Selected topics from graph theory

Midterm take-home exam
Date: April 11, 2016   Due: April 20, 2016.
(The solution can be submitted on paper or electronically.
I recommend pdf compiled from latex.)

1.1 PROBLEM. For every $k \geq 0$, construct a simplicial complex that is k-connected but not $(k+1)$-connected.

1.2 PROBLEM. Let P be the poset consisting of levels $a, a+1, \ldots, b$ of the Boolean algebra of subsets of an n-element set V. Recall that $\mathcal{C}(P)$ denotes the simplicial complex of all chains in P. Prove that $C(P)$ is $(b-a-1)$-connected.

1.3 PROBLEM. Let $A_1, \ldots, A_m$ be disjoint nonempty finite sets, and let $\mathcal{M}(A_1, \ldots, A_m)$ be the simplicial complex consisting of those subsets $X \subseteq V$ for which $|X \cap A_i| \leq 1$ for every $1 \leq i \leq m$.

(a) Prove that if $|A_i| = 2$ then $\mathcal{M}(A_1, \ldots, A_m) \cong S^{m-1}$.

(b) Prove that if $A'_i \subseteq A_i$, then $\mathcal{M}(A'_1, \ldots, A'_m)$ is a retract of $\mathcal{M}(A_1, \ldots, A_m)$.

(c) Prove that $\mathcal{M}(A_1, \ldots, A_m)$ is $(m-2)$-connected.

1.4 PROBLEM. Prove that the graph property of being planar is evasive (if the number of nodes is at least 5).

1.5 PROBLEM. Prove that contractibility is not equivalent to non-evasiveness. More exactly, there exists an evasive property of boolean functions (for infinitely many values of the number of variables) for which the associated simplicial complex is contractible.