -equivariant Poincaré dual of X . The ‘change of action’ formula implies that

$$\deg(X)x^d = h^*[X]_{\rho}.$$

It frequently happens that we can only find a homomorphism $h : GL(1) \rightarrow G$, such that $\rho \circ h(z)v = z^k v$ for some non zero integer k . Then the same way we obtain

$$k^d \deg(X)x^d = [X]_{\rho \circ h} = h^*[X]_{\rho}.$$

The calculation of h^* is fairly simple. Suppose that $m : GL(1)^r \rightarrow G$ is a (parametrized) maximal complex torus of G . Then by Borel’s theorem [?, §27] $H^*(BG)$ is naturally isomorphic to the Weyl-invariant subring of $H^*(BT)$ (T is the maximal torus of G), and $H^*(BT)$ can

be identified with the symmetric algebra of the character group of T . Hence $H^*(BG) = \mathbb{Q}[\alpha_1, \dots, \alpha_r]^W$, where the α_i 's generate the weight lattice of G (one can identify α_i with the i^{th} projection $\pi_i : GL(1)^r \rightarrow GL(1)$) and W is the Weyl group of G . Then $[X] = p(\alpha_1, \dots, \alpha_r)$ for some homogeneous polynomial p of degree d . We can assume that the homomorphism $h : GL(1) \rightarrow G$ factors through m , i.e. $h = m \circ \varphi$, where $\varphi(z) = (z^{k_1}, \dots, z^{k_r})$ for some integers k_1, \dots, k_r . Application of the ‘change of action’ formula (1) once more leads to

Proposition 2.1. *For the polynomial p , and integers k, k_1, \dots, k_r defined above*

$$\deg(X) = p(k_1/k, \dots, k_r/k).$$

This innocent-looking statement provides a uniform approach to calculate the degree of degeneracy loci whenever a Chern–class formula is known. A similar but more involved argument (see [?]) calculates the equivariant Poincaré dual of $\mathbb{P}X$ in $\mathbb{P}V$.

3. CALCULATION OF THE EQUIVARIANT POINCARÉ DUAL

Now we reduce the problem of computing an equivariant Poincaré dual to computing an integral. We will need G -equivariant characteristic classes: the equivariant Chern classes $c_i(E) \in H_G^{2i}(M)$ of a G -equivariant vector bundle $E \rightarrow M$ are defined via the Borel construction. We also use $e(E)$ for the top Chern class and call it the equivariant Euler class.

As in the case of ordinary cohomology, pushforward can be defined in the equivariant setting. An introduction can be found in [?]. Its properties are similar; for example if $\varphi : \tilde{X} \rightarrow Y$ is a G -equivariant resolution of the d -codimensional invariant subvariety $X \subset Y$, then $[X] = \varphi_* 1 \in H_G^{2d}(Y)$.

Now let G act on the vector space V , and let $Y = \mathbb{P}V$ with universal sub- and quotient bundles $S \rightarrow \mathbb{P}V$, $Q \rightarrow \mathbb{P}V$. Let $\varphi : \tilde{X} \rightarrow \mathbb{P}V$ be G -equivariant, such that its image is the G -invariant subvariety $\mathbb{P}X \subset \mathbb{P}V$. If φ is a resolution of $\mathbb{P}X$ then the composition of the embedding $i : \varphi^* S \rightarrow \tilde{X} \times V$ and the projection $\pi : \tilde{X} \times V \rightarrow V$ gives a G -equivariant resolution of X . Hence $[X] = \pi_* i_* 1$. We want to calculate $\pi_* i_* 1$: Since i is an embedding, $i_* 1$ is the (equivariant) Euler class of $\varphi^*(V \ominus S) = \varphi^*(Q)$; and π_* is integration along the fiber of π . Hence we obtain the following

Proposition 3.1. *Let G act on V . If $\varphi : \tilde{X} \rightarrow \mathbb{P}V$ is G -equivariant, then*

$$(2) \quad \pi_* i_* 1 = \int_{\tilde{X}} \varphi^*(e(Q)) \quad \in H_G^*(V) = H^*(BG).$$

If φ is also a resolution of the invariant subvariety $\mathbb{P}X \subset \mathbb{P}V$ then

$$(3) \quad [X] = \int_{\tilde{X}} \varphi^*(e(Q)) \quad \in H_G^*(V) = H^*(BG).$$

Remark 3.2. If $\varphi(\tilde{X})$ has smaller dimension than \tilde{X} then $\pi_* i_* 1 = \int_{\tilde{X}} \varphi^*(e(Q))$ is zero since this cohomology class is supported on $\varphi(\tilde{X})$ and its codimension is bigger than the rank of $\pi_* i_* 1$.

In our main result we will calculate the integral (3) using the Atiyah-Bott integral formula [?], that we recall now. (It seems that other localization formulas, like the Bott localization or the Jeffrey-Kirwan localization lead to less transparent formulas for the degree of a discriminant.)

Proposition 3.3 (Atiyah-Bott). *Suppose that M is a compact manifold and T is a torus acting smoothly on M and $C(M)$ is the set of components of the fix point manifold. Then for any cohomology class $\alpha \in H_T^*(M)$*

$$(4) \quad \int_M \alpha = \sum_{F \in C(M)} \int_F \frac{i_F^* \alpha}{e(\nu_F)}.$$

Here $i_F : F \rightarrow M$ is the inclusion, $e(\nu_F)$ is the T -equivariant Euler class of the normal bundle ν_F of $F \subset M$. The right side is considered in the fraction field of the polynomial ring of $H_T^*(point) = H^*(BT)$ (see more on details in [?]): part of the statement is that the denominators cancel when we add up the terms.

4. THE EQUIVARIANT COHOMOLOGY CLASS OF THE DUAL OF SMOOTH VARIETIES

Let the torus T act on the complex vector space V and let $X \subset V$ be a T -invariant cone. Assume moreover that $\mathbb{P}X \subset \mathbb{P}V$ is smooth, and that the projective dual $\mathbb{P}\check{X}$ of $\mathbb{P}X$ is a hypersurface. Our objective in this section is to find a formula for the T -equivariant rational cohomology class of the dual \check{X} of X . The result will be a cohomology class in $H^*(BT)$ which we identify with the symmetric algebra of the character group of T . Since we use localization techniques, our formulas will formally live in the fraction field of $H^*(BT)$, but part of the statements will be that in the outcome the denominators cancel.

Remark 4.1. Using the arguments of the preceding section the calculation of the equivariant cohomology class of X itself is straightforward; since we can take the identity map as a resolution. Thus from (3) and (4) one obtains a nontrivial special case of the **Duistermaat-Heckman formula**: *If the set $F(\mathbb{P}X)$ of fix points on $\mathbb{P}X$ is finite, then*

$$[X] = e(V) \sum_{f \in F(\mathbb{P}X)} ((\text{weight of } f) \cdot e(T_f \mathbb{P}X))^{-1}.$$

Let us consider the *incidence variety*:

$$\check{X} = \{(p, H) \in \mathbb{P}V \times \mathbb{P}\check{V} \mid H \text{ is tangent to } \mathbb{P}X \text{ at } p\}.$$

Since we assumed that $\mathbb{P}\check{X}$ is a hypersurface and $\mathbb{P}X$ is smooth, the second projection $\pi_2 : \check{X} \rightarrow \mathbb{P}\check{V}$ is an equivariant resolution of $\mathbb{P}\check{X} \subset \mathbb{P}\check{V}$ (see e.g. [?, Thm 1.10]). The first projection $\pi_1 : \check{X} \rightarrow \mathbb{P}X$ is the *conormal* bundle of $\mathbb{P}X$, i.e. the fiber of π_1 at $p \in \mathbb{P}X$ is the conormal space $\mathbb{P}\check{N}_p X$.

Let \check{S} and \check{Q} be the universal sub and quotient bundles over $\mathbb{P}\check{V}$. Equation (3) provides a formula for the class of \check{X}

$$(5) \quad [\check{X}] = \int_{\check{X}} \pi_2^*(e(\check{Q})) = \int_{\check{X}} \pi_2^*\left(\frac{e(\check{V})}{e(\check{S})}\right) = e(\check{V}) \int_{\check{X}} \frac{1}{c_1(\pi_2^* \check{S})}$$

Let $F(M)$ denote the fixed point set of G acting on a G -space M . The identification of the resolution \check{X} with the conormal bundle over $\mathbb{P}X$ implies that

$$F(\check{X}) = \{(\mu, \nu) \in F(\mathbb{P}V) \times F(\mathbb{P}\check{V}) \mid \mu \in F(\mathbb{P}X), \nu \in F(\mathbb{P}\check{N}_\mu X)\}.$$

Suppose now that $F(\mathbb{P}X)$ is finite (this is the case for the minimal orbit of an irreducible representation). For $\mu \in F(\mathbb{P}X)$ let μ also denote the corresponding weight, and let $\mathbb{P}_{\mu\nu}$

denote the projectivization of the ν -weight subspace of $\check{N}_\mu X$. Let its dimension be $m_{\mu\nu}$, and its inclusion into $\mathbb{P}\check{N}_\mu X$ be $i_{\mu\nu}$. Using Atiyah-Bott localization we obtain

$$[\check{X}] = e(\check{V}) \sum_{(\mu\nu) \in F(\check{X})} \int_{\mathbb{P}_{\mu\nu}} \frac{1}{c_1(i_{\mu\nu}^* \pi_2^* \check{S}) \cdot e(\mathbb{P}_{\mu\nu} \subset \check{X})},$$

where $e(A \subset B)$ means the Euler class of the normal bundle of A in B . Let $\mathcal{W}(M)$ denote the set of weights of the module M , and let $\mathcal{W}_{\mu\nu}$ denote the weights of $\check{N}_\mu X$ different from ν . The Euler class

$$e(\mathbb{P}_{\mu\nu} \subset \check{X}) = e(\mathbb{P}N_\mu^* X \subset \check{X})|_{\mathbb{P}_{\mu\nu}} \cdot \prod_{\gamma \in \mathcal{W}_{\mu\nu}} (\gamma + x)^{m_{\mu\gamma}} = e(T_\mu \mathbb{P}X) \prod_{\gamma \in \mathcal{W}_{\mu\nu}} (\gamma + x)^{m_{\mu\gamma}},$$

where x is the hyperplane class of $\mathbb{P}_{\mu\nu}$, i.e. $H_T^*(\mathbb{P}_{\mu\nu}) = H_T^*(pt)[x]/(x + \nu)^{m_{\mu\nu}}$. Since

$$\mathcal{W}(V) = \{\mu\} \cup \{\mu + \beta | \beta \in T_\mu \mathbb{P}X\} \cup \mathcal{W}(N_\mu \mathbb{P}X),$$

and therefore

$$\mathcal{W}(\check{V}) = \{-\mu\} \cup \{-\mu - \beta | \beta \in T_\mu \mathbb{P}X\} \cup \mathcal{W}(\check{N}_\mu \mathbb{P}X),$$

we obtain that

$$\begin{aligned} (6) \quad [\check{X}] &= \sum_{\mu, \nu} \int_{\mathbb{P}_{\mu\nu}} \frac{e(\check{V})}{(-x) \cdot e(T_\mu \mathbb{P}X) \prod_{\gamma \in \mathcal{W}_{\mu\nu}} (\gamma + x)^{m_{\mu\gamma}}} \\ &= \sum_{\mu} \left[(-\mu) \left(\prod_{\beta \in T_\mu \mathbb{P}X} \frac{-\beta - \mu}{\beta} \right) \sum_{\nu} \nu^{m_{\mu\nu}} \int_{\mathbb{P}_{\mu\nu}} \frac{1}{-x} \prod_{\gamma \in \mathcal{W}_{\mu\nu}} \left(\frac{\gamma}{\gamma + x} \right)^{m_{\mu\gamma}} \right]. \end{aligned}$$

Integration on $\mathbb{P}_{\mu\nu}$ means taking the coefficient of $y^{m_{\mu\nu}-1}$, where $y = x + \nu$; hence we substitute $y = x + \nu$:

$$(7) \quad [\check{X}] = \sum_{\mu} \left[(-\mu) \left(\prod_{\beta \in T_\mu \mathbb{P}X} \frac{\beta + \mu}{-\beta} \right) \sum_{\nu} \nu^{m_{\mu\nu}} \int_{\mathbb{P}_{\mu\nu}} \frac{1}{\nu - y} \prod_{\gamma \in \mathcal{W}_{\mu\nu}} \left(\frac{\gamma}{\gamma + y - \nu} \right)^{m_{\mu\gamma}} \right].$$

Now we need the following lemma.

Lemma 4.2. *Suppose $n > 0$ and m_1, \dots, m_n are positive integers. Then*

$$(8) \quad \frac{1}{\prod_{i=1}^n x_i^{m_i}} = \sum_{i=1}^n \int_{\mathbb{P}^{m_i-1}} \left(\frac{1}{x_i - y} \prod_{j \neq i} \frac{1}{(y + x_j - x_i)^{m_j}} \right)$$

is an identity of rational functions in x_1, \dots, x_n ; where $\int_{\mathbb{P}^k} \sum_{j=0}^{\infty} a_j(x) y^j = a_k(x)$.

Proof. We give a topological proof of the identity when the x_i 's are integers, from this the lemma follows. Let $U(1)$ act on the vector space W of dimension $\sum_{i=0}^n m_i > 1$, with weights x_0, \dots, x_n , and corresponding weight multiplicities m_i . Let W_i be the weight subspace corresponding to x_i . The Atiyah-Bott localization formula yields

$$0 = \int_{\mathbb{P}W} 1 = \sum_{i=0}^n \int_{\mathbb{P}W_i} \frac{1}{e(\mathbb{P}W_i \subset \mathbb{P}W)} = \sum_{i=0}^n \int_{\mathbb{P}W_i} \prod_{j \neq i} \frac{1}{(x_j + x)^{m_j}}.$$

Here $H_{U(1)}^*(\mathbb{P}W_i) = H^*(BU(1))[x]/(x_i + x)^{m_i}$ with x the hyperplane class. Substituting $y = x + x_i$ in the i 'th integral we obtain

$$0 = \sum_{i=0}^n \int_{\mathbb{P}^{m_i}} \prod_{j \neq i} \frac{1}{(y + x_j - x_i)^{m_j}},$$

where integration means as in the statement of the lemma. Writing $m_0 = 1$, $x_0 = 0$, and rearranging gives the formula in the lemma. \square

Multiplying (8) with $\prod x_i^{m_i}$ we conclude that the factor

$$\sum_{\nu} \nu^{m_{\mu\nu}} \int_{\mathbb{P}_{\mu\nu}} \frac{1}{\nu - y} \prod_{\gamma \in \mathcal{W}_{\mu\nu}} \left(\frac{\gamma}{\gamma + y - \nu} \right)^{m_{\mu\nu}}$$

in (7) is identically 1. Hence from (7) we obtain the following theorem.

Theorem 4.3. *Suppose that $\chi : T \rightarrow GL(V)$ is a representation of the torus T on the complex vector space V and $X \subset V$ is a T -invariant cone. Assume moreover that $\mathbb{P}X \subset \mathbb{P}V$ is smooth with finitely many T -fixed points and the projective dual $\mathbb{P}\check{X}$ is a hypersurface. Then the equivariant cohomology class represented by the cone \check{X} of the dual $\mathbb{P}\check{X}$ is*

$$[\check{X}] = - \sum_{\mu \in F(\mathbb{P}X)} \mu \prod_{\beta \in T_{\mu}\mathbb{P}X} \frac{\beta + \mu}{\beta}.$$

Remark 4.4. The lemma we used in the proof (that radically simplified formula (6)) follows from Lagrange interpolation if all $m_i = 1$. It can also be interpreted as follows: when localizing an integral with (4) on a projective space one can pretend that the weights of the action are different, with the only price to pay that the weights should be counted with multiplicities the dimension of the weight subspaces. We could have started our proof with this observation—in this case we would have only needed Lagrange interpolation at the end of the proof. Notice also that in the course of the proof we used Atiyah-Bott localization twice, in different directions. The net effect is a ‘partial localization’: localizing an equivariant integral on \check{X} to the fibers of the conormal bundle over the fixed points of $\mathbb{P}X$. Such a partial localization statement could be stated and proved, but localizing to the fixed points of the torus action is so well-known that we chose to use that approach.

Remark 4.5. The difference of the dimensions of a hypersurface and the variety $\mathbb{P}\check{X}$ is called the *defect* of $\mathbb{P}X$. Hence Theorem 4.3 deals with the defect 0 case. It is customary to define the *cohomology class* and the *degree* of the dual of $\mathbb{P}X$ to be 0 if $\mathbb{P}X$ has positive defect. Using this convention Theorem 4.3 remains valid without the condition on the defect. Indeed, by Remark 3.2 the right hand side, which is equal to the pushforward of 1, is automatically zero if the image of π_2 has smaller dimension.

5. COHOMOLOGY AND DEGREE FORMULAS FOR THE DISCRIMINANT

In this section we apply Theorem 4.3 to obtain formulas for the equivariant class and the degree of the discriminants of irreducible representations.

Let $\rho : G \rightarrow GL(V)$ be an irreducible representation of the complex connected reductive Lie group G on the complex vector space V and let $\mathbb{P}X \subset \mathbb{P}V$ be the (closed, smooth) orbit of $[v]$

where $v \in V$ is a vector corresponding to the highest weight λ . Let $D_\lambda \subset \check{V}$ and $\mathbb{P}D_\lambda \subset \mathbb{P}\check{V}$ be the duals of X and $\mathbb{P}X$, they are called the “discriminants” of the representation of highest weight λ . The discriminant is “usually” a hypersurface, a complete list of representations of semisimple Lie groups for which the discriminant is not a hypersurface (i.e. the defect is positive, c.f. Remark 4.5) can be found in [?], see also [?, Th.9.21].

Remark 5.1. For an irreducible representation of a reductive group G on V the action of the center of G is trivial on $\mathbb{P}V$ so only the semisimple part act. So theoretically it would be enough to state the degree formula for the discriminant only for semisimple groups. However we calculate the degree from the equivariant cohomology class and $H_G^2(\text{point}) = 0$ for any semisimple group G .

Let W be the Weyl group, let $R(G)$ be the set of root of G , and $R^-(G)$ the set of negative roots. The Killing form on the space of characters of the maximal torus is denoted by B . The Lie algebra of a reductive group is the sum of the semisimple part and the center, so the Killing form is the trivial extension of the Killing form of the semisimple part.

Theorem 5.2. Main Formula—Short Version. *With $G, W, B, V, \lambda, D_\lambda, R^-(G)$ as above, the equivariant Poincaré dual of D_λ in \check{V} is*

$$[D_\lambda] = - \sum_{\mu \in W\lambda} \mu \prod_{\beta \in T_\mu} \frac{\mu + \beta}{-\beta},$$

where

$$T_\lambda = \{\beta \in R^-(G) \mid B(\beta, \lambda) < 0\},$$

and $T_{w\lambda} = wT_\lambda$ for $w \in W$. Here we used the convention of Remark 4.5, i.e. the class of $[D_\lambda]$ is defined to be 0 if D_λ is not a hypersurface.

The proof is based on the following two lemmas:

Lemma 5.3. *The fixed point set $F(\mathbb{P}X)$ of the maximal torus $T \subset G$ is equal to the orbit of $[v] \in \mathbb{P}V$ for the action of the Weyl group W .*

Notice that the Weyl group $W = N_G(T)/T$ indeed acts on $\mathbb{P}X$ since T fixes $[v] \in \mathbb{P}V$.

Proof. It is enough to show that if $[v] \in \mathbb{P}V$ and $g[v]$ are both fixed points of T and v is a maximal weight vector, then there exists a $\beta \in N_G(T)$ such that $g[v] = \beta[v]$.

Let $G_{[v]}$ be the stabilizer of $[v]$. Then, by the assumption, T and $g^{-1}Tg$ are contained in $G_{[v]}$. These are maximal tori in $G_{[v]}$, so there is a $p \in G_{[v]}$ such that $g^{-1}Tg = p^{-1}Tp$ (see e.g. [?, p. 263]). Then $\beta = gp^{-1} \in N_G(T)$ and $g[v] = \beta[v]$. \square

Lemma 5.4. *The weights of the tangent space $T_f(\mathbb{P}X)$ as a T -space are*

$$T_f = \{\beta \in R^-(G) : B(\beta, w_f) < 0\}$$

for any $f \in F(\mathbb{P}X)$.

For semisimple G the proof can be found in [?] and a more detailed version in [?, p.36]. By the remark above on the Killing form of reductive groups the formula extends to the reductive case without change.

Proof of Theorem 5.2. It is enough to apply Theorem 4.3 to our situation, Lemma 5.3 determines the fixed points and Lemma 5.4 gives that

$$T_\lambda = \{\beta \in R^-(G) \mid B(\beta, \lambda) < 0\},$$

and the weights at other fixed points are obtained by applying the appropriate element of the Weyl group. \square

For the Lie group $G = GL_n(\mathbb{C})$ the maximal torus can be identified with the diagonal matrices $\text{diag}(z_1, \dots, z_n)$, $|z_i| = 1$, which is the product of n copies of S^1 's the natural way. Let L_i be the character of this torus, which is the identity on the i 'th S^1 factor, and constant 1 on the others. It is also the i 'th standard Chern-root of $H^*(BGL_n(\mathbb{C}))$ after appropriate identification.

Example 5.5. Consider the dual of $Gr_3(\mathbb{C}^8)$ in its Plücker embedding. This is the discriminant of the representation of $GL_8(\mathbb{C})$ with highest weight $L_1 + L_2 + L_3$. Let $\binom{n}{k}$ stand for the set of k -element subsets of $\{1, \dots, n\}$. By Theorem 5.2 the class of the discriminant is

$$- \sum_{S \in \binom{8}{3}} (L_{s_1} + L_{s_2} + L_{s_3}) \cdot \prod_{i \in S} \prod_{j \notin S} \frac{L_{s_1} + L_{s_2} + L_{s_3} + L_j - L_i}{(L_i - L_j)}.$$

This is a 58-term sum. However, we know that it must have the form of $v \cdot \sum_{i=1}^8 L_i$. A well chosen substitution will kill most of the terms. E.g. substitute $L_i = i - 13/3$, then all terms—labelled by 3-element subsets S of $\{1, \dots, 8\}$ —are zero, except for the last one corresponding to $S = \{6, 7, 8\}$. For the last term we obtain -8 . Hence, $-8 = v \cdot (1 - 13/3 + 2 - 13/3 + \dots + 8 - 13/3)$, which gives the value of $v = -6$. Thus we obtain the equivariant class of the dual of $Gr_3(\mathbb{C}^8)$ to be $-6(L_1 + \dots + L_8)$, and in turn its degree as $-6(-1/3 - \dots - 1/3) = 16$ (by Proposition 2.1).

Although similarly lucky substitutions cannot be expected in general, the $L_i = i$ substitution yields a formula for the degree of the dual variety of the Grassmannian $Gr_k(\mathbb{C}^n)$ in its Plücker embedding:

$$(9) \quad \deg(\check{Gr}_k(\mathbb{C}^n)) = \frac{2k}{n+1} \sum_{S \in \binom{n}{k}} l(S) \prod_{i \in S, j \notin S} \frac{l(S) + j - i}{i - j},$$

where $l(S) = \sum_{s \in S} s$.

To study how the degree depends on the highest weight λ for a fixed group G we introduce another expression for the degree where the sum is for all elements of the Weyl group instead of the orbit $W\lambda$.

For a dominant weight λ , let $O_\lambda = R^-(G) \setminus T_\lambda = \{\beta \in R^-(G) : B(\beta, \lambda) = 0\}$, and let the sign $\varepsilon(\lambda)$ of λ be $(-1)^{|O_\lambda|}$. Let $W_\lambda \leq W$ be the stabilizer subgroup of λ .

Theorem 5.6. Main Formula—Symmetric Version. *Under the conditions of Theorem 5.2 we have*

$$[D_\lambda] = -\frac{\varepsilon(\lambda)}{|W_\lambda|} \sum_{w \in W} w \left(\lambda \prod_{\beta \in R^-(G)} \frac{\lambda + \beta}{-\beta} \right).$$

Proof. Let w_1, \dots, w_m be left coset representatives of $W_\lambda \leq W$, i.e. the disjoint union of the $w_i W_\lambda$'s is W . Then

$$(10) \quad -\frac{\varepsilon(\lambda)}{|W_\lambda|} \sum_{w \in W} w \left(\lambda \prod_{\beta \in R^-(G)} \frac{\lambda + \beta}{-\beta} \right) = -\frac{\varepsilon(\lambda)}{|W_\lambda|} \sum_{i=1}^m w_i \left(\sum_{w \in W_\lambda} w \left(\lambda \prod_{\beta \in T_\lambda} \frac{\lambda + \beta}{-\beta} \prod_{\beta \in O_\lambda} \frac{\lambda + \beta}{-\beta} \right) \right) =$$

$$-\frac{\varepsilon(\lambda)}{|W_\lambda|} \sum_{i=1}^m w_i \left(\lambda \prod_{\beta \in T_\lambda} \frac{\lambda + \beta}{-\beta} \sum_{w \in W_\lambda} w \left(\prod_{\beta \in O_\lambda} \frac{\lambda + \beta}{-\beta} \right) \right),$$

since $wT_\lambda = T_\lambda$ (but wO_λ is not necessarily equal to O_λ). Now we need the following lemma.

Lemma 5.7.

$$\sum_{w \in W_\lambda} w \left(\prod_{\beta \in O_\lambda} \frac{\lambda + \beta}{-\beta} \right) = \varepsilon(\lambda) |W_\lambda|.$$

Proof. The polynomial

$$P(x) = \sum_{w \in W_\lambda} \prod_{\beta \in O_\lambda} \frac{x + w(\beta)}{-w(\beta)} \in \mathbb{Z}(L_i)[x]$$

is W_λ -invariant, and has the form

$$\frac{q_k + q_{k-1}x + \dots + q_0 x^k}{\prod_{\beta \in O_\lambda} \beta}$$

where $k = |O_\lambda|$ and q_i is a degree i polynomial on the orthogonal complement of λ . This means that the numerator has to be anti-symmetric under W_λ (which is itself a Weyl group of a root system), hence it must have degree at least the number of positive roots. That is, all q_i , $i < k$ must vanish. This means that $P(x)$ is independent of x , i.e. $P(\lambda) = P(0) = \varepsilon(\lambda) |W_\lambda|$, as required. \square

Using this Lemma, formula (10) is further equal to

$$-\sum_{i=1}^m w_i \left(\lambda \prod_{\beta \in T_\lambda} \frac{\lambda + \beta}{-\beta} \right),$$

which completes the proof of Theorem 5.6. \square

The advantage of the Short Version (Theorem 5.2) is that for certain λ 's (λ 's on the boundary of the Weyl-chamber) the occurring products have only few factors, while the advantage of the Symmetric Version (Theorem 5.6) is that it gives a unified formula for all λ 's of a fixed group. Now we will expand this latter observation.

Let G be semisimple and consider the representation with highest weight λ . Extend this action to an action of $G \times GL_1(\mathbb{C})$ with $GL_1(\mathbb{C})$ acting by scalar multiplication. Denoting the Chern root of $GL_1(\mathbb{C})$ by u we obtain that

$$(11) \quad [D_\lambda] = -\frac{\varepsilon(\lambda)}{|W_\lambda|} \sum_{w \in W} w \left((\lambda + u) \prod_{\beta \in R^-(G)} \frac{\lambda + u + \beta}{-\beta} \right).$$

Proposition 2.1 then turns this formula to a degree formula for the discriminant, by substituting $u = -1$:

$$(12) \quad \deg(D_\lambda) = -\frac{\varepsilon(\lambda)}{|W_\lambda|} \sum_{w \in W} w \left((\lambda - 1) \prod_{\beta \in R^-(G)} \frac{\lambda + \beta - 1}{-\beta} \right).$$

Part of the statement is that this formula is a constant, i.e. expression (11) is equal to a constant times u (although this can also be deduced from the fact that $H^2(BG) = 0$ for semisimple groups).

Corollary 5.8. *Let G be a semisimple Lie group, and \mathfrak{h} the corresponding Cartan subalgebra. There exists a polynomial $F_G : \mathfrak{h}^* \rightarrow \mathbb{Z}$, with degree the number of positive roots of G , such that*

$$\deg(D_\lambda) = \frac{\varepsilon(\lambda)}{|W_\lambda|} F_G(\lambda)$$

if λ is a dominant weight. $F_G(\lambda) = 0$ if and only if D_λ is not a hypersurface.

Proof.

$$(13) \quad F_G = - \sum_{w \in W} w \left((\lambda - 1) \prod_{\beta \in R^-(G)} \frac{\lambda + \beta - 1}{-\beta} \right),$$

and the last statement follows from Remark 4.5. □

Remark 5.9. The polynomial dependence of $\deg(D_\lambda)$ for *regular* weights λ (hence $\varepsilon(\lambda) = 1$, $W_\lambda = \{1\}$), as well as a formula for a special value of the polynomial (the value at the sum of the fundamental weights) is given in [?]. Since $\deg(D_\lambda)$ is always non-negative the value of $\varepsilon(\lambda)$ is determined by the sign of $F_G(\lambda)$. The positive defect cases are known but this corollary provides an alternative and uniform way to find them.

Example 5.10. A choice of simple roots $\alpha_1, \alpha_2, \dots, \alpha_r$ of G determines the fundamental weights w_1, w_2, \dots, w_r by $B(w_i, \frac{2\alpha_j}{B(\alpha_j, \alpha_j)}) = \delta_{i,j}$ where $B(., .)$ denotes the Killing form. We follow the convention of De Concini and Weyman [?] by writing F_G in the basis of fundamental weights (i.e. $y_1 w_1 + \dots + y_r w_r \mapsto F_G(y_1, \dots, y_r)$) and substituting $y_i = x_i + 1$. The advantage of this substitution is that in this way, according to [?], all the coefficients of the polynomial F_G become non-negative. Formula (13) gives the following polynomials for all semisimple Lie groups of rank at most 2 and for some of rank 3. [In these examples our convention for simple roots agrees with the one in the *coxeter/weyl* Maple package www.math.lsa.umich.edu/~jrs/maple.html written by J. Stembridge.]

$$\mathbf{A}_1, \quad \alpha = L_2 - L_1:$$

$$F = 2x_1.$$

$$\mathbf{A}_1 + \mathbf{A}_1, \quad \alpha = (L_2 - L_1, L'_2 - L'_1):$$

$$F = 6x_1x_2 + 2x_2 + 2x_1 + 2.$$

$$\mathbf{A}_2, \quad \alpha = (L_2 - L_1, L_3 - L_2):$$

$$F = 6(x_1 + x_2 + 1)(2x_1x_2 + x_1 + x_2 + 1) \text{ [?, ex 7.18].}$$

$$\mathbf{B}_2, \quad \alpha = (L_1, L_2 - L_1),$$

$$F = 20(2x_2^3x_1 + 3x_2^2x_1^2 + x_2x_1^3) + 12(2x_2^3 + 12x_2^2x_1 + 11x_2x_1^2 + x_1^3) + 24(3x_2^2 + 7x_2x_1 + 2x_1^2) + 8(9x_2 + 8x_1) + 24.$$

$$\mathbf{G}_2, \quad \alpha = (L_2 - L_1, L_1 - 2L_2 + L_3):$$

$$F = 42(18x_2^5x_1 + 45x_2^4x_1^2 + 40x_2^3x_1^3 + 15x_2^2x_1^4 + 2x_2x_1^5) + 60(9x_2^5 + 90x_2^4x_1 + 150x_2^3x_1^2 + 90x_2^2x_1^3 + 20x_2x_1^4 + x_1^5) + 110(27x_2^4 + 132x_2^3x_1 + 144x_2^2x_1^2 + 52x_2x_1^3 + 5x_1^4) + 8(822x_2^3 + 2349x_2^2x_1 + 1527x_2x_1^2 + 248x_1^3) + 6(60x_2^2 + 1972x_2x_1 + 579x_1^2) + 4(1025x_2 + 727x_1) + 916.$$

$$\mathbf{A}_1 + \mathbf{A}_1 + \mathbf{A}_1, \quad \alpha = (L_2 - L_1, L'_2 - L'_1, L''_2 - L''_1):$$

$$F = 24x_2x_1x_3 + 12(x_2x_1 + x_2x_3 + x_1x_3) + 8(x_2 + x_1 + x_3) + 4.$$

$$\mathbf{A}_1 + \mathbf{A}_2, \quad \alpha = (L_2 - L_1, L'_2 - L'_1, L'_3 - L'_2):$$

$$F = 60x_2^2x_1x_3 + 60x_2x_1x_3^2 + 36x_2^2x_1 + 36x_2^2x_3 + 144x_2x_1x_3 + 36x_2x_3^2 + 36x_1x_3^2 + 24x_2^2 + 72x_2x_1 + 96x_2x_3 + 72x_1x_3 + 24x_3^2 + 48x_2 + 36x_1 + 48x_3 + 24.$$

$$\mathbf{A}_3, \quad \alpha = (L_2 - L_1, L_3 - L_2, L_4 - L_3):$$

$$F = 420x_1x_3x_2(x_1+x_2)(x_2+x_3)(x_2+x_1+x_3) + 300(x_1^3x_2^2 + 4x_1^3x_2x_3 + x_1^3x_3^2 + 2x_1^2x_2^3 + 15x_1^2x_2^2x_3 + 12x_1^2x_2x_3^2 + x_1^2x_3^3 + x_1x_2^4 + 12x_1x_2^3x_3 + 15x_1x_2^2x_3^2 + 4x_1x_2x_3^3 + x_2^4x_3 + 2x_2^3x_3^2 + x_2^2x_3^3) + 220(3x_1^3x_2 + 3x_1^3x_3 + 12x_1^2x_2^2 + 33x_1^2x_2x_3 + 9x_1^2x_3^2 + 10x_1x_2^3 + 48x_1x_2^2x_3 + 33x_1x_2x_3^2 + 3x_1x_3^3 + x_2^4 + 10x_2^3x_3 + 12x_2^2x_3^2 + 3x_2x_3^3) + 8(42x_1^3 + 453x_1^2x_2 + 411x_1^2x_3 + 699x_1x_2^2 + 1566x_1x_2x_3 + 411x_1x_3^2 + 164x_2^3 + 699x_2^2x_3 + 453x_2x_3^2 + 42x_3^3) + 12(126x_1^2 + 491x_1x_2 + 407x_1x_3 + 239x_2^2 + 491x_2x_3 + 126x_3^2) + 16(133x_1 + 169x_2 + 133x_3) + 904.$$

A Maple computer program computing F for any semisimple Lie group is available at www.unc.edu/~rimanyi/progs/feherpolinom.mw.

Straightforward calculation gives that F_G for $G = A_1 + \dots + A_1$ (n times) is

$$F_{nA_1}(y_1, \dots, y_n) = \sum_{k=0}^n (-2)^{n-k} (k+1)! \sigma_k(y_1, \dots, y_n),$$

where σ_i is the i 'th elementary symmetric polynomial. This can also be derived from [?, Th.2.5, Ch.13].

6. COMBINATORICS OF THE DEGREE FORMULAS

Our main formulas, Theorems 5.2 and 5.6, can be encoded using standard notions from the combinatorics of symmetric functions. In Sections 6.1-6.4 we assume that $G = GL_n(\mathbb{C})$, and that the simple roots are $L_i - L_{i+1}$.

6.1. Symmetrizer operators. Let

$$\lambda^+ = \lambda \prod_{\beta \in R^-(G)} (\lambda + \beta), \quad \text{and} \quad \Delta = \prod_{1 \leq i < j \leq n} (L_i - L_j),$$

and recall the definition of the Jacobi-symmetrizer ([?]) of a polynomial $f(L_1, \dots, L_n)$:

$$J(f)(L_1, \dots, L_n) = \frac{1}{\Delta} \sum_{w \in S_n} \text{sgn}(w) f(L_{w(1)}, \dots, L_{w(n)}),$$

where $\text{sgn}(w)$ is the sign of the permutation w , i.e. (-1) raised to the power of the number of inversions in w . Then for $\lambda = \lambda_1 L_1 + \dots + \lambda_n L_n$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, from Theorem 5.6

we obtain

$$(14) \quad [D_\lambda] = -\frac{\varepsilon(\lambda)}{|W_\lambda|} J(\lambda^+),$$

and $\deg D_\lambda$ is obtained by multiplying this with the factor

$$(15) \quad \frac{-n}{|\lambda| \cdot \sigma_1}, \quad \text{where } |\lambda| := \sum \lambda_i \text{ and } \sigma_1 := \sum L_i.$$

The Jacobi symmetrizer is a special case of the divided difference operators ∂_w [?, Ch.7], corresponding to the maximal permutation $w_0 = [n, n-1, \dots, 2, 1]$ (i.e. $w_0(i) = n+1-i$). Shorter divided difference operators also turn up for certain representations. We will illustrate this with the case of the Plücker embedding of Grassmann varieties, i.e. $G = GL_n(\mathbb{C})$, $\lambda = L_1 + \dots + L_k$. Let $[k|n-k] \in S_n$ be the permutation $[n-k+1, n-k+2, \dots, n, 1, 2, \dots, n-k]$. Theorem 5.2 gives

$$[D_{L_1+\dots+L_k}] = -\partial_{[k|n-k]} \left((L_1 + \dots + L_k) \prod_{i=1}^k \prod_{j=k+1}^n (L_1 + \dots + L_k + L_j - L_i) \right),$$

and the degree of the discriminant of $Gr_k(\mathbb{C}^n)$ in its Plücker embedding is obtained by multiplying this with $-n/(k\sigma_1)$.

6.2. Scalar product. Now we show how to use the scalar product on function spaces to encode the formula of Theorem 5.6 in case of $G = GL_n(\mathbb{C})$. Following [?], we define the scalar product of the polynomials f, g in n variables L_1, \dots, L_n as

$$\langle f, g \rangle = \frac{1}{n!} \left[f \bar{g} \prod_{i \neq j} \left(1 - \frac{L_i}{L_j} \right) \right]_1,$$

where $\bar{g}(L_1, \dots, L_n) = g(1/L_1, \dots, 1/L_n)$ and $[h]_1$ is the constant term (i.e. the coefficient of 1) of the Laurent polynomial $h \in \mathbb{Z}[L_1^\pm, \dots, L_n^\pm]$. The Jacobi symmetrizer is basically a projection, thus for a degree $\binom{n}{2} + 1$ polynomial f we have

$$\frac{1}{n!} \sum_w \text{sgn}(w) f(L_{w(1)}, \dots, L_{w(n)}) = \frac{\langle f, \sigma_1 \Delta \rangle}{\langle \sigma_1 \Delta, \sigma_1 \Delta \rangle} \sigma_1 \Delta.$$

Here $\langle \sigma_1 \Delta, \sigma_1 \Delta \rangle$ can be calculated to be $n(2n-3)!!$, hence we obtain

$$[D_\lambda] = \frac{-\varepsilon(\lambda)n!}{|W_\lambda| \cdot n(2n-3)!!} \langle \lambda^+, \sigma_1 \Delta \rangle \sigma_1,$$

and hence the following form of our Main Formula:

Theorem 6.1. *The degree of the discriminant of the irreducible representation of $GL_n(\mathbb{C})$ with highest weight λ is*

$$\deg D_\lambda = \frac{\varepsilon(\lambda)}{|\lambda| \cdot |W_\lambda|} \frac{n!}{(2n-3)!!} \langle \lambda^+, \sigma_1 \Delta \rangle.$$

6.3. Permanent. For $(\nu_1, \dots, \nu_n) \in \mathbb{N}^n$ and $w \in S_n$ let $w(\nu) = (\nu_{w(1)}, \dots, \nu_{w(n)})$ and L^ν will denote the monomial $L_1^{\nu_1} \dots L_n^{\nu_n}$.

Lemma 6.2. *Let $\mu = (n, n-2, n-3, \dots, 2, 1, 0) \in \mathbb{N}^n$. If $\sum \nu_i = \binom{n}{2} + 1$ then we have*

$$J(L^\nu) = \begin{cases} \operatorname{sgn}(w)\sigma_1 & \text{if } \nu_i = \mu_{w(i)} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If $\nu_i = \nu_j$ then the terms of $J(L^\nu)$ turn up in cancelling pairs, hence $J(L^\nu) = 0$. This leaves only $\nu = \mu$ for possible non-zero J -value. Direct check shows $J(L^\mu) = \sigma_1$ (c.f. the well known identity $J(L^{(n-1, n-2, \dots, 2, 1, 0)}) = 1$). \square

The coefficient of a monomial L^ν in a polynomial f will be denoted by $c(f, L^\nu)$. Formula (14) and Lemma 6.2 yield that the class and degree of D_λ can be computed by counting coefficients.

Theorem 6.3. *The equivariant class of D_λ is*

$$[D_\lambda] = -\frac{\varepsilon(\lambda)}{|W_\lambda|} \sum_{w \in S_n} \operatorname{sgn}(w) c(\lambda^+, L^{w(\mu)}) \cdot \sigma_1,$$

and the degree of D_λ is obtained by multiplication with $-n/(|\lambda|\sigma_1)$.

Similar sums will appear later, hence we define the ν -permanent of a polynomial f as

$$\sum_{w \in S_n} \operatorname{sgn}(w) c(f, L^{w(\nu)}),$$

and denote it by $P(f, \nu)$.

The name *permanent* is justified by the following observation. Let $\nu = (\nu_1, \dots, \nu_n)$ be a partition. If the polynomial f is the product of $|\nu|$ linear factors $\sum_{j=1}^n a_j^{(i)} L_j$, $i = 1, \dots, |\nu|$, then $P(f, \nu)$ can be computed from the $n \times |\nu|$ matrix $(a_j^{(i)})$ as follows. For a permutation $w \in S_n$ we choose ν_1 entries from row $\nu(1)$ (i.e. $j = \nu(1)$), ν_2 entries from row $\nu(2)$, etc, such a way that no chosen entries are in the same column (hence they form a complete ‘rook arrangement’). The product of the chosen entries with sign $\operatorname{sgn}(w)$ will be a term in $P(f, \nu)$, and we take the sum for all $w \in S_n$ and all choices. For example the $(2, 1)$ -permanent of the product $(aL_1 + a'L_2)(bL_1 + b'L_2)(cL_1 + c'L_2)$, considering the matrix $\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$, is

$$abc' + ab'c + a'bc - a'b'c - a'bc' - ab'c'.$$

Theorem 6.4. *(Boole’s formula, [?, 7.1]) If $\lambda = aL_1$, then the degree of D_λ is $n(a-1)^{n-1}$.*

Proof. We have

$$\lambda^+ = aL_1 \prod_{i=2}^n ((a-1)L_1 + L_i) \cdot \prod_{2 \leq i < j \leq n} (L_j - L_i + aL_1),$$

and let

$$\lambda^{++} = aL_1 \prod_{i=2}^n ((a-1)L_1 + L_i) \cdot \prod_{2 \leq i < j \leq n} (L_j - L_i).$$

Observe that $P(\lambda^+, \mu) = P(\lambda^{++}, \mu)$, since in the difference each term comes once with a positive, once with a negative sign. In $P(\lambda^{++}, \mu)$ only L_1 has degree $\geq n$ (namely n), hence we have

$$P(\lambda^{++}, \mu) = a(a-1)^{n-1} P\left(\prod_{2 \leq i < j \leq n} (L_i - L_j), (n-2, n-3, \dots, 2, 1)\right)$$

which is further equal to $a(a-1)^{n-1}(-1)^{\binom{n-1}{2}}(n-1)!$ and Theorem 6.3 gives the result. \square

6.4. Hyperdeterminants. The discriminant of the standard action of the product group $\prod_{u=1}^k GL_{n_u}(\mathbb{C})$ on $\bigotimes_{u=1}^k \mathbb{C}^{n_u}$ is called *hyperdeterminant* because it generalizes the case of the determinant for $k=2$. In other words the hyperdeterminant is the dual of the Segre embedding $\prod_{u=1}^k \mathbb{P}(\mathbb{C}^{n_u}) \rightarrow \mathbb{P}(\bigotimes_{u=1}^k \mathbb{C}^{n_u})$. Gelfand, Kapranov and Zelevinsky [?] give a concrete description of the degree of the hyperdeterminants by giving a generator function. In the so-called boundary case when $n_k - 1 = \sum_{u=1}^{k-1} (n_u - 1)$ they get a closed formula. We show how to prove this using ν -permanents.

When considering representations of $\prod_{u=1}^k GL_{n_u}(\mathbb{C})$ ($n_1 \leq n_2 \leq \dots \leq n_k$) we will need k sets of variables (the k sets of Chern roots), we will call them $L_{(u),i}$, $u=1, \dots, k$, $i=1, \dots, n_u$. The hyperdeterminant is the discriminant corresponding to the representation with highest weight $\lambda = \sum_{u=1}^k L_{(u),1}$. If $\nu_{(u)} \in \mathbb{N}^{k_u}$, then a polynomial f in these variables has a $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$ -permanent, defined as

$$P(f, \nu) = \sum_{w^{(1)} \in S_{n_1}} \dots \sum_{w^{(k)} \in S_{n_k}} \left(\prod_{u=1}^k \text{sgn}(w^{(u)}) \right) c(f, \prod_u L_{(u)}^{w^{(u)}(\nu_{(u)})}).$$

Theorem 6.5 ([?, 14.2.B]). *If $n_k - 1 > \sum_{u=1}^{k-1} (n_u - 1)$ then the discriminant is not a hypersurface. If $n_k - 1 = \sum_{u=1}^{k-1} (n_u - 1)$ then its degree is $n_k! / \prod_{u=1}^{k-1} (n_u - 1)!$.*

Proof. Let $\mu^{(u)} = (\mu_{(1)}, \dots, \mu_{(k)})$, where $\mu_{(v)} = (n_v - 1, n_v - 2, \dots, 1, 0)$ for $v \neq u$, and $\mu_{(u)} = (n_u, n_u - 2, n_u - 3, \dots, 2, 1, 0)$. A straightforward generalization of the $k=1$ case gives

$$(16) \quad [D_\lambda] = -\frac{(-1)^{\sum \binom{n_u-1}{2}}}{\prod_{u=1}^k (n_u - 1)!} \sum_{u=1}^k \left(P(\lambda^+, \mu^{(u)}) \sum_{i=1}^{n_u} L_{(u),i} \right),$$

where

$$\lambda^+ = \lambda \left(\prod_{u=1}^k \prod_{i=2}^{n_u} (L_{(u),i} - L_{(u),1} + \lambda) \right) \left(\prod_{u=1}^k \prod_{2 \leq i < j \leq n_u} (L_{(u),j} - L_{(u),i} + \lambda) \right).$$

Just like in the proof of the Boole formula above, we can change λ^+ to

$$\lambda^{++} = \lambda \left(\prod_{u=1}^k \prod_{i=2}^{n_u} (L_{(u),i} - L_{(u),1} + \lambda) \right) \left(\prod_{u=1}^k \prod_{2 \leq i < j \leq n_u} (L_{(u),j} - L_{(u),i}) \right)$$

without changing the value of the permanent. Observe that the highest power of an $L_{(k),i}$ variable in λ^{++} is $\sum_{u=1}^{k-1} (n_u - 1)$, thus the last term in (16) is 0 if $n_k - 1 > \sum_{u=1}^{k-1} (n_u - 1)$. This proves the first statement of the theorem.

If we have $n_k - 1 = \sum_{u=1}^{k-1} (n_u - 1)$, then considering the power of $L_{(k),1}$ again, we get that

$$\begin{aligned}
P(\lambda^{++}, \mu_{(k)}) &= \\
P\left(\left(\sum_{u=1}^{k-1} L_{(u),1}\right)^{n_k-1} \prod_{u=1}^k \prod_{2 \leq i < j \leq n_u} (L_{(u),j} - L_{(u),i}), ((n_1 - 1, \dots, 1, 0), \dots \right. \\
&\quad \left. \dots, (n_{k-1} - 1, \dots, 1, 0), (n_k - 2, \dots, 1, 0))\right) = \\
\frac{(n_k - 1)!}{\prod_{u=1}^{k-1} (n_u - 1)!} P\left(\prod_{u=1}^k \prod_{2 \leq i < j \leq n_u} (L_{(u),j} - L_{(u),i}), ((n_1 - 2, \dots, 1, 0), \dots, (n_k - 2, \dots, 1, 0))\right) = \\
&\quad (-1)^{\Sigma (n_{u_2}^{-1})} \frac{(n_k - 1)!}{\prod_{u=1}^{k-1} (n_u - 1)!} \prod_{u=1}^k (n_u - 1)! = (-1)^{\Sigma (n_{u_2}^{-1})} (n_k - 1)!^2.
\end{aligned}$$

Hence, in the expansion $-[D_\lambda] = c_1 \sum L_{(1),i} + c_2 \sum L_{(2),i} + \dots + c_k \sum L_{(k),i}$ we have

$$c_k = (n_k - 1)!^2 / \prod_{u=1}^k (n_u - 1)! = (n_k - 1)! / \prod_{u=1}^{k-1} (n_u - 1)!$$

Proposition 2.1 then gives $\deg(D_\lambda)$ by substituting (e.g.) $L_{(u),i} = 0$ for $u < k$ and $L_{(k),i} = -1$ into $[D_\lambda]$, which proves the Theorem. \square

7. THE CASE $G = GL(n)$. SOME MORE EXPLICIT FORMULAE.

In this section G is the general linear group $GL(n)$. Then $R^-(G) = \{L_i - L_j : i > j\}$ and irreducible representations correspond to the weights $\lambda = \sum_{i=1}^n a_i L_i$ with $a_1 \geq a_2 \geq \dots \geq a_n$. For such a weight one has $T_\lambda = \{L_i - L_j : i > j, a_i < a_j\}$. Set $\sigma_1 := \sum_{i=1}^n L_i$ as above.

Notice that the expression of $[D_\lambda]$ from 5.2 is homogeneous in L_1, \dots, L_n of degree 1. Moreover, $[D_\lambda] \cdot \Delta$ is anti-symmetric of degree $\deg \Delta + 1$, hence it is the product of Δ and a symmetric polynomial of degree 1. Hence, by (15), for some constant $f_\lambda^{(n)}$ one has

$$(17) \quad [D_\lambda] = -f_\lambda^{(n)} \cdot \sigma_1 \quad \text{and} \quad \deg(D_\lambda) = \frac{n}{|\lambda|} \cdot f_\lambda^{(n)}.$$

By similar argument one deduces that for a free variable t one has

$$R_\lambda^{(n)}(t) := \sum_{\mu \in W_\lambda} (\mu + t) \prod_{\beta \in T_\mu} \frac{\mu + t + \beta}{-\beta} = A_\lambda^{(n)} t + f_\lambda^{(n)} \sigma_1$$

for another constant $A_\lambda^{(n)}$. Since the dual variety associated with the representation ρ_λ and the dual variety associated with the action $\rho_\lambda \times \text{diag}$ of $GL(n) \times GL(1)$ are the same, their degrees are the same too. Applying our main result for these two groups and actions, one gets

$$(18) \quad R_\lambda^{(n)}(t) = f_\lambda^{(n)} \cdot \left(\frac{n}{|\lambda|} \cdot t + \sigma_1 \right).$$

In fact, (18) is an entirely algebraic identity, and it is valid for *any* weight λ (i.e. not only for the dominant weights).

Using (18), we prove some identities connecting different weights.

Lemma 7.1. *For any $\lambda = \sum_i a_i L_i$ define $\bar{\lambda} := \sum_i a_i L_{n-i}$. Assume that either (i) $\lambda' = \lambda + a\sigma_1$, or (ii) $\bar{\lambda}' + \lambda = a\sigma_1$ for some $a \in \mathbb{Z}$. Then*

$$\deg(D_{\lambda'}) = \deg(D_\lambda).$$

Proof. In case (i), using (18), one gets $T_{\lambda+a\sigma_1} = T_\lambda$ and $f_{\lambda+a\sigma_1}^{(n)} = f_\lambda^{(n)}(\frac{na}{|\lambda|} + 1)$, hence $f_{\lambda+a\sigma_1}^{(n)}/|\lambda+a\sigma_1| = f_\lambda^{(n)}/|\lambda|$. Then apply (17). For (ii) notice that $T_{\lambda'} = \{-\bar{\beta} : \beta \in T_\lambda\}$, hence $-[D_{\lambda'}]$ equals

$$\sum_{\mu \in W_\lambda} \overline{a\sigma_1 - \lambda} \prod_{\beta \in T_\mu} \frac{\overline{a\sigma_1 - \lambda} + \overline{-\beta}}{\beta} = - \sum_{\mu \in W_\lambda} (\lambda - a\sigma_1) \prod_{\beta \in T_\mu} \frac{\lambda - a\sigma_1 + \beta}{-\beta} = -f_\lambda^{(n)}\left(-\frac{na}{|\lambda|} + 1\right)\sigma_1.$$

Therefore, $f_{\lambda'}^{(n)} = f_\lambda^{(n)}(\frac{na}{|\lambda|} - 1)$, or $f_{\lambda'}^{(n)}/|\lambda'| = f_\lambda^{(n)}/|\lambda|$.

Geometrically the lemma corresponds to the fact that tensoring with the one dimensional representation or taking the dual representation does not change the discriminant. \square

It follows from the lemma that we can always assume that the last coefficient a_n of λ is zero.

Example 7.2. The case of $\lambda = (a+b)L_1 + b(L_2 + \dots + L_{n-1})$ with $a, b > 0$ and $n \geq 3$.

Set $\lambda' := (a+b)L_1 + a(L_2 + \dots + L_{n-1})$. Then $\bar{\lambda}' + \lambda = (a+b)\sigma_1$, hence $\deg(D_\lambda) = \deg(D_{\lambda'})$ by 7.1. In particular, $\deg(D_\lambda)$ is a *symmetric* polynomial in variables (a, b) . In the sequel we deduce its explicit form. For this notice that

$$T_\lambda = \{L_i - L_1, L_n - L_i : 2 \leq i \leq n-1\} \cup \{L_n - L_1\}.$$

It is convenient to write λ as $aL_1 - bL_n + b\sigma_1$. First, assume that $a \neq b$. Using (17) and Lemma 7.1 one gets

$$\deg(D_\lambda) = \frac{n}{|\lambda|} f_\lambda^{(n)} = \frac{n}{a+b(n-1)} f_{aL_1-bL_n}^{(n)} \cdot \left(\frac{nb}{a-b} + 1\right) = \frac{n}{a-b} \cdot f_{aL_1-bL_n}^{(n)},$$

where $f_{aL_1-bL_n}^{(n)} \cdot \sigma_1$ equals

$$(19) \quad \sum_{i \neq j} (aL_i - bL_j) \frac{aL_i - bL_j + L_j - L_i}{L_i - L_j} \prod_{k \neq i, j} \frac{aL_i - bL_j + L_k - L_i}{L_i - L_k} \cdot \frac{aL_i - bL_j + L_j - L_k}{L_k - L_j}.$$

Clearly, $f_{aL_1-bL_n}^{(n)} = \lim_{L_n \rightarrow \infty} (f_{aL_1-bL_n}^{(n)} \cdot \sigma_1) / L_n$. (In the sequel we write simply \lim for $\lim_{L_n \rightarrow \infty}$.) In order to determine this limit, we separate L_n in the expression E of (19). We get three types of contributions: $E = I + II + III$, where I contains those terms of the sum $\sum_{i \neq j}$ where $i, j \leq n-1$; II contains the terms with $i = n$; while III those terms with $j = n$.

One can see easily that $\lim I$ is finite (for any fixed L_1, \dots, L_{n-1}), hence $\lim I/L_n = 0$.

The second term (after re-grouping) is

$$II = \frac{S}{\prod_{j \leq n-1} L_n - L_k},$$

where

$$S := \sum_{j \leq n-1} \frac{\prod_{k \leq n-1} (aL_n - bL_j - L_n + L_k)(aL_n - bL_j + L_j - L_k)}{\prod_{n \neq k \neq j} L_k - L_j}.$$

By similar argument as above, S can be written as $S = \sum_{i=0}^n P_i \cdot L_n^{n-i}$, where P_i is a symmetric polynomial in L_1, \dots, L_{n-1} of degree i (and clearly also depends on a, b and n). In particular, $\lim II/L_n$ is the ‘constant’ $P_0 = P_0(a, b, n)$.

Set $t_1 := aL_n - bL_j - L_n$ and $t_2 := aL_n - bL_j + L_j$. Notice that if we modify the polynomial $P = \prod_{k \leq n-1} (t_1 + L_k)(t_2 - L_k)$ by any polynomial situated in the ideal generated by the symmetric polynomials $\sigma_1^{(n-1)} := \sum_{i=1}^{n-1} L_i, \sigma_2^{(n-1)}, \dots$ (in L_1, \dots, L_{n-1}), then the modification has no effect in the ‘constant’ term P_0 . In particular, since $P = (t_1^{n-1} + \sigma_1^{(n-1)} t_1^{n-2} + \dots)(t_2^{n-1} - \sigma_1^{(n-1)} t_2^{n-2} + \dots)$, in the expression of S , the polynomial P can be replaced by $t_1^{n-1} t_2^{n-1}$. Therefore, if $[Q(t)]_{t^i}$ denotes the coefficient of t^i in $Q(t)$, then one has:

$$P_0(a, b, n) = \left[\sum_{j \leq n-1} \frac{(aL_n - bL_j - L_n)^{n-1} (aL_n - bL_j + L_j)^{n-1}}{\prod_{n \neq k \neq j} L_k - L_j} \right]_{L_n^n}.$$

Set

$$P_{a,b}(t) = At^2 + Bt + C := (at - t - b)(at - b + 1).$$

Then

$$P_0(a, b, n) = \left[P_{a,b}(t)^{n-1} \right]_{t^n} \cdot \sum_{j \leq n-1} \frac{L_j^{n-2}}{\prod_{n \neq k \neq j} L_k - L_j}.$$

Notice that the last sum is exactly $(-1)^{n-2}$ by Lemma 4.2 (or its special case, the Lagrange interpolation formula).

A very similar computation provides $\lim III/L_n$, and using the symmetry of $P_{a,b}$ one finds that

$$\deg(D_\lambda) = \frac{(-1)^n n}{a-b} \left([P_{a,b}(t)^{n-1}]_{t^n} - [P_{a,b}(t)^{n-1}]_{t^{n-2}} \right) = \frac{(-1)^n n}{a-b} \left([P_{a,b}(t)^{n-1}]_{t^n} - [P_{b,a}(t)^{n-1}]_{t^n} \right).$$

If ∂ denotes the ‘divided difference’ operator $\partial Q(a, b) := (Q(a, b) - Q(b, a))/(a - b)$, then the last expression reads as

$$\deg(D_\lambda) = (-1)^n n \cdot \partial \left[P_{a,b}(t)^{n-1} \right]_{t^n}.$$

By Corollary 5.8, this is valid for $a = b$ as well. By a computation (using the multinomial formula):

$$(20) \quad \deg(D_\lambda) = n!(a+b-1) \sum_{i=1}^{[n/2]} \frac{A^{i-1} C^{i-1} (-B)^{n-2i}}{i!(i-1)!(n-2i)!}.$$

This has the following factorization:

$$\deg(D_\lambda) = n(n-1)(a+b-1)(-B)^{n-2[n/2]} \cdot \prod_{i=1}^{[n/2]-1} (B^2 + \xi_i AC),$$

where

$$\prod_{i=1}^{[n/2]-1} (t + \xi_i) = t^{[n/2]-1} + \frac{(n-2)!}{2!1!(n-4)!} t^{[n/2]-2} + \frac{(n-2)!}{3!2!(n-6)!} t^{[n/2]-3} + \dots.$$

E.g., for small values of n one has the following expressions for $\deg(D_\lambda)$:

$$\begin{aligned} n = 3 & \quad 6(a+b-1)(-B) \\ n = 4 & \quad 12(a+b-1)(B^2+AC) \\ n = 5 & \quad 20(a+b-1)(-B)(B^2+3AC) \\ n = 6 & \quad 30(a+b-1)(B^2+\xi AC)(B^2+\bar{\xi}AC), \text{ where } \xi^2+6\xi+2=0. \end{aligned}$$

For $n = 3$, $\deg(D_\lambda) = 6(a+b-1)(2ab-a-b+1)$ is in fact the universal polynomial $F_{GL(3)}$ (which already was determined in Example 5.10 case A_2 , and from which one can determine the degrees of all the irreducible $GL(3)$ -representations by Corollary 5.8), cf. also with 7.18 of [?].

If we write $u := a - 1$ and $v := b - 1$, then all the coefficients of the polynomials $-B, A, C$, expressed in the new variables (u, v) , are non-negative. Using (20), this remains true for $\deg(D_\lambda)$ as well; a fact compatible with [?].

For $a = 1$ the formula simplifies drastically (since $A = 0$): $\deg(D_\lambda) = n(n-1)b^{n-1}$. Symmetrically for $b = 1$. For $a = b = 1$ we recover the well known formula $\deg(D_\lambda) = n(n-1)$ for the degree of the discriminant of the adjoint representation. (A matrix is in the discriminant of the adjoint representation if it has multiple eigenvalues i.e. the equation of the discriminant is the discriminant of the characteristic polynomial.)

Example 7.3. The case of $\lambda = aL_1 + bL_2$ with $a > b \geq 1$ and $n \geq 3$.

Since $T_\lambda = \{L_i - L_1, L_i - L_2 : 3 \leq i \leq n\} \cup \{L_2 - L_1\}$, one has the following expression for $f_\lambda^{(n)} \cdot \sigma_1$:

$$(21) \quad \sum_{i \neq j} (aL_i + bL_j) \frac{aL_i + bL_j + L_j - L_i}{L_i - L_j} \prod_{k \neq i, j} \frac{aL_i + bL_j + L_k - L_i}{L_i - L_k} \cdot \frac{aL_i + bL_j + L_k - L_j}{L_j - L_k}.$$

We compute $f_\lambda^{(n)}$ as in 7.2. For this we write the expression E from (21) as $I + II + III$, where I, II and III are defined similarly as in 7.2. The computations of the first two contributions are similar as in 7.2: $\lim I/L_n = 0$ while

$$\lim II/L_n = \left[(at - t + b)^{n-1} (at + b - 1)^{n-1} \right]_{t^n}.$$

On the other hand, III is slightly more complicated. For this write $t_1 := aL_i + bL_n - L_i$, $t_2 := aL_i + bL_n - L_n$, and $\Pi := \prod_{n \neq k \neq i} L_i - L_k$. Moreover, we define the relation $R_1 \equiv R_2$ whenever $\lim R_1/L_n = \lim R_2/L_n$. Then (for t_1 using the usual ‘trick’ as above by separating terms from the ideal \mathcal{I}_{sym} generated by $\sigma_i^{(n-1),s}$):

$$\begin{aligned} -III &= \prod_{k \leq n-1} \frac{1}{L_n - L_k} \cdot \sum_{i \leq n-1} \frac{\prod_{k \leq n-1} (t_1 + L_k) \cdot (aL_i + bL_n + L_n - L_i) \cdot \prod_{n \neq k \neq i} (t_2 + L_k)}{\Pi} \\ &\equiv \sum_{i \leq n-1} \frac{t_1^{n-1}}{\Pi \cdot L_n^{n-1}} \cdot (aL_i + bL_n + L_n - L_i) \cdot \prod_{n \neq k \neq i} (t_2 + L_k) \\ &= \sum_{i \leq n-1} \frac{t_1^{n-1}}{\Pi \cdot L_n^{n-1}} \left((b+1)L_n + (a-1)L_i \right) \frac{t_2^{n-1} + \sigma_1^{(n-1)} t_2^{n-2} + \dots}{t_2 + L_i} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i \leq n-1} \frac{t_1^{n-1}}{\prod \cdot L_n^{n-1}} \left((b+1)L_n + (a-1)L_i \right) \cdot \left(\frac{t_2^{n-1} - (-L_i)^{n-1}}{t_2 + L_i} + \sigma_1^{(n-1)} \frac{t_2^{n-2} - (-L_i)^{n-2}}{t_2 + L_i} + \dots \right) \\
&\quad + \sum_{i \leq n-1} \frac{t_1^{n-1}}{\prod \cdot L_n^{n-1}} \left((b+1)L_n + (a-1)L_i \right) \frac{(-L_i)^{n-1} + \sigma_1^{(n-1)}(-L_i)^{n-2} + \dots}{t_2 + L_i}
\end{aligned}$$

From the first sum we can again eliminate terms from \mathcal{I}_{sym} , while the second sum is a rational function in L_n , where the numerator and denominator both have degree n , hence this expression is $\equiv 0$. In particular,

$$III \equiv - \sum_{i \leq n-1} \frac{t_1^{n-1}}{\prod \cdot L_n^{n-1}} \left((b+1)L_n + (a-1)L_i \right) \cdot \frac{t_2^{n-1} - (-L_i)^{n-1}}{t_2 + L_i}.$$

Therefore, $\deg(D_\lambda)$ equals

$$\frac{n}{a+b} \left[(at-t+b)^{n-1}(at+b-1)^{n-1} - (bt+a-1)^{n-1}(bt+t+a-1) \cdot \frac{(bt-t+a)^{n-1} - (-1)^{n-1}}{bt-t+a+1} \right]_{tn}.$$

For $n=3$ we recover the polynomial $F_{GL(3)}$ (cf. 5.8) – computed by 7.2 as well.

Also, for arbitrary n but for $b=1$ the formula becomes simpler:

$$\deg(D_\lambda) = \frac{n}{(a+1)^2} \left((n-1)a^{n+1} - (n+1)a^{n-1} + 2(-1)^{n-1} \right).$$

This expression coincides with Tevelev's formula from [?], see also [?, 7.14]. (Notice that in [?, 7.2C], Tevelev provides a formula for arbitrary a and b , which is rather different from ours.)

Example 7.4. Specializations of $\lambda = aL_1 + bL_2$.

Assume that λ_s is a specialization of λ (i.e. λ_s is either aL_1 – obtained by specialization $b=0$, or it is $aL_1 + aL_2$ – by taking $b=a$). In this case, by Corollary 5.8 one has

$$\deg(D_{\lambda_s}) = \frac{\varepsilon(\lambda)|W_\lambda|}{\varepsilon(\lambda_s)|W_{\lambda_s}|} \cdot (\deg(D_\lambda)|_{\text{specialized}}).$$

In the first case, $\deg(D_{aL_1+bL_2})|_{b=0}$ is $(-1)^n n(n-1)(a-1)^{n-1}$, and the correction factor is $(-1)^n(n-1)$, hence we recover Boole's formula

$$\deg(D_{aL_1}) = n(a-1)^{n-1}.$$

(Evidently, this can be easily deduced by a direct limit computations – as above – as well.)

In the second case, $\deg(D_{aL_1+bL_2})|_{b=a}$ should be divided by -2 . We invite the reader to verify that this gives

$$\deg(D_{aL_1+aL_2}) = \frac{n}{2a} \left[(t-1)(at+a-1)^{n-1} \cdot \frac{(at-t+a)^{n-1} - (-1)^{n-1}}{at-t+a+1} \right]_{tn}.$$

For $a=1$ we recover the well known formula of Holme [?] for the degree of the dual variety of the Grassmannian $Gr_2(\mathbb{C}^n)$:

$$\deg(D_{L_1+L_2}) = \frac{n}{2} \cdot \frac{1 - (-1)^{n-1}}{2}.$$

Example 7.5. The case of $\lambda = L_1 + L_2 + L_3$, $n \geq 4$.

This case was studied by Lascoux [?]. Using K -theory he gave an algorithm to calculate the degree of the dual of the Grassmannian $Gr_3(\mathbb{C}^n)$ and calculated many examples. In Proposition 7.6 we give a closed formula for the degree.

We write $\binom{n}{k}$ for the subsets of $\{1, 2, \dots, n\}$ with k elements. For any $S \in \binom{n}{k}$ set $L_S := \sum_{i \in S} L_i$. Since $T_\lambda = \{L_i - L_j : i > 3, j \leq 3\}$, one has

$$f_\lambda^{(n)} \cdot \sigma_1 = \sum_{S \in \binom{n}{3}} L_S \prod_{\substack{i \notin S \\ j \in S}} \frac{L_S + L_i - L_j}{L_j - L_i}.$$

Similarly as in the previous examples, we separate L_n , and we write the above expression as the sum $I + II$ of two terms, I corresponds to the subsets S with $n \notin S$, while II to the others. It is easy to see that $\lim_{L_n \rightarrow \infty} I/L_n = 0$. In order to analyze the second main contribution, it is convenient to introduce the following expression (in variables L_1, \dots, L_n and a new free variable t):

$$R^{(n)}(t) := \sum_{\mathcal{J} = \{j_1, j_2\} \in \binom{n}{2}} (t + L_{\mathcal{J}})(t - L_{j_1})(t - L_{j_2}) \prod_{i \notin \mathcal{J}} \frac{(t + L_{j_1} + L_i)(t + L_{j_2} + L_i)(L_{\mathcal{J}} + L_i)}{(L_{j_1} - L_i)(L_{j_2} - L_i)}.$$

This is a homogeneous expression in variables (L, t) of degree $n + 1$, hence it can be written as

$$R^{(n)}(t) = \sum_{k=0}^{n+1} P_k^{(n)} t^{n+1-k},$$

where $P_k^{(n)}$ is a symmetric polynomial in variables L . The point is that

$$II = \frac{R^{(n-1)}(L_n)}{\prod_{i \leq n-1} L_n - L_i}, \text{ hence } \lim II/L_n = P_0^{(n-1)}.$$

Therefore,

$$f_\lambda^{(n)} = P_0^{(n-1)} \text{ and } \deg(D_\lambda) = n P_0^{(n-1)} / 3.$$

Next, we concentrate on the leading coefficient $P_0^{(n)}$ of $R^{(n)}(t)$. For two polynomials R_1 and R_2 of degree $n + 1$, we write $R_1 \equiv R_2$ if their leading coefficients are the same. Set $t_r := t + L_{j_r}$ for $r = 1, 2$; and for each k write (over the field $\mathbb{C}(L)$)

$$t_r^k = Q_{r,k}(t_r)(t_r + L_{j_1})(t_r + L_{j_2}) + A_{r,k} t_r + B_{r,k}$$

for some polynomial $Q_{r,k}$ of degree $k - 2$ and constants $A_{r,k}$ and $B_{r,k}$. Then

$$\prod_{i \notin \mathcal{J}} (t + L_{j_r} + L_i) = \frac{t_r^n}{(t_r - L_{j_1})(t_r - L_{j_2})} + \sum_{k=1}^n \sigma_k Q_{r,n-k} + \sum_{k=1}^n \sigma_k \frac{A_{r,n-k} t_r + B_{r,n-k}}{(t_r - L_{j_1})(t_r - L_{j_2})}.$$

In $P_0^{(n)}$ the first sum has no contribution since it is in the ideal generated by the (non-constant) symmetric polynomials, the second sum has no contribution either, since its limit is zero when $t \rightarrow \infty$. Therefore, $R^{(n)}(t)$ is \equiv with

$$\sum_{\mathcal{J} \in \binom{n}{2}} (t + L_{\mathcal{J}})(t - L_{j_1})(t - L_{j_2}) \cdot \frac{(t + L_{j_1})^n}{(t + 2L_{j_1})(t + L_{\mathcal{J}})} \cdot \frac{(t + L_{j_2})^n}{(t + 2L_{j_2})(t + L_{\mathcal{J}})} \cdot \prod_{i \notin \mathcal{J}} \frac{L_{\mathcal{J}} + L_i}{(L_{j_1} - L_i)(L_{j_2} - L_i)},$$

which equals

$$\sum_{\mathcal{J} \in \binom{(n)}{2}} \frac{1}{t + L_{\mathcal{J}}} \cdot \frac{(t - L_{j_1})(t + L_{j_1})^n}{t + 2L_{j_1}} \cdot \frac{(t - L_{j_2})(t + L_{j_2})^n}{t + 2L_{j_2}} \cdot \prod_{i \notin \mathcal{J}} \frac{L_{\mathcal{J}} + L_i}{(L_{j_1} - L_i)(L_{j_2} - L_i)}.$$

Let us define the—binomial-like—coefficients $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ by the expansion (near $t = \infty$):

$$(22) \quad \frac{(t-1)(t+1)^n}{t+2} = \sum_{k \leq n} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} t^k,$$

Using

$$\frac{1}{t+1} = \frac{1}{t} - \frac{1}{t^2} + \frac{1}{t^3} - \dots,$$

it provides the expansions

$$\frac{(t - L_{j_r})(t + L_{j_r})^n}{t + 2L_{j_r}} = \sum_{k \leq n} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} t^k L_{j_r}^{n-k}, \quad \text{and} \quad \frac{1}{t + L_{\mathcal{J}}} = \frac{1}{t} - \frac{L_{\mathcal{J}}}{t^2} + \frac{L_{\mathcal{J}}^2}{t^3} - \dots.$$

Therefore,

$$(23) \quad P_0^{(n)} = \sum_{\substack{k \leq n, l \leq n \\ k+l \geq n+2}} (-1)^{k+l+n} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} n \\ l \end{smallmatrix} \right\} \mathcal{L}_{n-k, n-l}^{(n)},$$

where

$$\mathcal{L}_{n-k, n-l}^{(n)} := \frac{1}{2} \cdot \sum_{\mathcal{J} = \{j_1, j_2\} \in \binom{(n)}{2}} L_{\mathcal{J}}^{k+l-n-2} (L_{j_1}^{n-k} L_{j_2}^{n-l} + L_{j_1}^{n-l} L_{j_2}^{n-k}) \cdot \prod_{i \notin \mathcal{J}} \frac{L_{\mathcal{J}} + L_i}{(L_{j_1} - L_i)(L_{j_2} - L_i)}.$$

(Here we already symmetrized \mathcal{L} in order to be able to apply in its computation the machinery of symmetric polynomials. Using this index-notation for $\mathcal{L}_{n-k, n-l}^{(n)}$ has the advantage that in this way this expression is independent of n , as we will see later.) Next, we determine the constants $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ and $\mathcal{L}_{n-k, n-l}^{(n)}$. The index-set of the sum of (23) says that we only need these constants for $k \geq 2$ and $l \geq 2$.

The constants $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$. The identity

$$\frac{(t+1)^n(t-1)}{t+2} = (t+1)^n + 3 \sum_{i \geq 1} (-1)^i (t+1)^{n-i}$$

provides

$$(24) \quad \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \binom{n}{k} + 3 \sum_{i \geq 1} (-1)^i \binom{n-i}{k} \quad \text{for any } 0 \leq k \leq n.$$

Notice that these constants satisfy ‘Pascal’s triangle rule’: $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}$. This law together with the ‘initial values’ $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = 1$ for n even and $= -2$ for n odd, and with $\left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1$ determines completely all $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ for $0 \leq k \leq n$. E.g., the first values are:

5	-2	-1	-2	1	2	1
4	1	-2	0	1	1	
3	-2	0	0	1		
2	1	-1	1			
1	-2	1				
0	1					
n/k	0	1	2	3	4	5

The constants $\mathcal{L}_{n-k,n-l}^{(n)}$. For $a \geq 0$, $b \geq 0$, $n \geq 3$ and $a + b = n - 2$ we set

$$X_{a,b}^{(n)} := \sum_{\mathcal{J}=\{j_1,j_2\} \in \binom{[n]}{2}} (L_{j_1}^a L_{j_2}^b + L_{j_1}^b L_{j_2}^a) \cdot \prod_{i \notin \mathcal{J}} \frac{L_{\mathcal{J}} + L_i}{(L_{j_1} - L_i)(L_{j_2} - L_i)}.$$

Then $X_{a,b}^{(n)} = X_{b,a}^{(n)}$. By the above homogeneity argument $X_{a,b}^{(n)}$ is a constant. For $a > 0$ and $b > 0$ we separate L_n and substitute $L_n = 0$, and we get

$$(25) \quad X_{a,b}^{(n)} = X_{a-1,b}^{(n-1)} + X_{a,b-1}^{(n-1)}.$$

Therefore, $X_{a,b}^{(n)}$ can be determined from this ‘Pascal rule’ and the values $X_{n-1,0}^{(n)}$ (for $n \geq 3$). Next we compute these numbers. We write $X_{n-2,0}^{(n)}$ as $I + II + III$, where I corresponds to $\mathcal{J} \in \binom{[n-1]}{2}$, and

$$II = \sum_{j \leq n-1} L_j^{n-2} \prod_{j \neq i \neq n} \frac{L_n + L_j + L_i}{(L_n - L_i)(L_j - L_i)}, \quad III = \sum_{j \leq n-1} L_n^{n-2} \prod_{j \neq i \neq n} \frac{L_n + L_j + L_i}{(L_n - L_i)(L_j - L_i)}.$$

Clearly, $X_{n-2,0}^{(n)} = \lim_{L_n \rightarrow 0} (I + II + III)$. It is easy to see that $\lim I = 0$ and (by 4.2) $\lim II = 1$. Moreover, using (18), we get

$$III = \frac{L_n^{n-2}}{\prod_{i \leq n-1} (L_n - L_i)} \cdot \left(-\frac{1}{2} R_{2L_1}^{(n-1)}(L_n) + \frac{3}{2} L_n \sum_{j \leq n-1} \prod_{j \neq i \neq n} \frac{L_j + L_i}{L_j - L_i} \right),$$

hence

$$(26) \quad X_{n-2,0}^{(n)} = 1 - \frac{n-1}{2} + \frac{3}{4} (1 + (-1)^n).$$

Now, we return back to $\mathcal{L}_{n-k,n-l}^{(n)}$. The binomial formula for $(L_{j_1} + L_{j_2})^{k+l-n-2}$ and (25) gives

$$2\mathcal{L}_{n-k,n-l}^{(n)} = \sum_{i=0}^{k+l-n-2} \binom{k+l-n-2}{i} X_{l-2-i,n-l+i}^{(n)} = X_{l-2,k-2}^{(k+l-2)}.$$

Let $\rangle_{a,b}^{(n)}$ be defined (for $a, b \geq 0$ and $a + b = n - 2$) by the Pascal rule and initial values $\rangle_{n-2,0}^{(n)} = X_{n-2,0}^{(n)}$ and $\rangle_{0,n-2}^{(n)} = 0$. Symmetrically, define $\langle_{a,b}^{(n)} = \rangle_{b,a}^{(n)}$, hence $\rangle_{a,b}^{(n)} + \langle_{a,b}^{(n)} = X_{a,b}^{(n)}$. It is really surprising that X is another incarnation of the constants $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ (for $k \geq 1$). Indeed, comparing the initial values of $\rangle_{a,b}^{(n)}$ and $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ we get that

$$\rangle_{l-2,k-2}^{(k+l-2)} = \left\{ \begin{smallmatrix} k+l-5 \\ k-1 \end{smallmatrix} \right\} \quad (l \geq 2, k \geq 2).$$

Therefore, we proved the following fact.

Proposition 7.6. *For any $n \geq 3$ and $2 \leq k \leq n$ consider the constants defined by (24), or by (22). Consider the weight $\lambda = L_1 + L_2 + L_3$ of $GL(3)$ which provides the Plücker embedding of $Gr_3(\mathbb{C}^n)$. Then the degree of the dual variety of $Gr_3(\mathbb{C}^n)$ is*

$$\deg(D_\lambda) = \frac{n}{3} \cdot \sum_{\substack{k \leq n-1, \\ 4 \leq l \leq n-1, \\ k+l \geq n+1}} (-1)^{k+l+n-1} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ l \end{Bmatrix} \begin{Bmatrix} k+l-5 \\ k-1 \end{Bmatrix}.$$

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