The volume of convex bodies in high dimension

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* denotes parts not discussed in the course due to lack of time.

1 Convex sets and bodies

Convex sets, convex hull. Convex bodies, polytopes, polyhedra, simplices, parallelepipeds. Unit balls of euclidean and other norms. Supporting hyperplanes and extreme points.

2 Volume

2.1 Definition and examples

Additivity, monotonicity, normalization. Uniqueness. Volume of simplices, parallelepipeds, balls, sections of balls.

2.2 Strange behavior in high dimension

Volume near the surface, volume of balls.

2.3 Simple properties

Homothetical and linear transformations.

3 Surface

3.1 Parallel bodies

Theorem 1 $\mathrm{vol}(K + rB)$ *is a polynomial in* r *with positive coefficients.*

Proof. True for a polytope. Approximation by polytope. □

Definition of surface: coefficient of r.

3.2 The space of convex bodies

Hausdorff distance of two sets $S, T \subseteq \mathbb{R}^n$:

$$d^H(S,T) = \inf\{t : \ S \subseteq T + tB, \ T \subseteq S + tB\}.$$

Theorem 2 *(Blaschke Compactness Theorem) Let* $\mathcal{K}$ *be the set of closed compact subsets of a convex body* $K_0 \subseteq \mathbb{R}^n$*. Then the metric space* $(\mathcal{K}, d^H)$ *is compact.*

Volume and surface are continuous and monotone functions on $(\mathcal{K}, d^H)$.

3.3 Symmetrization

Volume is preserved. Surface and moment of inertia are decreased.

3.4 Isoperimetry

Theorem 3 *(Isoperimetric inequality) Among all convex bodies with given volume, the ball minimizes the surface.*

Proof. Existence of extremal body by compactness. Conclusion by symmetrization. □

4 The Brunn-Minkowski Inequality

Definition: Minkowski sum.

Theorem 4 *Let $K \subseteq \mathbb{R}^n$ be a convex body and let H_t denote the hyperplane defined by $x_1 = t$. Let $[a, b]$ denote the interval where $H_t \cap K \neq 0$. Define*

$$f(t) = \mathrm{vol}_{n-1}(H_t \cap K).$$

Then $f(t)^{1/(n-1)}$ is a concave function on the interval $[a, b]$.

Proof. By symmetrization. □

Corollary 5 *(The Brunn-Minkowski Inequality) Let $K, K' \subseteq \mathbb{R}^n$ be convex bodies. Then the function $\mathrm{vol}(tK + (1-t)K')^{1/n}$ is a concave function of t for $0 \leq t \leq 1$.*

4.1 *Mixed volumes

4.2 Localization Lemma

4.2.1 Statement

The following lemma was proved in [9, 15]

Lemma 6 (Localization Lemma) *Let g and h be upper semi-continuous integrable functions on $\mathbb{R}^n$ such that*

$$\int\limits_{\mathbb{R}^n} g(x)\,dx > 0 \qquad textand \qquad \int\limits_{\mathbb{R}^n} h(x)\,dx > 0.$$

Then there exist two points $a, b \in \mathbb{R}^n$ and a linear function $\ell : [0,1] \to \mathbb{R}_+$ such that

$$\int\limits_0^1 \ell(t)^{n-1} g((1-t)a + tb)\,dt > 0 \qquad \text{and} \qquad \int\limits_0^1 \ell(t)^{n-1} g((1-t)a + tb)\,dt > 0.$$

4.2.2 Applications I: Isoperimetry in convex bodies

Theorem 7 *Let K be a convex body in $\mathbb{R}^n$ with diameter d. Let $K = K_1 \cup K_2 \cup K_3$ be a partition of K into three measurable sets such that the distance of K_1 and K_3 is at least ε. Then*

$$\mathrm{vol}(K_3) \geq \frac{2\varepsilon}{d - \varepsilon} \min\{\mathrm{vol}(K_1), \mathrm{vol}(K_2)\}.$$

4.2.3 *Applications II

Brunn-Minkowski Inequality, isoperimetric inequality.

Corollary 8 *Let K be a convex body in $\mathbb{R}^n$, and let a be its center of gravity. Let H be a hyperplane through a. Then the volume of K on each side of H is at least a fraction of $1/e$ of the total volume.*

5 *Polarity

5.1 *Santalo Inequality

5.2 *The Mayer Conjecture and the Bourgain–Milman Inequality

6 How to specify a convex body?

6.1 Oracles

Membership oracle: a black box which works as follows: if we plug in a vector $x \in \mathbb{Q}^n$, it returns "YES" or "NO". Its answers must be consistent with the interpretation that "YES" means $x \in K$ for some convex body K.

Separation oracle: we plug in a vector $x \in \mathbb{Q}^n$, it returns "YES" or "NO", and in this latter case, it returns a hyperplane with rational coefficients separating x from K.

*Weak separation and membership oracles.

6.2 Guarantees

Inscribed and circumscribed balls must be given.

7 The Ellipsoid Method

7.1 Basic version

Application to optimization.

7.2 *Shallow Cut version

Application to Löwner–John ellipsoids.

8 Negative results

8.1 Bad approximability in the oracle model

Lemma 9 *Let S be a set of k points in the unit ball $B \subseteq \mathbb{R}^n$. Then*

$$\mathrm{vol}(\mathrm{cone}(S)) \leq \frac{k}{2^n}\mathrm{vol}(B).$$

Corollary 10 (Elekes) *For any polynomial time algorithm which assigns to every convex body K (given by a separation oracle) an upper bound $w(K)$ on $\mathrm{vol}(K)$, there exists a body K in every dimension n for which $w(K) > 2^n\mathrm{vol}(K)$.*

The lower bound on the error can even be improved:

Theorem 11 (Bárány–Füredi) *For any polynomial time algorithm which assigns to every convex body K (given by a separation oracle) an upper bound $w(K)$ on $\mathrm{vol}(K)$, there exists a body K in every dimension n for which $w(K) > n^{n/10}\mathrm{vol}(K)$.*

Note: There is a polynomial time algorithm which assigns to every convex body K (given by a separation oracle) a number $w(K)$ such that

$$\mathrm{vol}(K) \leq w(K) \leq n^{3n/2}\mathrm{vol}(K).$$

(Cf. Homework or Löwner–John ellipsoids.)

8.2 *NP-hardness for polytopes

The results of Khachiyan.

8.3 *Lower bounds for randomized algorithms

The results of Vempala.

9 Randomized volume algorithms

9.1 Preliminary sketch

The Dyer–Frieze–Kannan algorithm [6] (mentioned).

Ball walk, the role of stepsize, connection with isoperimetry in convex bodies. The O^* notation.

9.2 General Markov chains and mixing times

Let $(\Omega, \mathcal{A})$ be a σ-algebra. For every $u \in \Omega$, let P_u be a probability measure on Ω, and assume that for every $A \in \mathcal{A}$, the value $P_u(A)$ is measurable as a function of u. We call the triple $\mathcal{M} = (\Omega, \mathcal{A}, \{P_u : u \in \Omega\})$ a *Markov chain*. A Markov chain, together with an *initial distribution* σ on Ω, defines a *random walk*, i.e. a sequence of random variables $w^0, w^1, w^2, \dots$ with values from Ω such that w^0 is chosen from distribution σ and w^{i+1} is chosen from distribution P_{w_i} (independently from $w_0, \dots, w_{i-1}$). We denote by σ^k the distribution of w^k.

A probability measure π on $(\Omega, \mathcal{A})$ is a *stationary distribution* for the Markov chain if choosing w^0 from this distribution, w^1 will have the same distribution (then, of course, so does every w^i). This is equivalent to saying that for all $A \in \mathcal{A}$,

$$\int_\Omega P_u(A)\, d\pi(u) = \pi(A).$$

We assume that the Markov chain has a stationary distribution π, and we fix one.

A Markov chain is *time-reversible* if (roughly speaking) for any two sets $A, B \in \mathcal{A}$, it steps from A to B as often as from B to A. Formally, this means that

$$\int_B P_u(A)\, d\pi(u) = \int_A P_u(B)\, d\pi(u).$$

We call a Markov chain *lazy* if $P_u(u) \geq 1/2$ at each node.

We define the *ergodic flow* $\Phi : \mathcal{A} \to [0,1]$ of the Markov chain by

$$\Phi(A) = \int_A P_u(\Omega \setminus A)\, d\pi(u).$$

The *conductance* of the Markov chain is

$$\Phi = \inf_{0<\pi(A)<1} frac\Phi(A)\pi(A)\pi(\Omega \setminus A.$$

The main theorem we prove [17]:

Theorem 12 *Let $M = \sup_A \sigma(A)/\pi(A)$. Then for every $A \subseteq \Omega$,*

$$|\sigma^k(A) - \pi(A)| \leq \sqrt{M} \left(1 - \frac{1}{2}\Phi^2\right)^k.$$

9.3 Conductance and isoperimetry

Lemma 13 *Let K be a convex body in $\mathbb{R}^n$ with diameter d. Let $x, y \in K$ such that $|x - y| < 1/\sqrt{n}$. Suppose that* $\mathrm{vol}(B \cap K), \mathrm{vol}(B \cap K') \geq \frac{1}{2}\mathrm{vol}(B)$. *Then* $\mathrm{vol}((x + B) \cap (y + B) \cap K) \geq \frac{1}{5}\mathrm{vol}(B)$.

9.4 *Polynomial running time bound

Preconditioning, weak dependence, rounding.

9.5 The fastest known algorithm (sketch)

9.5.1 Logconcave functions and distributions

Definition: A function $f : \mathbb{R}^n \to \mathbb{R}_+$ is *logconcave*, if $f(\alpha x + (1 - \alpha)y) \geq f(x)^\alpha f(y)^{1-\alpha}$ for all $x, y \in \mathbb{R}^n$ and $0 \leq \alpha \leq 1$.

Examples: x^n, e^x, $e^{-|x|}$, e^{-x^2}, $mathbf1_K$ for a convex body. If $f : \mathbb{R}^n \to \mathbb{R}_+$ and $g : \mathbb{R}^n \to \mathbb{R}_+$ are logconcave functions, then so are fg and $\min(f, g)$.

Theorem 14 (Leindler–Prékopa inequality) *Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a logconcave function and $1 \leq k \leq n$. Then*

$$g(x_1, \ldots, x_k) = \int_{\mathbb{R}^{n-k}} f(x_1, \ldots, x_n)\, dx_{k+1} \ldots dx_n$$

is logconcave.

Corollary 15 *If $f : \mathbb{R}^n \to \mathbb{R}_+$ and $g : \mathbb{R}^n \to \mathbb{R}_+$ are logconcave functions, then so is their convolution*

$$(f \circ g)(x) = \int_{\mathbb{R}^n} f(y)g(x - y)\, dy.$$

Corollary 16 *Let K and L be convex bodies in $\mathbb{R}^n$. Then $f(x) = \mathrm{vol}(K \cap (x + L))$ is logconcave.*

Corollary 17 *The local conductance ℓ is logconcave.*

Suppose that $\int_{\mathbb{R}^n} f\,dx$ is finite, and define a probability measure by

$$\pi_f(A) = \int_A f\,dx \Big/ \int_{\mathbb{R}^n} f\,dx$$

Let $D(f) = \mathsf{E}(|X - Y|)$, where X and Y are two independent random points from the distribution π_f. Then we have the following isoperimetric inequality for logconcave distributions.

Theorem 18 *Let f be a logconcave density function on $\mathbb{R}^n$ with support K. For any partition of $\mathbb{R}^n$ into three measurable sets S_1, S_2, S_3,*

$$\pi_f(S_3) \geq \mathrm{const}\frac{d(S_1, S_2)}{D(f)}\pi_f(S_1)\pi_f(S_2).$$

9.5.2 Product estimator for integration (simulated annealing)

Let $\mathcal{M} = (\Omega, \mathcal{A}, \pi\})$ be a probability space and let $f : \Omega \to \mathbb{R}_+$ be an integrable function, with $c_1 = \int f$ and $c_2 = \int f^2$. We give a general randomized algorithm to compute $I = \int f\,d\pi$.

A natural approach is to generate independent points $w_i \in \Omega$ $(1 \leq i \leq N)$ from π, and then use $(1/N)\sum_{i=1}^{N} f(w_i)$ as an approximation of the integral. But by elementary statistics N has to be at least c_2/c_1^2, which may be very large in quite natural problems. We are going to show that it can be reduced to (roughly) the logarithm of this number [16, 18].

The idea is to write, for an appropriate $m \geq 1$,

$$\int f\,d\pi = \prod_{j=1}^{m} \int_\Omega f^{\frac{j}{m}}\,d\pi \Big/ \int_\Omega f^{\frac{j-1}{m}}\,d\pi = \prod_{j=1}^{m} \int f^{\frac{1}{m}}\,d\pi_{j-1},$$

where π_j is the probability measure defined by

$$\pi_j(A) = \int_A f^{\frac{j}{m}}\,d\pi \Big/ \int_\Omega f^{\frac{j-1}{m}}\,d\pi$$

Here $f^{\frac{1}{m}}$ is much smoother than f. Assuming we have a Markov chain sampler for π, sampling from the distributions π_j can be done using the Metropolis algorithm. There is no general mixing time bound, however.

In more concrete geometric situations one can make use of the additional structure to design better sampling algorithms.

9.5.3 Hit-and-run algorithm

The *hit-and-run walk* in a convex body $K \subseteq \mathbb{R}^n$ is defined as follows. If the current point is v, we generate the next by selecting a random line ℓ through v (uniformly over all directions), and choose the next point of the chain uniformly from the segment $K \cap \ell$. Hit-and-run was introduced by R.L. Smith [24].

Let f be a logconcave distribution in $\mathbb{R}^n$. For any line ℓ in $\mathbb{R}^n$, let $\mu_{\ell,f}$ be the measure induced by f on ℓ, i.e.

$$\mu_{\ell,f}(S) = \int_{p+tu \in S} f(p+tu)dt,$$

where p is any point on ℓ and u is a unit vector parallel to ℓ. The probability measure $P_\ell(S) = \mu_\ell(S)/\mu_\ell(\ell)$ is the *distribution induced by f on ℓ*.

We use the following generalization of the *hit-and-run* walk in a convex body.

- Pick a uniformly distributed random line ℓ through the current point.
- Move to a random point y along the line ℓ chosen from the distribution P_ℓ.

The first step is easy to implement. For example, we can generate n independent random numbers $U_1, \ldots, U_n$ from the standard normal distribution, and use the vector $(U_1, \ldots, U_n)$ to determine the direction of the line.

For the second step, use binary search to find the point p on ℓ where the function is maximal, and the points a and b on both sides of p on ℓ where the value of the function is $\varepsilon f(p)$. We allow a relative error of ε, so the number of oracle calls is only $O(\log(1/\varepsilon))$.

Then select a uniformly distributed random point y on the segment $[a, b]$, and independently a uniformly distributed random real number in the interval $[0, 1]$. Accept y if $f(y) > rf(p)$; else, reject y and repeat.

The distribution of the point generated this way is closer to the desired distribution than ε in total variation distance, and the expected number of function evluations needed is $O(\log(1/\varepsilon))$.

9.5.4 The pencil construction

The following was proved in [18].

Theorem 19 *The volume of a convex body in $\mathbb{R}^n$, given by a separation oracle, can be computed in $O^*(n^4)$ time.*

Proof. Let K be the given body in $\mathbb{R}^n$ and $\varepsilon > 0$. Let C denote the cone in $\mathbb{R}^{n+1}$ defined by

$$C = \{x \in \mathbb{R}^{n+1} \ : \ x_0 \geq 0, \sum_{i=1}^{n} x_i^2 \leq x_0^2/4\}$$

where $x = (x_0, x_1 \ldots, x_n)$. We define a new convex body $K' \in \mathbb{R}^{n+1}$ as follows (Figure 1):

$$K' = \Big([0, 2D] \times K\Big) \cap C.$$

For each real number $a > 0$, let

$$Z(a) = \int_{K'} e^{-ax_0}\, dx$$

Then $Z(0) = \mathrm{vol}(K')$ and for $a \geq 2n$ the value of $Z(a)$ is essentially the same as the integral over the whole cone:

$$Z(a) \approx \int_C e^{-ax_0}\, dx = n!\pi_n a^{-(n+1)}.$$

For $m = O^*(n)$ and

$$a_i = 2n(1 - \frac{1}{\sqrt{n}})^i \quad \text{for } i = 1, \ldots, m,$$

we write

$$Z(0) \approx Z(a_m) = Z(a_0) \prod_{i=0}^{T-1} \frac{Z(a_{i+1})}{Z(a_i)}.$$

Here we estimate the ratios $Z(a_{i+1})/Z(a_i)$ by the method os Section 9.5.2, using hit-and-run to sample from the probability distributions over K' with density proportional to $e^{-a_i x_0}$. $\square$

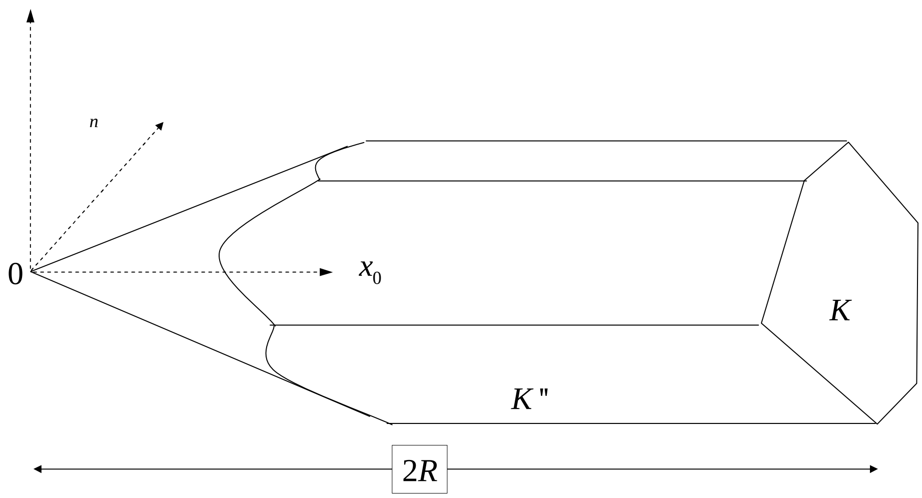

Figure 1: The pencil construction.

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