

Graph homomorphisms: definitions

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June 2008

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1 Graph parameters

A *graph parameter* is a real valued function defined on isomorphism types of graphs (including the graph K_0 with no nodes and edges). A *simple graph parameter* is defined only on isomorphism types of simple graphs (i.e., on graphs with no loops or multiple edges). A graph parameter t is *multiplicative* if $t(G) = t(G_1)t(G_2)$ whenever G is the disjoint union of G_1 and G_2 . We say that a graph parameter is *normalized* if its value on K_1 , the graph with one node and no edge, is 1. Note that if a graph parameter is multiplicative and not identically 0, then its value on K_0 is 1.

2 Homomorphism numbers

For two finite graphs G and H , let $\text{hom}(G, H)$ denote the number of homomorphisms (adjacency-preserving maps) from G to H . We often normalize these homomorphism numbers, and consider the *homomorphism densities* $t(F, G) = \text{hom}(F, G)/|V(G)|^{|V(F)|}$, which is the probability that a random map of $V(F)$ into $V(G)$ is a homomorphism. This definition can be extended to the case when G has nodeweights α_v and edgeweights β_{uv} :

$$\text{hom}(F, G) = \sum_{\phi: V(F) \rightarrow V(G)} \prod_{u \in V(F)} \alpha_{\phi(u)}(G) \prod_{uv \in E(F)} \beta_{\phi(u), \phi(v)}(G).$$

We also define a certain “hardcore” version. Let $S(F, G)$ denote the set of those maps $\phi: V(F) \rightarrow V(G)$ for which $|\phi^{-1}(i) - \alpha_i(G)|V(F)|| \leq 1$ for all $i \in V(G)$. Let

$$\text{hom}^*(F, G) = \sum_{\phi \in S(F, G)} \prod_{u \in V(F)} \alpha_{\phi(u)}(G) \prod_{uv \in E(F)} \beta_{\phi(u), \phi(v)}(G)$$

and

$$\mathcal{E}(F, G) = \max_{\phi \in S(F, G)} \prod_{uv \in E(F)} \beta_{\phi(u), \phi(v)}(G).$$

Suppose that the edges of a graph F are partitioned into two sets E' and E'' , called “positive” and “negative”. The triple $\widehat{F} = (V, E', E'')$ will be called a *signed graph*. Then we define

$$\text{hom}(\widehat{F}, G) = \sum_{\phi: V(F) \rightarrow V(G)} \prod_{u \in V(F)} \alpha_{\phi(u)}(G) \prod_{uv \in E'} \beta_{\phi(u), \phi(v)}(G) \prod_{uv \in E''} (1 - \beta_{\phi(u), \phi(v)}(G)). \quad (1)$$

If all edges are positive, then $\text{hom}(\widehat{F}, G) = t(F, G)$. If G is a simple unweighted graph and all edges of F are negative, then $\text{hom}(\widehat{F}, G) = t(F, \overline{G})$ (where \overline{G} is the complement of G , with loops). If F is a simple graph on $[k]$ and \widehat{F} is defined on the complete graph on $[k]$

In general, $\text{hom}(\widehat{F}, G)$ can be expressed as

$$\text{hom}(\widehat{F}, G) = \sum_{Y \subseteq E''} (-1)^{|Y|} \text{hom}((V, E' \cup Y), G). \quad (2)$$

3 Graph algebras

3.1 Quantum graphs

A *quantum graph* is defined as a formal linear combination of graphs with real coefficients. Every signed graph $\widehat{F} = (V, E', E'')$ will be viewed as the quantum graph

$$\sum_{Y \subseteq E''} (-1)^{|Y|} (V, E' \cup Y).$$

The definition of $\text{hom}(F, G)$ extends to quantum graphs linearly: if $f = \sum_{i=1}^n \lambda_i F_i$ and $g = \sum_{j=1}^m \mu_j G_j$, then

$$\text{hom}(f, g) = \sum_{i=1}^n \sum_{j=1}^m \lambda_i \mu_j \text{hom}(F_i, G_j).$$

This extension is in line with (2).

3.2 Partially labeled graphs

A *partially labeled graph* is a finite graph in which some of the nodes are labeled by different integers. A *k-labeled graph* is a partially labeled graph in which the set of labels is $[k]$. A 0-labeled graph is just an unlabeled graph.

Let G_1 and G_2 be two partially labeled graphs. Their *product* G_1G_2 is defined as follows: we take their disjoint union, and then identify nodes with the same label. Clearly this multiplication is associative and commutative, and the product of two k -labeled graphs is k -labeled. For two 0-labeled graphs, F_1F_2 is their disjoint union.

3.3 Colored graphs

A *colored graph* is a finite graph whose nodes are colored by integers. A *k-colored graph* ($k \geq 1$) is a colored graph where the set of colors used is $[k]$. A 1-colored graph can be thought of as an uncolored graph.

Let G_1 and G_2 be two partially graphs. Their *product* $G_1 \times G_2$ is defined as follows: its nodes are all pairs (u_1, u_2) where $u_1 \in V(G_1)$, $u_2 \in V(G_2)$, and u_1 and u_2 have the same color. The nodes (u_1, u_2) and (v_1, v_2) are connected by m_1m_2 edges, where m_i is the multiplicity of the edge u_iv_i in G_i ($i = 1, 2$). Clearly this multiplication is associative and commutative. For 1-colored graphs, we get the standard categorical product (weak product) of the two graphs. A *colored quantum graph* is defined as a formal linear combination of colored graphs with real coefficients.

3.4 Connection matrices

Let f be any graph parameter and fix an integer $k \geq 0$. We define two (infinite) symmetric matrices $M(f, k)$ and $N(f, k)$.

The rows and columns of $M(f, k)$ are indexed by (isomorphism types of) k -labeled graphs. The entry in the intersection of the row corresponding to G_1 and the column corresponding to G_2 is $f(G_1G_2)$.

The rows and columns of $N(f, k)$ are indexed by (isomorphism types of) k -colored graphs. The entry in the intersection of the row corresponding to G_1 and the column corresponding to G_2 is $f(G_1 \times G_2)$.

We call the matrices $M(f, k)$ and $N(f, k)$ the *left- and write-connection matrices* of the graph parameter f . The ranks of these matrices, as a function of k , are called the *left- and right-connection rank function* of the parameter. (This function may have infinite values.) We call the graph parameter *reflection positive* (from the left or from the right) if all the corresponding connection matrices are positive semidefinite.

3.5 The algebras of partially labeled graphs and colored graphs

A *partially labeled [k-labeled] quantum graph* is defined as a formal linear combination of partially labeled [k-labeled] graphs with real coefficients.

Let \mathcal{G}_k denote the (infinite dimensional) vector space of all k -labeled quantum graphs. We can turn \mathcal{G}_k into an algebra by using F_1F_2 introduced above as the product of two generators,

and then extending this multiplication to the other elements linearly. Clearly \mathcal{G}_k is associative and commutative. The graph O_k on k nodes with no edges is the multiplicative identity in \mathcal{G}_k .

Every graph parameter f can be extended linearly to quantum graphs, and defines an inner product on \mathcal{G}_k by

$$\langle x, y \rangle := f(xy). \quad (3)$$

Let $\mathcal{N}_k(f)$ denote the kernel of this inner product, i.e.,

$$\mathcal{N}_k(f) := \{x \in \mathcal{G}_k : f(xy) = 0 \forall y \in \mathcal{G}_k\}.$$

Then we can define the factor algebra

$$\mathcal{G}_k/f := \mathcal{G}_k/\mathcal{N}_k(f).$$

For $x, y \in \mathcal{G}_k$, we write $x \equiv y \pmod{f}$ if $x - y \in \mathcal{N}_k(f)$.

The parameter is reflection positive if and only if the inner product is positive semidefinite on \mathcal{G}_k (equivalently, positive definite on \mathcal{G}_k/f).

The notion of quantum colored graphs and the algebra of them can be defined analogously.

3.6 Connectors and contractors

For a 2-labeled graph F , let F' denote the graph obtained by identifying the two labeled nodes. The map $F \mapsto F'$ maps 2-labeled graphs to 1-labeled graphs. We can extend it linearly to get an algebra homomorphism $x \mapsto x'$ from \mathcal{G}_2^0 into \mathcal{G}_1 .

The graph parameter f is *contractible*, if for every $x \in \mathcal{G}_2$, $x \equiv 0 \pmod{f}$ implies $x' \equiv 0 \pmod{f}$; in other words, $x \mapsto x'$ factors to a linear map $\mathcal{G}_2/f \rightarrow \mathcal{G}_1/f$. We say that $z \in \mathcal{G}_2$ is a *contractor for f* if $f(xz) = f(x')$ for every $x \in \mathcal{G}_2$. We say that $z \in \mathcal{G}_2$ is a *connector for f* , if $z \equiv K_2 \pmod{f}$.

4 Graphons

Let \mathcal{W} denote the space of all bounded symmetric measurable functions $W : [0, 1]^2 \rightarrow \mathbb{R}$ (i.e., $W(x, y) = W(y, x)$ for all $x, y \in [0, 1]$). Let \mathcal{W}_0 denote the set of all functions $W \in \mathcal{W}$ such that $0 \leq W \leq 1$. Two functions $W, W' \in \mathcal{W}$ are called *isomorphic*, if there is a third function $U \in \mathcal{W}$ and measure preserving maps $\phi, \phi' : [0, 1] \rightarrow [0, 1]$ such that

$$W(x, y) = U(\phi(x), \phi(y)) \quad \text{and} \quad W'(x, y) = U(\phi'(x), \phi'(y)).$$

Equivalence classes of functions in \mathcal{W}_0 under isomorphism are called *graphons*.

A function $W \in \mathcal{W}$ is called a *stepfunction*, if there is a partition $S_1 \cup \dots \cup S_k$ of $[0, 1]$ into measurable sets such that W is constant on every product set $S_i \times S_j$. The number k is the *number of steps* of W .

For every weighted graph G , we define a stepfunction $W_G \in \mathcal{W}_0$ as follows. Let $V(G) = [n]$. Split $[0, 1]$ into n intervals J_1, \dots, J_n of length $\lambda(J_i) = \alpha_i/\alpha_G$. For $x \in J_i$ and $y \in J_j$, let

$$W_G(x, y) = \beta_{ij}(G).$$

For every $W \in \mathcal{W}$ and simple graph $F = (V, E)$, define

$$t(F, W) = \int_{[0,1]^V} \prod_{ij \in E} W(x_i, x_j) \prod_{i \in V} dx_i$$

and for every weighted graph H with $V(H) = [q]$ and $\alpha_H = 1$

$$\mathcal{E}(W, H) = \inf_{(S_i)} \prod_{i,j=1}^q \beta_{ij}(H)^{W(S_i, S_j)},$$

where (S_1, \dots, S_q) ranges over all partitions of $[0, 1]$ into measurable sets with $\lambda(S_i) = \alpha_i$, and

$$W(S_i, S_j) = \int_{S_i \times S_j} W(x, y) dx dy.$$

If $\widehat{F} = (V, E', E'')$ is a signed graph, then we define

$$t(\widehat{F}, W) = \int_{[0,1]^V} \prod_{ij \in E'} W(x_i, x_j) \prod_{ij \in E''} (1 - W(x_i, x_j)) \prod_{i \in V} dx_i. \quad (4)$$

Similarly as in the case of homomorphism functions, $t(\widehat{F}, W)$ can be expressed as

$$t(\widehat{F}, W) = \sum_{Y \subseteq E''} (-1)^{|Y|} t((V, E' \cup Y), W). \quad (5)$$

5 Graphings

Let \mathbf{G}_d denote the set of connected countable graphs with all degrees bounded by d , rooted at a node, and \mathbf{G}'_d , the set of these rooted graphs with an edge from the root also specified. Let \mathcal{A}_d denote the σ -algebra on \mathbf{G}_d generated by subsets obtained by fixing a finite neighborhood of the root, and \mathcal{A}'_d , the analogous σ -algebra of subsets of \mathbf{G}'_d . For every probability measure ρ on $(\mathbf{G}'_d, \mathcal{A}'_d)$, we get a probability measure ρ^* on $(\mathbf{G}_d, \mathcal{A}_d)$ by forgetting the specified edge. We say that π fits ρ , if the Radon-Nykodim derivative $\frac{d\rho^*}{d\pi}$ is equal to the degree of the root for almost all graphs in \mathcal{G}_d . We say that ρ is *symmetric*, if the map $\mathbf{G}'_d \rightarrow \mathbf{G}'_d$ obtained by shifting the root node to the other endnode of the root edge is measure preserving. We say that π is *unimodular*, if it fits a measure preserving ρ .

Let G be a graph with node set $[0, 1]$, with all degrees bounded by d , and assume that for every (Lebesgue) measurable set B , its neighborhood $N(B)$ in G is also measurable. For every set $A \subseteq [0, 1]$ and $x \in [0, 1]$, let $d_A(x)$ denote the number of neighbors of x in B . We say that G is *measure preserving*, if for any two measurable sets A, B ,

$$\int_A d_B(x) dx = \int_B d_A(x) dx.$$

Let ϕ_1, \dots, ϕ_k be measure preserving involutions on $[0, 1]$, then the tuple $([0, 1], \phi_1, \dots, \phi_k)$ is called a *graphing*.

From every graphing $([0, 1], \phi_1, \dots, \phi_k)$ we get a measure preserving graph with bounded degree by connecting x and y in $[0, 1]$ if there is an i such that $y = \phi_i(x)$. From every measure

preserving graph G with bounded degree we get a unimodular measure on rooted graphs with the same degree bound by selecting an $x \in [0, 1]$ uniformly at random, and considering the connected component of G , with root x . These constructions are surjective but not bijective (cf. Elek [31]).

6 Distance of two graphs

For an unweighted graph G and sets $S, T \subseteq V(G)$, let $e_G(S, T)$ denote the number of edges in G with one endnode in S and the other in T (the endnodes may also belong to $S \cap T$; so $e_G(S, S)$ is twice the number of edges spanned by S). For two graphs G and G' on the same set of nodes, we define

$$d_{\square}(G, G') = \frac{1}{n^2} \max_{S, T \subseteq V(G)} |e_G(S, T) - e_{G'}(S, T)|.$$

Note that we are dividing by n^2 and not by $|S| \times |T|$, so (for simple graphs) the contribution of a pair S, T is at most $|T| \times |S|/n^2$.

If G and G' are unlabeled unweighted graphs on different node sets but of the same cardinality n , then we define

$$\hat{\delta}_{\square}(G, G') = \min_{\tilde{G}, \tilde{G}'} d_{\square}(\tilde{G}, \tilde{G}'), \quad (6)$$

where \tilde{G} and \tilde{G}' range over all labelings of G and G' by $1, \dots, n$, respectively.

Let G and G' be weighted graphs with (say) $V(G) = [n]$, $V(G') = [n']$, and assume that the sum of nodeweights is 1 (just scale the nodeweights of each graph). A *fractional overlay* of G and G' is a nonnegative $n \times n'$ matrix X such that

$$\sum_{u=1}^{n'} X_{iu} = \alpha_i(G)$$

and

$$\sum_{i=1}^n X_{iu} = \alpha_u(G').$$

Let $\mathcal{X}(G, G')$ denote the set of all fractional overlays. We define

$$\delta_{\square}(G, G') = \min_{X \in \mathcal{X}(G, G')} \max_{S, T \subseteq V \times V'} \left| \sum_{\substack{(i, u) \in S \\ (j, v) \in T}} X_{iu} X_{jv} (\beta_{ij}(G) - \beta_{uv}(G')) \right|. \quad (7)$$

This notion of a distance extends to graphons as follows. We consider on \mathcal{W} the *cut norm*

$$\|W\|_{\square} = \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} W(x, y) dx dy \right|$$

where the supremum is taken over all measurable subsets S and T , and the *cut distance*

$$\delta_{\square}(U, W) = \inf_{\phi} \|U - W^{\phi}\|_{\square},$$

where ϕ ranges over all invertible measure preserving maps from $[0, 1] \rightarrow [0, 1]$, and $W^\phi(x, y) = W(\phi(x), \phi(y))$. If G and G' are weighted graphs, then

$$\delta_{\square}(G, G') = \delta_{\square}(W_G, W_{G'}).$$

Similar construction can be applied to other norms, e.g., from

$$\|W\| = \int_{[0,1]^2} |W(x, y)| dx$$

we get

$$\delta_1(U, W) = \inf_{\phi} \|U - W^\phi\|_1.$$

7 Internal metric

Given a graphon W , we define $d_W(y, z)$ ($y, z \in [0, 1]$) as the L_1 distance between the functions $W(\cdot, y)$ and $W(\cdot, z)$. We define the *dimension* of W as the infimum of numbers $d > 0$ such that all of $[0, 1]$ but a set of measure ε can be covered by $O(\varepsilon^{-d})$ sets of diameter at most ε (measured in d_W).

8 Edge coloring model

Let G be a finite graph. An *edge coloring model* is determined by a finite set C and a mapping $h : \mathbb{Z}_+^C \rightarrow \mathbb{R}_+$, which we call the *node evaluation function*. Here C is the set of possible edge colors; for any coloring of the edges, we think of $h(a)$ as the value of a node incident with $a(c)$ edges with the color c ($c \in C$). In terms of statistical physics, an edge coloring is a state of the system, and $\log h(a)$ is the contribution of a node (incident with $a(c)$ edges with the color c) to the energy of the state.

To be more precise, for an edge-coloring $\phi : E(G) \rightarrow C$ and node v , let $a_{\phi, v}(c)$ denote the number of edges e incident with v with $\phi(e) = c$. So $a_{\phi, v} \in \mathbb{Z}_+^C$ is the “local view” of node v . The weight of the assignment ϕ is defined by

$$w(\phi) = \prod_{v \in V(G)} h(a_{\phi, v}),$$

and the *edge coloring parameter*, by

$$\text{col}(G, h) = \sum_{\phi: E(G) \rightarrow C} w(\phi).$$

(It will be also useful to allow a single edge with no endpoints; we call this graph the *circle*, and denote it by \bigcirc . By definition, $\text{col}(\bigcirc, h) = |C|$.)

We can define edge-connection matrices that are analogous to the connection matrices defined before: Instead of gluing graphs together along nodes, we glue them together along edges. To be precise, we define a *k-broken graph* as a k -labeled graph in which the labeled nodes have degree one. (It is best to think of the labeled nodes not as nodes of the graph, but rather as points

where the k edges sticking out of the rest of the graph are broken off.) We allow that both ends of an edge be broken off.

For two k -broken graphs G_1 and G_2 , we define $G_1^*G_2$ by gluing together the corresponding broken ends of G_1 and G_2 . These ends are not nodes of the resulting graph any more, so $G_1^*G_2$ is different from the graph G_1G_2 we would obtain by gluing together G_1 and G_2 as k -labeled graphs. One very important difference is that while G_1G_2 is k -labeled, $G_1^*G_2$ has no broken edges any more, and so it is not k -broken. This fact leads to considerable difficulties in the treatment of edge models.

For every graph parameter f and integer $k \geq 0$, we define the *edge-connection matrix* $M'(f, k)$ as follows. The rows and columns are indexed by isomorphism types of k -broken graphs. The entry in the intersection of the row corresponding to G_1 and the column corresponding to G_2 is $f(G_1^*G_2)$. Note that for $k = 0$, we have $M(f, 0) = M'(f, 0)$, but for other values of k , connection and edge-connection matrices are different. We say that f is *edge reflection positive*, if $M'(f, k)$ is positive semidefinite for every $k \geq 0$.

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