FLATS IN MATROIDS AND GEOMETRIC GRAPHS

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1. INTRODUCTION

This paper was intended to deal with the covering problems in graphs. It has turned out, however, that their study becomes much simpler if a more general structure, which we shall call geometric graph, is considered. Some problems on the covering number of graphs can be translated then to Helly-type problems concerning flats in matroids. The solution of these Helly-type problems (which is complete for the representable matroids only) has required some operations on matroids which generalize the Kronecker product of matrices or versions of it. Analogous Helly-type problems on flats in projective spaces has led to the problem of generalizing the exterior calculus to matroids. The description of some of these constructions needs operations like "Cutting a set of flats by a hyperplane in general position". So the barycentre of the paper has shifted to a calculus of flats in geometries.

In Chapter 2 we deal with matroids, and flats of matroids, over a given, large commutative field. This restriction is required by the unfortunate fact that I have been able to carry out some of the key constructions for such matroids only. But let us remark that most, if not all, applications of matroid theory can be described using representable matroids. Also the clearness and convenience that we have the whole projective space to play around with provides some compensation for the loss of generality.

First we describe the geometry of the operations of "placing points on the flats in general position" and "cutting them by a hyperplane in general position". These operations have been used by several authors, but probably not formulated this way. Then we form the tensor powers, symmetric tensor powers and antisymmetric tensor powers (exterior or Grassmann algebra) of the linear space containing our matroid. This yields interesting matroid structures on the set of ordered k-tuples, unordered k-tuples and k-flats of our matroid, respectively ($k=1,2,\dots$)

Chapter 3 discusses the possibility to generalize these constructions to arbitrary matroids. A collection of flats in a matroid gives a hypermatroid, i.e. a set with a submodular monotone function on the subsets. The first two constructions mentioned

above generalize to arbitrary hypermatroids. On the other hand, we have very weak results in the direction of constructing various tensor powers of matroids.

If we think of a graph we usually visualize its points as points in the plane or - sometimes - in the space, and its edges as lines connecting these points. However, little use is made of this geometric picture. Let us try to consider graphs whose points are points in a geometry or, slightly more generally, elements of a matroid. Call such graphs <u>geometric</u> and <u>pregeometric</u>, respectively.

We may generalize various problems in graph theory, replacing the word "set" by "independent set", or "flat" and the word "cardinality" by "rank" in an appropriate manner. For example, the covering problem: "Determine the minimum number of points covering the edges of a graph!" becomes: "Determine the minimum rank of a set (or, equivalently, flat) of points covering all edges of a graph!". An example of such an investigation is Rado's classical generalization of the Kőnig-Hall Theorem.

Since Rado's Theorem has many applications which require more than just the Kőnig-Hall case, this example already provides some justification of this kind of investigation. It is hoped that this paper will give some further examples.

After listing some graph theoretical problems in which geometric graphs seem to play role and which are, therefore, candidates for applications of this notion, we discuss the covering problem in detail. The fact that we shall be able to move in a wider class of graphs will enable us to give a unified treatment of a large part of the theory of τ-critical graphs and also prove some conjectures. The same method gives some Helly-type theorems on flats in geometries.

Acknowledgements. The way matroids are studied in this paper has been motivated to a large extent by the excellent paper of Mason [15]. I am also indebted to many collegues for the discussions on the topic of this paper, in particular to A. Huhn, P.T. Nagy and A. Recski.

2. FLATS IN THE PROJECTIVE SPACE

2.1. Free selection of points

We consider a projective space P over a (commutative) field K of characteristic 0. In most of the considerations this latter assumption could be replaced by the assumption that K is large enough, but sometimes it will be convenient to be able to divide by natural numbers.

Let $\mathcal{F}$ be a collection of flats in P; for each $F \in \mathcal{F}$, let x_F be a point on F. We say that the collection $X = \{x_F\}$ is *in general position* relative to $\mathcal{F}$ if for every $F \in \mathcal{F}$, any flat spanned by members of $\mathcal{F}$ and points of $X - x_F$ containing x_F must contain the whole flat F. It is easy to show:

Proposition 2.1. It is possible to select one point from each flat of a finite collection $\mathcal{F}$ of flats such that these points are in general position relative to $\mathcal{F}$.

Selecting one point from each flat of a finite collection $\mathcal{F}$ of flats we get a point-set associated with the collection. It is natural to expect that if the selected points are in general position relative to $\mathcal{F}$ then the geometry of this set is well-determined by the collection. The following important formula describes the geometry of these points. It turns out that one does not even need to know all information about the relative position of the flats, only the ranks of unions of them.

Theorem 2.1. Let $\mathcal{F}$ be a finite collection of flats in P and let us select one point x_F from each F such that the collection $\{x_F\} = X$ is in general position relative to $\mathcal{F}$. Then

$$r(X) = \min_{\mathcal{G} \subseteq \mathcal{F}} (r(\cup \mathcal{G}) + |\mathcal{F} - \mathcal{G}|) .$$

Proof: The inequality $\leq$ is trivial, so we only have to find a $\mathcal{G} \subseteq \mathcal{F}$ with equality. We use induction on $|X|$, and distinguish two cases.

Assume first that there is an $F \in \mathcal{F}$ such that $r(X-x_F) = r(X) - 1$. Let $\mathcal{F}' = \mathcal{F} - F$. By the induction hypothesis there exists a subcollection $\mathcal{G} \subseteq \mathcal{F}'$ such that

$$r(X-x_F) = r(\cup \mathcal{G}) + |\mathcal{F}' - \mathcal{G}| .$$

Then

$$r(X)=1+r(X-x_F)=1+r(\cup \mathcal{G})+|\mathcal{F}'-\mathcal{G}|=r(\cup \mathcal{G})+|\mathcal{F}-\mathcal{G}| .$$

Secondly, assume that $r(X-x_F)=r(X)$ for all $F \in \mathcal{F}$. Then x_F is contained in the span of $X-x_F$ and so, by the assumption that X is in general position, the whole flat F is contained in the span of $X-x_F$. Thus the span of X contains all flats in $\mathcal{F}$, and so

$$r(X) = r(\cup \mathcal{F}) .$$

□□□

In combinatorial applications it is usually not sufficient to know that the rank of the set in general position is large, but one would like to pick some more special points. The following simple theorem provides a way to push the points in more special position.

Theorem 2.2. Let $\mathcal{F}$ be a collection of flats in P and let $x_F \in F$ for each $F \in \mathcal{F}$. Let B_F be a basis of F. Then it is possible to replace each x_F by an element y_F of B_F such that $r(\{y_F : F \in \mathcal{F}\}) \geq r(\{x_F : F \in \mathcal{F}\})$.

The easy proof is omitted.

2.2. Cutting hyperplane

Let $\mathcal{F}$ be a collection of flats in P, and H a hyperplane. We want to define when H is in general position. It is not quite easy to exclude all the specialities of position that a hyperplane may have, but for our pruposes it will be sufficient if we require the following: whenever X,Y and Z are sets are unions of flats of $\mathcal{F}$ and

$$\overline{(X \cap H) \cup Y} \cap \overline{(X \cap H) \cup Z} \subseteq H ,$$

we have

$$\overline{(X \cap H) \cup Y} \cap \overline{(X \cap H) \cup Z} \subseteq \overline{X \cap H} .$$

Proposition 2.2. For any finite collection of flats there exists a hyperplane in general position.

If $X \cap H$ is fixed then the condition for being in general position excludes a finite number of positions of H. From this a proof of proposition 2.2 can be obtained. We omit details.

The following formula describes some of the geometry of the collection $\mathcal{F}'$ of flats obtained by intersecting $\mathcal{F}$ by a hyperplane in general position.

Theorem 2.3. Let $\mathcal{F}$ be a collection of non-empty flats in P and H a hyperplane in general position relative to $\mathcal{F}$. Let $\mathcal{F}' = \{F \cap H : F \in \mathcal{F}\}$. Then

$$r(\cup \mathcal{F}') = \min \sum_{r=1}^{m} (r(\cup \mathcal{G}_i) - 1) ,$$

where $\{\mathcal{G}_1, \dots \mathcal{G}_m\}$ ranges over all partitions of $\mathcal{F}$

Proof: The inequality $\leq$ is trivial, so what we really have to do is to establish a partition $\{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ for which equality holds. Let $Q = \cup\mathcal{F} \cap H$. For every $F \in \mathcal{F}$ we have $r(F \cap Q) = r(F \cap H) = r(F) - 1$ and hence $r(F \cup Q) = r(Q) + 1$. Let $A_1, \dots, A_m$ be the flats of the form $\overline{F \cup Q}$ $(F \in \mathcal{F})$, and $\mathcal{G}_i = \{F \in \mathcal{F} : F \subseteq A_i\}$. Clearly $\mathcal{G}_1, \dots, \mathcal{G}_m$ is a partition of $\mathcal{F}$. Put $G_i = \cup \mathcal{G}_i$, $Q_i = G_i \cap H$. We claim that

$$r(Q_1 \cup \dots \cup Q_m) = r(Q_1) + \dots + r(Q_m),$$

i.e. that Q is direct sum of $Q_1, \dots, Q_m$. This will prove the theorem. It suffices to show that

$$\overline{Q-Q_i} \cap \overline{Q-Q_j} \subseteq \overline{Q-Q_i-Q_j}$$

for any choice of indices $1 \leq i < j \leq m$. Set $X = \cup(\mathcal{F} - \mathcal{G}_i - \mathcal{G}_j)$, then

$$\overline{(X \cap H) \cup G_i} \cap \overline{(X \cap H) \cup G_j} \subseteq A_i \cap A_j \subseteq H$$

and hence by the definition of "general position",

$$\overline{(X \cap H) \cup G_i} \cap \overline{(X \cap H) \cup G_j} \subseteq \overline{X \cap H}$$

Now here $\overline{(X \cap H) \cup G_i} \supseteq \overline{Q-Q_j}$, $\overline{(X \cap H) \cup G_j} \supseteq \overline{Q-Q_i}$ and $\overline{X \cap H} = \overline{Q-Q_i-Q_j}$. Hence the assertion.

□□□

Corollary: If $\cup \mathcal{F}'$ is connected (as a geometry) then

$$r(\cup \mathcal{F}') = r(\cup \mathcal{F}) - 1.$$

One might want to cut the family $\mathcal{F}$ by an r-flat in general position (where $r < n-1$). This can be achieved by cutting n-r times by hyperplanes. We do not go into details.

The most interesting case is when all flats in $\mathcal{F}$ are lines. In this case the intersection is a geometry. Several matroid constructions have this in the background. For example if $\mathcal{F}$ consists of all lines spanned by a set of points then the intersection geometry is the Dilworth truncation of the original, as introduced by Mason [15].

2.3. Tensor product

Let V be a linear space of dimension n over K and W its dual (i.e. W is the linear space of linear functions defined on V with values in K). A k-contravariant tensor or, shortly, a k-tensor is a function $w(x_1, \dots, x_k)$ with values in K, defined for $x_i \in W$, and linear in each variable. For $k=1$, 1-tensors can be identified with the elements of V. The k-tensors form a linear space V^k of dimension n^k.

Given a k-tensor $w(x_1, \dots, x_k)$ and an m-tensor

$w'(x_1,\ldots,x_m)$, their <u>tensor product</u> $w \otimes w'$ is defined as the $(k+m)$-tensor

$$(w \otimes w')(x_1,\ldots,x_{k+m}) = w(x_1,\ldots,x_k)w(x_{k+1},\ldots,x_{k+m}) .$$

In particular if $v_i \in V$ $(i=1,\ldots,k)$ then $v_1 \otimes \ldots \otimes v_k$ is defined and belongs to V^k. This way the cartesian power $V \times \ldots \times V$ is embedded in V^k.

Let $S_1,\ldots,S_k$ be subsets of V, then $S=S_1 \times \ldots \times S_k$ is thus embedded in V^k and hence it is supplied with a matroid structure (S,r^k). This matroid will be called a <u>tensor product</u> of the matroids (S_i,r) (where r denotes the rank in V). One has to formulate this definition carefully since the geometry of (S,r^k) is not uniquely determined by the matroids (S_i,r_i) but it also depends on the way they are embedded in V.

Let, for example, both (S_1,r_1) and (S_2,r_2) be "four-point lines", i.e. consist of four coplanar non-zero vectors. The product (described as a geometry in the projective 3-space) will consist of the 16 intersection points of two quadruples of generatrices of a doubly ruled surface (Figure 1). However, the fact whether or not the points labelled 1,2,3,4 are coplanar depends on whether or not the cross-ratios of the original quadruples of points are equal.

We remark that if two matroids are represented by

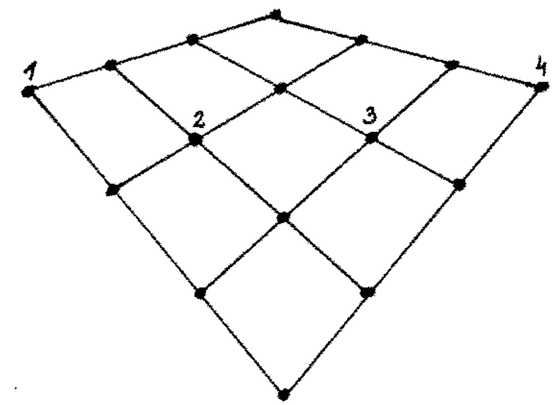

Figure 1

the columns of two matrices then their corresponding tensor product is represented by the Kronecker product of these matrices.

The example above suggests that even though the tensor product matroid (S,r) is not determined by the factors, many important geometric properties of it will be. The following are easily proved:

<u>Proposition 2.3.</u> Let A_i be an independent subset of S_i. Then $A_1 \times \ldots \times A_k$ is independent in any tensor product.

<u>Proposition 2.4.</u> Let $X_i \subseteq S_i$. Then $\overline{X_1 \times \ldots \times X_k} = \overline{X_1} \times \ldots \times \overline{X_k}$.

<u>Proposition 2.5.</u> Let F_i be a flat in (S_i,r_i). Then $F_1 \times \ldots \times F_k$ is a flat in (S,r) with rank $r_1(F_1)\ldots r_k(F_k)$.

Proposition 2.6. The natural projection of S onto S_i is a strong map.

Proposition 2.7. Given any non-loops $s_i \in S_i$, the subset $\{x_1 \otimes \dots \otimes x_k : x_i = s_i \text{ for all } i \neq i_o\}$ is isomorphic to S_{i_o}.

2.4. Symmetric tensor product

Let us consider the case when $S_1 = \dots = S_k = S$. The tensor product $V \otimes \dots \otimes V$ yields a matroid structure on the set of ordered k-tuples of elements of S. If we want to have a matroid structure on the unordered k-tuples, we have to consider symmetric tensors.

A k-tensor is called symmetric if permutation of its variables leaves its value invariant. From any k-tensor $w(x_1,\dots,x_k)$ we can construct a symmetric one by averaging over all permutations of the variables:

$$\widetilde{w}(x_1,\dots,x_k) = \frac{1}{k!} \sum_{\pi} w(x_{\pi(1)},\dots,x_{\pi(k)}).$$

The formula

$$u \circ v = \widetilde{u \otimes v}$$

defines a new multiplication among symmetric tensors. It is easily seen that this multiplication is associative and that $u \circ v \neq 0$ if $u,v \neq 0$.

Let $S^{(k)}(V)$ denote the linear space of symmetric k-tensors. It is easy to see that $S^{(k)}(V)$ has dimension $\binom{n+k-1}{k}$. With any k-subset $v_1,\dots,v_k$ of vectors, we may associate the symmetric k-tensor $v_1 \circ \dots \circ v_k$. This defines a matroid structure on the set of k-subsets of V, and hence, it defines a matroid $(S^{(k)}, r^{(k)})$ on the set of ordered k-tuples of elements of any matroid (S,r) embedded in V. Similarly as above, this matroid structure is not uniquely determined by the matroid structure of (S,r), but many important geometric properties of it are. We state one of these:

Proposition 2.8. For any flat F in (S,r), the k-tuples meeting F form a flat in $(S^{(k)}, r^{(k)})$.

2.5. Exterior Product

Let V be an n-dimensional linear space over a (commutative) field F. A polyvector of degree r or, shortly, an r-vector is defined as an r-tensor which is antisymmetric, i.e. has the property that exchanging any two variables just changes the sign of its value. It is clear that all r-vectors form a linear space $A^{(r)}(V)$ of dimension $\binom{n}{r}$. 1-vectors are just vectors; 2-vectors can be identified with antisymmetric matrices. The determinant is an n-vector (the only one up to a scalar factor).

Given any r-tensor $w(x_1,\dots,x_r)$, we can obtain a polyvector by the formula

$$\hat{w}(x_1,\dots,x_r) = \frac{1}{r!}\sum_{\pi} (\mathrm{sg}\,\pi)\, w(x_{\pi(1)},\dots,x_{\pi(r)}) ,$$

where π ranges over all permutations of $1,\dots,r$. If w itself is antisymmetric then $\hat{w} = w$.

The <u>exterior</u> <u>product</u> $u \wedge v$ is defined by

$$u \wedge v = \widehat{u \otimes v} .$$

The following facts are both simple and well-known.

<u>Proposition 2.9.</u> The exterior multiplication of polyvectors is associative and distributive; exchanging the two factors either leaves the product invariant or changes its sign, depending on the parity of the product of the degrees.

<u>Proposition 2.10.</u> Let $x_1,\dots,x_r$ be 1-vectors. Then $x_1 \wedge x_2 \wedge \dots \wedge x_r = 0$ iff $x_1,\dots,x_r$ are linearly dependent.

<u>Proposition 2.11.</u> Let $x_1,\dots,x_r,y_1,\dots,y_r$ be 1-vectors. Then the polyvectors $x_1 \wedge \dots \wedge x_r$ and $y_1 \wedge \dots \wedge y_r$ are parallel iff the spans of $\{x_1,\dots,x_r\}$ and $\{y_1,\dots,y_r\}$ are the same.

Now let (S,r) be any matroid embedded in V. Let A be a flat of (S,r) and $\{a_1,\dots,a_k\}$ a basis of A. Let us associate the k-vector $a_1 \wedge \dots \wedge a_k$ with the k-flat A. It follows from proposition 2.11. that this k-vector is determined by A up to a scalar factor. Therefore the matroid (F_k,r_k), induced on the set F_k of k-flats, is well-defined. Its structure of course depends on the way (S,r) is embedded in V; but again, many geometric properties of (F_k,r_k) are determined by the matroid structure of (S,r) alone. Before stating one of these we exploit exterior multiplication a bit more.

Let A be an r-flat and B an s-flat in (S,r); let a and b be a corresponding r-vector and k-vector, respectively. By proposition 2.10.

$$a \wedge b = \begin{cases} 0, \text{ if } r(A) + r(B) > r(A\cup B), \\ \text{an r-vector corresponding to } \overline{A\cup B}, \text{ if } \\ r(A) + r(B) = r(A\cup B). \end{cases}$$

Thus if we introduce a loop O_k in (F_k,r_k), called the <u>zero flat of rank k</u>, then $\wedge$ defines an operation on the flats of (S,r). Note that this operation is associative and it depends only on the matroid structure of (S,r). Put $S_k = F_k \cup \{O_k\}$.

<u>Proposition 2.12.</u> For any flat $A \in S_k$, the map $X \longmapsto A \wedge X$ is a strong map of (S_t,r_t) into (S_{t+k}, r_{t+k}).

<u>Proposition 2.13.</u> For any flat A of (S,r), the k-flats B such that $r(A\cup B) < r(A) + r(B)$ form a flat in (S_k,r_k).

3. HYPERMATROIDS AND FLATS IN MATROIDS

3.1. Points and planes in general position

The combinatorial properties of sets of points in linear spaces are described by the notion of a matroid. If we want to find a similar description of collections of flats in linear or projective spaces, or more generally matroids, we need the notion of a hypermatroid.

A <u>hypermatroid</u> is a set S with an integral valued function f defined on its subsets, which is non-negative, monotone and submodular, i.e. satisfies the relations

(i) $f(\emptyset) = 0$,
(ii) if $X \subseteq Y$ then $f(X) \leq f(Y)$,
(iii) $f(X \cap Y) + f(X \cup Y) \leq f(X) + f(Y)$.

If S is a collection of flats in a matroid, and for $X \subseteq S$, $f(X)$ is the rank of $\cup X$, then it is easy to see that we obtain a hypermatroid. Somewhat later we shall show:

<u>Proposition 3.1.</u> Every hypermatroid is representable as a collection of flats in some matroid.

Let us point out that a collection of flats in a (say) projective space can have "combinatorial" properties which are not reflected by the corresponding hypermatroid. For example, taking any 3 lines in the plane we get the same hypermatroid, regardless whether or not they pass through the same point. On the other hand, sometimes it may be useful not to represent a hypermatroid as flats in a matroid: for example, we might want to speak about "three mutually skew lines in the space" without any specified points on them.

An important construction associating a matroid with every hypermatroid is due to Edmonds [7]. Geometrically it means putting a point on each flat in general position. Precisely, to put a point x on an element y of a hypermatroid (S,f) means to construct the hypermatroid (S+x,f) where f remains the same on the subsets of S and

$$f(X+x) = \begin{cases} f(X), \text{ if } f(X+y) = f(X), \\ f(X)+1 \text{ if } f(X+y) > f(X) . \end{cases}$$

It is easy to verify that this is indeed a hypermatroid. Also we can check that if we repeat this construction by putting a point in general position on another member of S then the order of the two extensions is irrelevant. So if we repeat this construction we end up with a hypermatroid $(S \cup S', f)$, where there is a one-to-one correspondence between S and S', and each element $x \in S'$ has $f(x) \leq 1$. By the same argument as in Chapter 2 we get that

<u>Proposition 3.2.</u> Let $(S \cup S', f)$ be obtained from the hypermatroid (S,f) by putting a point on each member of (S,f) in general position. Then

$$f(S') = \min_{X \subseteq S} (f(X) + |S-X|).$$

Let us take each member y of S f(y) times. This means that we replace y by a set A_y of f(y) new elements, and define $S' = \bigcup_y A_y$, $f'(X) = f(\{y: X \cap A_y \neq \emptyset\})$. Place a point on each member of the hypermatroid (S',f'), in general position, and let F_y be the flat of the resulting matroid spanned by the points placed on members of A_y. Then the flats F_y form a hypermatroid isomorphic to the original one, as one easily checks from proposition 3.2. This implies proposition 3.1.

Let us consider now cutting by a hyperplane in general position. This also could be described as an extension of the polymatroid. Note, however, that one has to add all "intersections" of the hyperplane and the original flats, not just the hyperplane, because the important properties of the "general position" could not be defined otherwise. We do not go into the details of this, but define instead the "intersection with a hyperplane in general position" as the hypermatroid (S,f'), where

$$f'(X) = \min_{Y_1 \cup \ldots \cup Y_m = X} \sum_{i=1}^{m} (f(Y_i)-1), \quad (X \subseteq S) .$$

(cf. Theorem 2.2). It is not difficult to check that this formula in fact defines a hypermatroid.

If (S,f) is the hypermatroid formed by the lines of matroid, then (S,f') is a geometry, called the <u>Dilworth truncation</u> of the matroid (see Mason [15]).

3.2. Product of matroids

Let (S_1,r_1) and (S_2,r_2) be two matroids. We say that the matroid (S,r) is a <u>product</u> of (S_1,r_1) and (S_2,r_2) if the following conditions hold:

(i) $S = S_1 \times S_2$;

(ii) for every $x \in S_1$ the mapping $y \mapsto (x,y)$ is an isomorphic embedding of (S_2,r_2) into (S,r), and vice versa;

(iii) for any two flats F_1 of (S_1,r_1) and F_2 of (S_2,r_2), the set $F_1 \times F_2$ is a flat in (S,r).

Using the notion of strong maps, these conditions can be rephrased. On the set $S_1 \times S_2$ we obtain a matroid $(S_1 \times S_2, r_1)$ as the direct sum of S_2 copies of (S_1,r_1); we define $(S_1 \times S_2, r_2)$ analogously. Then the matroid (S,r) is a product of (S_1,r_1) and (S_2,r_2) if and only if $S = S_1 \times S_2$ and the identity maps: $(S_1 \times S_2, r_i) \longrightarrow (S,r)$, as well as the natural projections: $(S,r) \longrightarrow (S_i,r_i)$, are strong.

It is easy to prove that

Proposition 3.3. Let (S,r) be a product of the matroids (S_1,r_1) and (S_2,r_2). Then

$$r_1(S_1) + r_2(S_2) - 1 \leq r(S) \leq r_1(S_1)r_2(S_2) .$$

A product of two matroids for which the first (second) equality holds in Proposition 1 will be called a lower (upper) product, respectively.

Proposition 3.4. Any two matroids have a lower product.

Proof. Consider first the direct sum of (S_1,r_1) and (S_2,r_2). This can be visualized by placing (S_1,r_1) and (S_2,r_2) on two non-intersecting flats of rank r_1 and r_2, respectively, in the projective space of rank r_1+r_2. Let H be a hyperplane in general position, and identify the pair (x_1,x_2) $(x_i \in S_i)$ with the point where the line through x_1 and x_2 intersects H. This way the set $S_1 \times S_2$ is embedded in the hyperplane H. (Figure 2).

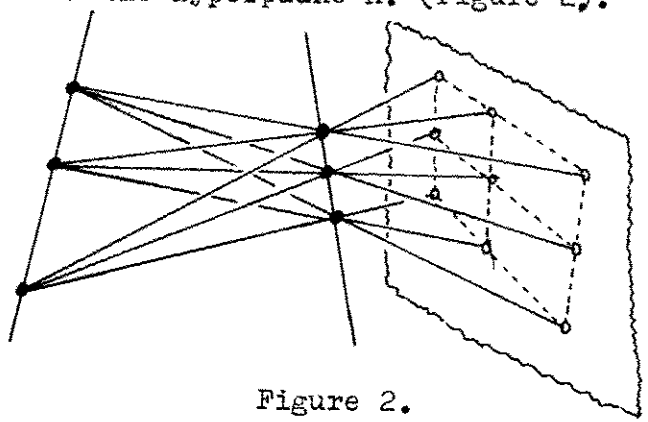

Figure 2.

Let (S,M) be the matroid formed by these points of H.

This is a product of (S_1,M_1) and (S_2,M_2). Since it is contained in H, its rank is at most r_1+r_2-1 . By Proposition 1, its rank is exactly r_1+r_2-1 and so it is a lower product.

In [15] Mason proposes another "product", which is obtained by putting points in general position on the lines x_1x_2 $(x_i \in S_i)$. If $S_1 = S_2$ then the diagonal of Mason's product is just the well-known sum of the two matroids. The diagonal of the lower product is another "sum".

I do not know whether an upper product of any two matroids exists. If it does, it should be a very useful tool. By the results of Chapter 2, it exists if the two matroids are representable over the same commutative field.

The situation is similar if we consider symmetric product. By the results of Chapter 2, if (S,r) is a matroid representable over a commutative field then a matroid structure with nice properties can be defined on the set of its k-subsets. One may ask whether for every matroid (S,r) a matroid $(S^{(k)}\ r^{(k)})$, called its k^{th} symmetric power, exists with the following properties:

(i) $S^{(k)}$ is the set of k-subsets of S;

(ii) if F is any flat in S then the set of all k-subsets of S which meet F is a flat in $(S^{(k)}, r^{(k)})$;

(iii) $r^{(k)}(S^{(k)}) = \binom{r(S)+k-1}{k}$.

3.3. Exterior calculus in matroids

Let $\mathcal{L}(+,\cdot)$ be a atomic lattice with rank function r. For each r=0,1,... take new element 0_r, and define $r(0_r) = r$. Define an operation $\wedge$ called <u>exterior multiplication</u> on $\mathcal{L} \cup \{0_0, 0_1, \dots\}$ as follows: if r(x)=s, r(y)=t then

$$x \wedge y = \begin{cases} 0_{r+t} & \text{if } x=0_s \text{ or } y=0_t \text{ or } r(x+y)<s+t \\ x+y & \text{otherwise.} \end{cases}$$

<u>Proposition 3.5.</u> Exterior multiplication is associative iff the lattice if geometric.

The proof is straightforward. Let us assume in the sequel that $\mathcal{L}$ is geometric, i.e. it is the lattice of flats or a matroid (S,r). Call 0_r the 0-flat of rank r, then the exterior product of any two flats is a flat whose rank is the sum of the ranks of the two factors. Thus the operation of exterior multiplication in linear spaces can be extended to arbitrary matroids.

For flats in projective spaces over commutative fields, however, there is a further important tool, and this is the geometry structure on the set Q_r of r-flats or, equivalently, the matroid structure on the set $Q_r \cup \{0_r\}$. The question whether an analogous structure can be defined on the flats of an arbitrary matroid is open. More precisely, let a <u>Grassmann graduated matroid</u> of (S,r) be defined as a sequence (S_k, r_k) of matroids such that

(i) the elements of S_k are the k-flats, 0_k having rank 0, the other k-flats having rank 1.

(ii) for every flat A in (S,r) of rank k, and for every t, the mapping $A \mapsto A \wedge B$ is a strong map of (S_t, r_t) into (S_{t+k}, r_{t+k}).

These two assumptions imply very strong properties of the Grassmann graduated matroid. It follows that if B is a basis in (S,r) then the elements of the form $b_1 \wedge \dots \wedge b_k$, where $b_1, \dots, b_k$ are distinct elements of B, form a basis of (S_k, r_k). Consequently, $r_k(S_k) = \binom{r(S)}{k}$. Another important property is that if F is a flat in (S,r) then those k-flats G for which $r(F \cup G) < r(F) + k$, form a flat in (S_k, r_k).

It follows from the results in Chapter 2 that every matroid representable over a commutative field has an associated Grassmann graduated matroid. In the next chapter we shall see applications of this fact. It seems to be an extremely question whether every matroid has a Grassmann graduated matroid.

4. GEOMETRIC GRAPHS

4.1. Geometries induced by graphs

There are several graph theoretical investigations which yield matroid structures on the points of graphs. We shall discuss these briefly. It may be hoped that these investigations can be generalized to geometric graphs. One example where, to a certain extent, such a generalization has been carried out is the covering problem, discussed in the next section. In this case, the generalization yields simple and unified treatment of the known results, and leads to the solution of some unsolved problems for ordinary graphs. Maybe the generalization to geometric graphs will have similar effect on the other problems listed below.

a) Transversal matroids. Given a bipartite graph G with bipartition $\{A,B\}$, we define the transversal matroid induced by G on A as follows: a set $X \subseteq A$ is independent iff it can be covered by independent edges, i.e. iff its points can be matched with distinct points of B. It follows easily e.g. by Theorem 2.2 that this is indeed a matroid and has the following geometric description:

Proposition 4.1. Let G be a bipartite graph with bipartiton $\{A,B\}$. Let the points of B form a free matroid and place each point of A in general position on the flat spanned by its neighbors. Then the resulting matroid on A is the transversal matroid induced by G. Moreover, if we contract A we obtain the dual of the transversal matroid induced by G on B.

Calculating the rank of the transversal matroid by Theorem 2.1 we obtain the Kőnig-Hall theorem.

Proposition 2.1 indicates that we may have an arbitrary geometry on B instead of the free one. This important generalization of the Kőnig-Hall theorem is due to Rado [18].

b) Matching matroids. Edmonds [6] generalized the construction yielding transversal matroids to arbitrary graphs. Let G be an arbitrary graph. We define a matroid, called the matching matroid of G, on the set V(G) by calling a subset independent iff it can be covered by independent edges. It turns out that this is indeed a matroid and that it is in fact a transversal matroid. The Tutte-Berge formula for the number of independent edges of G and Edmonds' algorithm to determine this number concern, in this formulation, the rank of the matching matroid. It is often convenient to consider the dual matching matroid, in which a subset of points is independent iff its removal does not decrease the maximum number of independent edges.

For later reference we cite the following theorem [13]:

<u>Theorem 4.1.</u> If G is a bipartite graph, $X \subseteq V(G)$, $|X| = \vartheta$ and G has the property that deleting any edge of it the restriction of the dual matching matroid to X changes, then at most $2^{4\vartheta}$ points of G have degree ≥ 3.

c) <u>Gammoids.</u> Let G be a graph and $p_1, \dots, p_m$ points of G. Call a subset $X \subseteq V(G)$ independent, if there exist disjoint paths in G matching this subset with some of $p_1, \dots, p_m$. It was shown by Pym [17] and Perfect [16] that this way one obtains a matroid. These matroids are called <u>strict gammoids</u>, and their restrictions <u>gammoids</u>. Brualdi [5] generalized this construction by considering a pregeometric graph, and then defining a new matroid on the set of its points by calling a subset independent if it can be connected to a set of points independent in the original matroid by disjoint paths. He also noted the interesting connection that the duals of strict gammoids are transversal matroids and vice versa.

These results are in a similar relationship to Menger's theorem to the relationship between Rado's theorem and the Kőnig-Hall theorem. It might be interesting to generalize other problems concerning connectivity of graphs, e.g. the deep results of Halin and Mader [14] on critically connected graphs, to geometric graphs.

d) <u>Planar graphs.</u> Let G be a 3-connected planar graph. Then it can be realized as the graph of a convex polytope. Such a realization yields a matroid structure on the set of points (of rank 4). It is not clear whether any use can be made of this observation.

4.2. Geometric graphs and their matroids

Recall that a graph together with a matroid on the set of its points is called a <u>pregeometric graph</u>. If the underlying matroid is a geometry we call the graph <u>geometric</u>.

Let e be an edge of a pregeometric graph G. By <u>deleting</u> e we mean deleting it from the graph, keeping the underlying matroid invariant. But to define its <u>contraction</u> we have to alter the underlying matroid as follows. Let us delete e. Put a point y in general position on the line spanned by e and then contract y. Then the two endpoints of e become parallel points and they can be identified.

Let x be a point in a pregeometric graph. <u>Deleting</u> x means deleting it from the graph (together with all edges adjacent to it) and also from the matroid. <u>Contracting</u> x means to delete it from the graph and contract it in the matroid. In the case when

G is an ordinary graph i.e. the underlying matroid is free these two operations are the same.

It is quite natural in the context of this paper to try to define matroids on the set E(G) of edges of a pregeometric graph G. First we generalize the usual circuit-matroid of the graph. Let G be a geometric graph, (V(G),r) its underlying matroid and E(G) its set of edges. Then each edge E of the graph spans a line $\overline{E}$ of the matroid. Cutting this bundle of lines by a hyperplane in general position we obtain a matroid, which we shall call the circuit-matroid of G. If G is an ordinary graph (i.e. its points are independent) then this matroid is the usual circuit matroid. Note that the above construction provides a very graphic definition of the circuit matroid (Figures 3a and 3b).

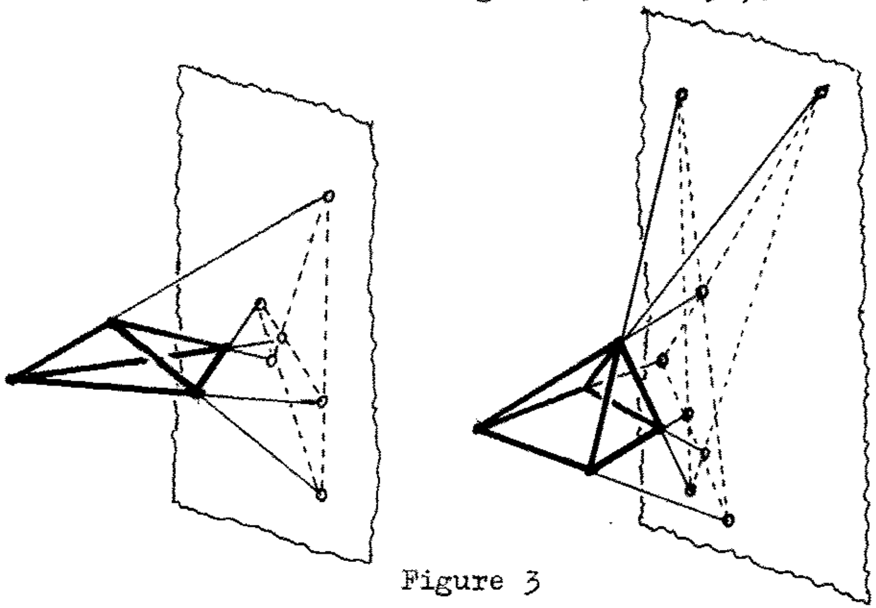

Figure 3

Another way to associate a matroid with the bundle of lines spanned by the edges is to select one point from each edge in general position. This again results in a matroid on the set of edges. In the ordinary case a set of edges is independent iff every connected component of it contains at most one circuit.

If the symmetric square of the underlying matroid exists, then the set of edges can be considered as a subset of this, and this way we obtain another matroid. Similarly we obtain a matroid from the antisymmetric square. These two matroids are trivial (free) if the underlying matroid of the graph is free, but we obtain interesting matroids if we consider appropriate truncations of the underlying matroid.

4.3. The covering problem for graphs; results

Let G be a pregreometric graph. Define its covering number as the minimum rank of a set of points which cover all edges. Denote this number by $\tau(G)$.

If G is defined on the free matroid then, trivially, $\tau(G)$ is the ordinary covering number, i.e. the minimum number of points covering all edges. It is easy to see that if we delete those points of G which have rank 0 and identify those pairs of points which form circuits, the covering number of the resulting geometric graph is the same as the covering number of G. Therefore in most problems we may restrict ourselves to geometric (rather than pregeometric) graphs. Similarly, we may delete the isolated points of G.

It is well known that even the case when the underlying matroid is free (i.e. when we are considering ordinary graphs) is a "hard" combinatorial problem, i.e. no efficient algorithm or useful minimax formula determining $\tau(G)$ can be expected. One aproach to gain information about this invariant of graphs is to consider the graphs which are τ-critical, i.e. which have the property that the deletion of any edge results in a graph whose covering number is smaller. Since isolated points play no role, we shall assume there are no isolated points in the τ-critical graphs.

The notion of τ-critical graphs was introduced by A.A. Zykov in 1949 [20]. The main directions in the study of their structure were started by P. Erdős and T. Gallai in 1961 [8]. Among others, they proved the following result.

Theorem 4.1. The number of points in a τ-critical graph G is at most $2\tau(G)$. Equality holds only for the graphs consisting of disjoint edges.

Let us remark that this result can be formulated as follows:

Theorem 4.1. If each subgraph spanned by 2t+2 points of a graph G can be covered by t points then G can be covered by t points.

In particular, the class of graphs which can be covered by k points can be characterized by a finite number of excluded subgraphs. Another important corollary is that a bipartite graph is not critical unless it consists of disjoint edges. Sharpening this result Andrásfai [1] proved that given any non-isolated edge in a τ-critical graph, there is an odd circuit containing it. A result of similar nature was found by Beineke, Harary and Plummer [2], who showed that any two adjacent edges of a τ-critical graph are contained in an odd circuit. Berge [3] proved a common generalization of these two results by showing:

Theorem 4.2. Any two adjacent edges of a τ-critical graph are contained in a chordless odd circuit.

A second question that Theorem 1 raises is the following: how large can the number of edges of a τ-critical graph be? The bound $\binom{2\tau(G)}{2}$ implied by Theorem 1 is not best possible; Erdős, Hajnal and Moon [9] proved that the number of edges in a τ-critical graph is at most $\binom{\tau(G)+1}{2}$.

We shall generalize this result:

<u>Theorem 4.3.</u> Let G be a τ-critical graph and A a stable subset of V(G). Then the number of edges not covered by A is at most $\binom{\tau(G)-|A|+1}{2}$.

This bound is attained e.g. for the complete graphs.

The third, and probably richest, direction of research was suggested by Gallai. Set $\delta(G) = 2\tau(G) - |V(G)|$. For τ-critical graphs without isolated points this number is non-negative and it equals to 0 only for graphs consisting of disjoint edges. What kind of structural properties of the graph G can be proved if $\delta(G)$ is considered fixed? A basic result is this direction is due to Hajnal: the degree of any point of a τ-critical graph without isolated points is at most $\delta(G)+1$. This result was slightly sharpened by L. Surányi [19] and the author:

<u>Theorem 4.4.</u> Let A be a stable set of points of a τ-critical graph G. Let d be the degree of any point in A. Then A has at least $|A|+d-1$ neighbors.

Using that $d \geqq 1$ and also the Kőnig-Hall Theorem, another important result of Hajnal can be obtained: Any stable set of points in a τ-critical graph can be covered by disjoint edges.

Hajnal's first-mentioned theorem easily implies that the only connected τ-critical graphs with $\delta=1$ are the odd circuits. Andrásfai [1] proved that the connected τ-critical graphs with $\delta=2$ are precisely those, which can be obtained from the complete 4-graph by subdividing each edge by an even number of points. Gallai formulated the following general conjecture. It is easy to see that if we subdivide each edge of a connected τ-critical graph by an even number of points, the resulting graph will be τ-critical with the same δ. Conversely, if we "iron out" two adjacent points of degree 2 in a τ-critical graph, we get a τ-critical graph with the same δ. Now Gallai conjectured that

for any fixed δ, there exist a finite set of connected τ-critical graphs with this prescribed δ, such that all connected τ-critical graphs with this δ can be obtained from them by subdividing each edge by an even number of points.

This conjecture was proved by L. Surányi [19] and the author for $\delta=3$, and more recently by the author for all values of δ [13]:

<u>Theorem 4.5.</u> Let G be a connected τ-critical graph. Then the number of G with degree $\geqq 3$ is at most $2^{5\delta^2}$.

This bound is certainly not sharp, but to sharpen it would certainly need significant new ideas. The best lower bound, provided by a non-trivial construction of L. Surányi, is only $c \cdot \delta^2$ [19].

4.4. Proofs; the covering problem for geometric graphs

In this section, unless the attribute "ordinary" is used, graph means pregeometric graph.

<u>Proposition 4.2.</u> Assume that the point x of a graph G is an isthmus in the underlying matroid, i.e. it is not contained in the span of the other points. Let G' arise by moving x to a point x' in general position on a flat containing the neighbors of x. Then $\tau(G') = \tau(G)$.

<u>Proof</u>: Trivially $\tau(G') \leq \tau(G)$. On the other hand, let T be a set covering the edges of G'. If $x' \notin T$ then T covers all edges of G as well and hence $r(T) \geq \tau(G)$. If $x' \in T$ but $x' \notin \overline{T-x'}$ then $r(T) = r(T-x')+1 = r(T-x'+x) \geq \tau(G)$. Finally, if $x' \in \overline{T-x'}$ then, by the general position of x', all neighbors of x are contained in $\overline{T-x'}$ and therefore $\overline{T-x'}$ covers all edges of G. Thus $r(T) = r(\overline{T-x'}) \geq \tau(G)$.

□□□

Similar arguments show:

<u>Proposition 4.3.</u> Assume that the point x of the graph G is contained in the flat spanned by its neighbors. Let G' arise from G by contracting x. Then $\tau(G') = \tau(G) - 1$.

An immediate consequence of these observations is

<u>Proposition 4.4.</u> If a graph G is τ-critical then performing either one of the operations in propositions 4.2 and 4.3, the resulting graph is τ-critical.

Considering now sufficient conditions under which edges can be eliminated, first we remark the following:

<u>Proposition 4.5.</u> Let (x,y) be an edge of the graph G and assume that y is contained in the span of the other neighbors of x. Then $\tau(G-(x,y)) = \tau(G)$.

The proof is immediate. For τ-critical graphs we obtain:

<u>Proposition 4.6.</u> The neighbors of any point of a τ-critical graph are independent in the underlying matroid.

(At this point, we can obtain a short proof of Theorem 4.3. Let G be a τ-critical ordinary graph, A a stable set in G, B the set of neighbors of A and $a \in A$. Eliminate the points of A-a one by one, by moving each one to a point in general position on the flat spanned by its neighbors and then contracting it. It is clear that at each time, the remaining points of A-a are isthmusses in the matroid, so the operations can be performed. By proposition 4.4, we obtain τ-critical graphs at each time. The rank of B decreases by 1 at each step, so at the end it becomes $|B|-|A|+1$. But the neighbors of $\underline{a}$ are independent in this final graph, and so their number is $|B|-|A|+1$.)

Proposition 4.5 is void if the graph is ordinary, i.e. the underlying matroid is free. Using tensor calculus we can obtain the following more general criterion.

Proposition 4.7. Let G be a graph and $r^{(2)}$ the rank function of the symmetric square of the underlying matroid. Assume that an edge e is not an isthmus in $(E(G),r^{(2)})$. Then $\tau(G-e) = \tau(G)$.

Proof: Let T be a set covering all edges of G-e. The edges meeting T form a flat in $(E(G),r^{(2)})$ by definition, and this flat contains all elements of $E(G)-e$. But since e is not an isthmus in $(E(G),r^{2})$, it follows that this flat contains e, i.e. e meets T. Thus T covers all edges of G, and therefore $|T| \geq \tau(G)$.

□□□

Theorem 4.6. Let G be a τ-critical graph such that the underlying matroid is representable over a commutative field. Then

$$|E(G)| \leq \binom{\tau(G)+1}{2}.$$

Proof: We may assume that the underlying matroid has rank $\tau(G)$, since otherwise we may consider its truncation of rank $\tau(G)$. By proposition 4.7, the edges from an independent subset of the symmetric square of the underlying matroid.

□□□

(At this point, Theorem 4.4 is easily proved. Let G be a τ-critical ordinary graph and A a stable set of its points. Eliminate the points of A by proposition 4.4. The resulting graph G' is τ-critical and has $\tau(G') = \tau(G) - |A|$. So Theorem 4.6 implies the bound on the number of edges of G' as stated.)

We are in the position now to prove Theorem 4.5. Let G be an ordinary τ-critical graph, B a minimum set of points covering its edges and $A = V(G) - B$. Let G_1 denote the graph obtained from G by deleting all edges spanned by B.

Move each point of A to a point in general position on the flat spanned by its neighbors and then contract it. By proposition 4.1, this results in a graph G' whose underlying matroid is the restriction of the dual matching matroid of G_1 to B. Moreover, we know by propositions 4.2 and 4.3 that

$$\tau(G') = \tau(G) - |A| = \delta(G),$$

and by theorem 4.4 that $|E(G')| \leq \binom{\delta(G)+1}{2}$.

Let e be any edge of G_1. Eliminate the points of A from G-e similarly as above. Then we obtain a graph G" whose underlying matroid is the restriction of the dual matching matroid of G_1-e to B. Moreover, propositions 4.2 and 4.3 imply that

(1) $\tau(G") = \tau(G-e) - |A| < \tau(G')$.

Let B' be the set of non-isolated points of G'. By the above, $|B'| \leq \delta(G)(\delta(G) + 1)$. (1) implies that the restriction of the underlying matroids of G' and G" to B' cannot be the same. But these are the same as the restrictions of the dual matching matroids of G_1 and G_1-e, respectively, to B'. Thus G_1 has the property that deleting any edge of it, the restriction of its dual matching matroid to B' changes. By Theorem 4.1, this implies that all but $2^{4\delta^2+4\delta}$ points of G_1 have degree ≤ 2. Consequently, at most $2^{4\delta^2+4\delta}+|B'| < 2^{5\delta^2}$ points have degree ≥ 3 in G .

□□□

4.5. The covering problem for hypergraphs and flats

The upper bound on the number of edges in τ-critical graphs given by Erdős, Hajnal and Moon was generalized by Bollobás to hypergraphs; we formulate it slightly differently as follows [4]:

<u>Theorem 4.7.</u> If every $\binom{r+t}{t}$ edges of an r-uniform hypergraph H can be covered by t points then all edges of H can be covered by t points.

In fact, Bollobás proved a more general result. The following result which turned out equivalent to Bollobás's result, was conjectured by Ehrenfeucht and Mycielski and proved by Katona [12] and Jaeger - Payan [11] :

<u>Theorem 4.7'</u> Let $A_1,\dots,A_m,B_1,\dots,B_m$ be sets such that $|A_i| = r$, $|B_i| = t$, and $A_i \cap B_j = \emptyset$ iff i=j. Then $m \leq \binom{r+t}{t}$.

To obtain theorem 4.7 from this result, we may assume that deleting any edge of H the rest can be covered by t points, but H itself cannot. Let $A_1,\dots,A_m$ be the edges of H, and let B_i be a t-set covering all edges of $H-A_i$. Then theorem 4.7' implies that $m \leq \binom{r+t}{t}$, which contradicts the assumption of theorem 4.7.

Using tensor calculus, we shall prove two generalizations of theorem 4.7':

<u>Theorem 4.8.</u> Let $A_1,\dots,A_m$ be flats of rank r in a matroid representable over a commutative field, and let $B_1,\dots,B_m$ be t-element sets of elements of this matroid. Assume that $A_i \cap B_j = \emptyset$ iff i = j. Then $m \leq \binom{r+t}{t}$.

<u>Theorem 4.9.</u> Let $A_1,\dots,A_m$ be r-flats and $B_1,\dots,B_m$ t-flats in a projective space over a commutative field. Assume that $A_i \cap B_j = \emptyset$ iff i=j. Then $m \leq \binom{r+t}{t}$.

By the same argument as above, these results yield the following theorems.

<u>Theorem 4.10.</u> Let H be an r-uniform hypergraph and assume that there is a matroid, representable over a commutative field, defined on its points. Assume that every $\binom{r+t}{r}$ edges of H can be covered by a set of rank t. Then all edges of H can be covered by a set of rank t.

<u>Theorem 4.11.</u> Let $\mathcal{F}$ be a collection of r-flats in a matroid representable over a commutative field. Assume that every $\binom{r+t}{r}$ members of $\mathcal{F}$ can be rep-

resented by t points. Then all members of $\mathcal{F}$ can be represented by t points.

<u>Theorem 4.12.</u> Let $\mathcal{F}$ be a collection of r-flats in a projective space over a commutative field. Assume that for any $\binom{r+t}{t}$ members of $\mathcal{F}$ there exists a t-flat meeting all of them. Then there exists a t-flat meeting all members of $\mathcal{F}$.

The sharpness of these theorems is shown by the set of all r-sets formed, or all r-flats spanned, by r+t independent points.

<u>Proof of Theorem 4.8:</u> Let (S,r) be the given matroid, and consider the t^{th} symmetric power $(S^{(t)},r^{(t)})$. The sets $B_1,\dots,B_m$ are elements of $(S^{(t)},r^{(t)})$. We claim they are independent. Let $1 \leq i \leq m$. The t-subsets of S which meet A_i form a flat F_i in $(S^{(t)},r^{(t)})$. By the assumption, F_i does not contain B_i but it contains all the other B_j's as elements. This proves that $B_1,\dots,B_m$ are independent. Therefore, their number is at most the rank of $(S^{(t)},r^{(t)})$.

□□□

<u>Proof of Theorem 4.9:</u> We may assume that the rank of the whole space is r+t, since otherwise we can project the space from a point in general position on a hyperplane, without changing the assumptions, and repeat this until the rank goes down to r+t. Let $\hat{A}_1,\dots,\hat{A}_m$ and $\hat{B}_1,\dots,\hat{B}_m$ be r- and t-vectors belonging to the flats $A_1,\dots,A_m$, $B_1,\dots,B_m$, respectively. We claim that $\hat{A}_1,\dots,\hat{A}_m$ are linearly independent. Assume that

$$\sum_{i=1}^{m} \lambda_i \hat{A}_i = 0 .$$

Take the exterior product of this equation with $\hat{B}_j$, then it follows that $\lambda_j(A_j \wedge B_j) = 0$, whence $\lambda_j = 0$ (since $A_i \wedge B_j = 0$ iff $i \neq j$). Thus $A_1,\dots,A_m$ are linearly independent. Hence their number is at most the dimension of the space of r-vectors.

□□□

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4.4. Proofs; the covering problem for geometric graphs

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Proof: Trivially $\tau(G') \leq \tau(G)$. On the other hand, let T be a set covering the edges of G'. If $x' \notin T$ then T covers all edges of G as well and hence $r(T) \geqq \tau(G)$. If $x' \in T$ but $x' \notin \overline{T-x'}$ then $r(T) = r(T-x')+1 = r(T-x'+x) \geqq \tau(G)$. Finally, if $x' \in \overline{T-x'}$ then, by the general position of x', all neighbors of x are contained in $\overline{T-x'}$ and therefore $\overline{T-x'}$ covers all edges of G. Thus $r(T) = r(\overline{T-x'}) \geqq \tau(G)$.

□□□

Similar arguments show:

Proposition 4.3. Assume that the point x of the graph G is contained in the flat spanned by its neighbors. Let G' arise from G by contracting x. Then $\tau(G') = \tau(G) - 1$.

An immediate consequence of these observations is

Proposition 4.4. If a graph G is τ-critical then performing either one of the operations in propositions 4.2 and 4.3, the resulting graph is τ-critical.

Considering now sufficient conditions under which edges can be eliminated, first we remark the following:

Proposition 4.5. Let (x,y) be an edge of the graph G and assume that y is contained in the span of the other neighbors of x. Then $\tau(G-(x,y)) = \tau(G)$.

The proof is immediate. For τ-critical graphs we obtain:

Proposition 4.6. The neighbors of any point of a τ-critical graph are independent in the underlying matroid.

(At this point, we can obtain a short proof of Theorem 4.3. Let G be a τ-critical ordinary graph, A a stable set in G, B the set of neighbors of A and $a \in A$. Eliminate the points of $A-a$ one by one, by moving each one to a point in general position on the flat spanned by its neighbors and then contracting it. It is clear that at each time, the remaining points of $A-a$ are isthmusses in the matroid, so the operations can be performed. By proposition 4.4, we obtain τ-critical graphs at each time. The rank of B decreases by 1 at each step, so at the end it becomes $|B|-|A|+1$. But the neighbors of $\bar{a}$ are independent in this final graph, and so their number is $|B|-|A|+1$.)