Selected topics in graph theory

Homework # 2
Date: April 15, 2017 Due: April 24, 2017
*(The solution can be submitted on paper or electronically.
I recommend pdf compiled from latex.)*

2.1 Problem. Let G_1 and G_2 be two simple graphs with $\delta_\square(G_1, G_2) = 0$. Prove that there is a simple graph G and $n_1, n_2 \geq 1$ such that $G_1 \cong G(n_1)$ and $G_2 \cong G(n_2)$.

2.2 Problem. If G_1 and G_2 are two simple graphs such that for every simple graph F we have $\hom(F, G_1) = \hom(F, G_2)$, then $G_1 \cong G_2$.

2.3 Problem. Starting with a single node, at each step we create a new node, flip a coin, and connect the new node either to all previous nodes, or to none of them, depending on the outcome of the coin flip. Prove that with probability 1, the sequence of graphs we obtain converges to the graphon

$$W(x,y) = \begin{cases} 1, & \text{if } x + y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

2.4 Problem. For a graphon W, define the following quantities analogous to the minimum degree and maximum degree of a graph:

$$\delta(W) = \min_{x \in [0,1]} \int_0^1 W(x,y)\, dy, \qquad \Delta(W) = \max_{x \in [0,1]} \int_0^1 W(x,y)\, dy,$$

Prove that for any tree T with k edges, $\delta(W)^k \leq t(T, W) \leq \Delta(W)^k$.

2.5 Problem. For two simple graphs G and H on node sets $V(G) = \{1, \dots, n\}$ and $V(H) = \{1, \dots, k\}$, respectively, define

$$f(G,H) = \max \sum_{ij \in E(H)} e_G(S_i, S_j),$$

where the maximum extends over all partitions $V(G) = S_1 \dot\cup \dots \dot\cup S_k$.
(a) What is $f(G, K_2)$?
(b) Let $\widehat{H}$ denote the weighted complete graph with loops on $V(H)$, where the edges of H are weighted with 2 and the edges of the complement, as well as the loops, are weighted by 1. Prove that

$$\left| \log \hom(G, \widehat{H}) - f(G,H) \right| \leq n \log k.$$