Selected topics in graph theory

Homework # 1
Date: March 18, 2017 Due: March 27, 2017.
*(The solution can be submitted on paper or electronically.
I recommend pdf compiled from latex.)*

1.1 Problem. The *half-graphs* H_n are bipartite graphs defined as follows.

$$V(H_n) = \{1, \dots, n\} \cup \{1', \dots, n'\}$$

$$E(H_n) = \{(i, j') : \ i \leq j\}.$$

Prove that for every simple graph F, $t(F, H_n)$ has a limit as $n \to \infty$. Find a combinatorial "meaning" for the limiting value.

1.2 Problem. Let A be an $n \times n$ real matrix and let $\mathcal{P} = V_1 \dot\cup \dots \dot\cup V_k$ be a partition of $\{1, \dots, n\}$. Recall that the matrix $A_{\mathcal{P}}$ is defined as the $n \times n$ matrix in which

$$(A_{\mathcal{P}})_{ij} = \frac{1}{|V_s|\,|V_t|} \sum_{x \in V_s, y \in V_t} A_{xy} \qquad (i \in V_s,\ j \in V_t).$$

Prove that $\|A_{\mathcal{P}}\|_\square \leq \|A\|_\square$.

1.3 Problem. For a symmetric $n \times n$ matrix A, define a modified cut norm by

$$\|A\|_\blacksquare = \max_{S \subseteq \{1, \dots, n\}} \left| \sum_{i,j \in S} A_{ij} \right|.$$

Prove that

$$\frac{1}{2}\|A\|_\square \leq \|A\|_\blacksquare \leq \|A\|_\square.$$

1.4 Problem. Prove that for every graphon W,

$$t(C_6, W)^2 \leq t(C_4, W)t(C_8, W).$$

(Here C_m denotes the cycle of length m.)

1.5 Problem. Prove that

$$\|A\|_\square \leq \|A\|_1 \leq n^2\|A\|_\square$$

for every $n \times n$ matrix A. Improve the factor n^2 to $2n$. [Bonus problem] Improve the factor n^2 to $10\sqrt{n}$.