ACCUMULATION POINTS OF GRAPHS OF BAIRE-1 AND BAIRE-2 FUNCTIONS

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ABSTRACT. During the last few decades E. S. Thomas, S. J. Agronsky, J. G. Ceder, and T. L. Pearson gave an equivalent definition of the real Baire class 1 functions by characterizing their graph. In this paper, using their results, we consider the following problem: let T be a given subset of $[0,1]\times\mathbb{R}$. When can we find a function $f:[0,1]\to\mathbb{R}$ such that the accumulation points of its graph are exactly the points of T? We show that if such a function exists, we can choose it to be a Baire-2 function. We characterize the accumulation sets of bounded and not necessarily bounded functions separately. We also examine the similar question in the case of Baire-1 functions.

1. Introduction

In the last sixty years, certain classes of real functions have been characterized with a description of their graphs. In the case of Baire-1 functions it is worth mentioning the article of E. S. Thomas and the article of Agronsky, Ceder, and Pearson (see [1] and [2]): in the former one an equivalent definition of bounded Baire-1 functions was given, in the latter this result was generalized for the not necessarily bounded case. In this paper we also investigate a property of graphs of Baire-1 and Baire-2 functions. The problem is the following: if T is a given subset of $[0,1]\times\mathbb{R}$, when does there exist a Baire-1 or Baire-2 function $f:[0,1]\to\mathbb{R}$ such that the accumulation points of its graph are exactly the points of T?

We answer these questions in two steps in both cases. It is easier to understand the theorems and the proofs if we also require f to be bounded, thus we start with this case.

2. Notation

Throughout this paper we use the following notation: the graph of the real function f is denoted by G. Analogously, the graph of f_0 is G_0. If f is a real function, the set of accumulation points of G is L_f. The vertical line given by the equation $x=r$ is denoted by v_r. If H is a set of $\mathbb{R}^2$, and r is a real number, the intersection of v_r and H is denoted by $H(r)$. For simplicity, if $(r, y) \in H$, we say that $y \in H_r$. The open ball with center r and radius ε is $B(r, \varepsilon)$. We use this notation for one-dimensional neighborhoods in $\mathbb{R}$, and also for two-dimensional neighborhoods in $\mathbb{R}^2$. We clarify this ambiguity by making clear if the center is a point of $\mathbb{R}$ or of $\mathbb{R}^2$. The interval $[0, 1]$ is denoted by I. The cardinality of a set H is $\#(H)$. The diameter of a set H is $\text{diam}(H)$. Finally, if a set $A \subseteq I$ is the subset of the domain of f, and $a \in A$, sometimes we refer to the point $(a, f(a))$ as a point of G above A.

3. Preliminary Results

In the introduction we have already mentioned the result of Agronsky, Ceder, and Pearson. This theorem will be a very useful tool for us, so it is appropriate to recall it. We need the following definition:

Definition 3.1. *An open set $S \subseteq \mathbb{R}^2$ is an open strip if for every $r \in R$ the set $S(r)$ is an open interval.*

In [2, Theorem 2.2] a characterization of Baire-1 functions was given by using this definition:

Proposition 3.1. *Let $f : I \to \mathbb{R}$ be a function. It is Baire-1 if and only if there is a sequence (S_n) of open strips such that $\cap_{n=1}^{\infty} S_n = G$.*

As we will see, this theorem is a truly useful tool if our goal is to show that a certain function is Baire-1. Besides that we will also apply the following lemma, which handles a variant of our original problem.

Lemma 3.1. *For a given closed set $T \subseteq I \times \mathbb{R}$, there exists a countable set $A \subseteq I$ such that there is a function $f : A \to \mathbb{R}$ satisfying $L_f = T$.*

Proof. Let $T_i = (I \times [-i, i]) \cap T$ for all $i \in \mathbb{N}$. Then every T_i is compact. Let us consider an open ball of radius one around each point of T_1. These open balls cover T_1, hence it is possible to choose a finite covering. Let us take a point in each chosen open ball such that the x coordinates of these points are pairwise different. Let us denote the set of these points by H_1, and the set of their x coordinates by A_1.

Now, similarly, let us consider open balls with radius $\frac{1}{2}$ around each point of T_2 and choose a finite covering, then finally take points in these chosen neighborhoods and define H_2 and A_2 analogously. We can continue this procedure by induction: in the n^{th} step we consider the $\frac{1}{n}$-neighborhoods of the points of T_n, and we define the finite sets H_n and A_n using these open balls.

Let $A = \cup_{n=1}^{\infty} A_n$ and $H = \cup_{n=1}^{\infty} H_n$. These are countable sets. Let f be the function that assigns to every $x \in A$ the y coordinate of the chosen point above x. Then this point of the graph is clearly a point of H. We would like to prove that $L_f = T$ for this function f. We do this by verifying two containments.

(1) $T \subseteq L_f$. Let us consider any point P of T. By definition, $P \in T_k$ for a suitable k positive integer. Thus for every n larger than k there exists a point $x_n \in A_n$ such that the distance of $(x_n, f(x_n))$ and P does not exceed $\frac{1}{n}$. Therefore, there exists a sequence of distinct points in G that converges to P, hence $T \subseteq L_f$.

(2) $L_f \subseteq T$. Let us consider any point P of L_f. Since it is an accumulation point of G, there exists a sequence (p_n) in G converging to P and containing each of its terms only once. Now if k is given, for sufficiently large n the point p_n is in H_m with $m \geq k$. It means that the distance of p_n and T does not exceed $\frac{1}{k}$. Thus there are points of T arbitrarily close to the sequence (p_n). Therefore, the limit of (p_n) is in T, since T is closed. Hence $P \in T$ and $L_f \subseteq T$.

□

Remark 3.1. *The above proof shows that there are only finitely many points of the graph G that are more than ε apart from T for a given $\varepsilon > 0$. Later we will use this slightly stronger result.*

4. Functions of Baire Class 2

As we have promised, we consider the bounded case first. It is obvious that if $L_f = T$, then T must be a compact set, being bounded and closed. There is another condition needed: $T(x)$ is never empty for $x \in I$. Indeed, if (x_n) is a sequence that converges to x, $(x_n \neq x)$, the sequence formed by the points $(x_n, f(x_n))$ is a bounded sequence in $\mathbb{R}^2$, and its limit is in T, thus $T(x) \neq \emptyset$.

We point out that until this point we have not used the Baire-2 property of the function f. Despite that, as we will see, these conditions are also sufficient:

Theorem 4.1. *Suppose $T \subseteq I \times \mathbb{R}$. There exists a bounded Baire-2 function $f : I \to \mathbb{R}$ such that $L_f = T$ if and only if*

- *T is compact,*
- *$T(x)$ is nonempty for $x \in I$.*

Proof. Before beginning the formal proof, we give a short sketch. First, we construct a function f_0 such that $f_0(x) \in T(x)$ for every $x \in I$. After this step, we apply Proposition 3.1 to prove that f_0 is a Baire-1 function. Finally, we use Lemma 3.1 to modify f_0 on a countable set A to obtain a bounded Baire-2 function f such that $L_f = T$.

Put $f_0(x) = \max(T(x))$ for every $x \in I$. Since $T(x)$ is nonempty, this definition makes sense. The function f_0 is Baire-1; this is a well-known fact since f_0 is upper semicontinuous and every upper semicontinuous function is Baire-1. Nevertheless, it is useful to find a direct proof which uses Proposition 3.1 to understand better how this theorem works.

We define a nested sequence of open strips, (S_n). First, we construct a subset S'_n of S_n, that is the union of certain neighborhoods of points of G_0. Let the radius of such an open ball be $\varepsilon_{x,n}$, where $\varepsilon_{x,n}$ satisfies the following three conditions: $\varepsilon_{x,n} \leq \frac{1}{n}$ and $\varepsilon_{x,n} \leq \varepsilon_{x,n-1}$ for every $n \geq 2$. It is obviously possible. Moreover, we have a bit more complicated so-called overlapping condition related to the projection of the open balls $B((x, f_0(x)), \varepsilon_{x,n})$ to the x-axis. Specifically:

$$\forall x \in I, \forall n \in \mathbb{N}, \forall r \in \mathbb{R}, r \in B(x, \varepsilon_{x,n}) \text{ we have } f_0(r) - f_0(x) < \frac{1}{n} .$$

Such $\varepsilon_{x,n}$ can be chosen. If not, then there is a sequence (x_k) that converges to x and $f_0(x_k) \geq f_0(x) + \frac{1}{n}$ for every k. In this case $(f_0(x_k))$ is a bounded sequence, so it has a convergent subsequence. As a consequence, the sequence $(x_k, f_0(x_k))$ has a limit point in the plane whose first coordinate is x, and whose second coordinate is larger than $f_0(x) = \max(T(x))$ by at least $\frac{1}{n}$. Since T is closed, it is a contradiction.
Thus for every $n \in \mathbb{N}$ and $x \in I$, we can choose some $\varepsilon_{x,n}$ satisfying all three of our conditions. By taking the union of the neighborhoods $B((x, f_0(x)), \varepsilon_{x,n})$, we obtain an open set S'_n containing G_0 for every n. Also $S'_n \subseteq S'_{n-1}$ for every $n \geq 2$, since S'_n is the union of open balls with the same centers and smaller radii. However, it is not sufficient for us: our aim is to construct open strips. But this problem can be solved easily. Specifically, there is a simple way to extend an arbitrary open set H' to an open strip H: for every x, let $H(x) = (\inf(H'(x)), \sup(H'(x))$. Figure 1 demonstrates such an extension, in a case

where H' is the union of a few open disks: H is the open set bounded by the dashed lines. It is plain to see that the set H made this way is an open strip which contains H'. We also use this method to construct $S_n(x)$ by extending $S'_n(x)$. The property $S_n \subseteq S_{n-1}$ is obviously preserved during the extension.

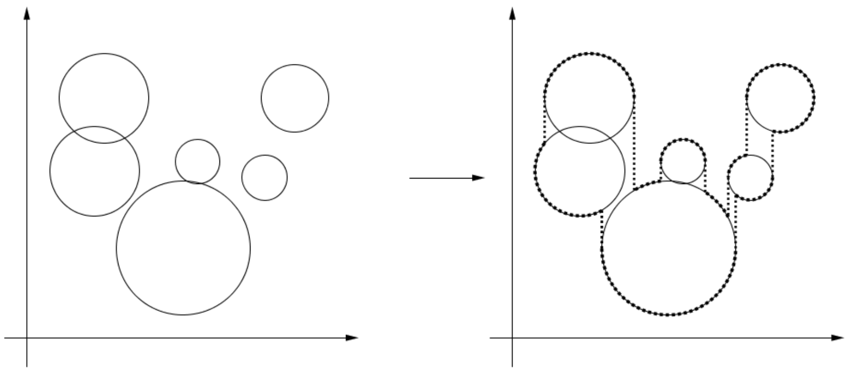

FIGURE 1. Extending an open set into an open strip

To apply Proposition 3.1, we have to verify that $S = \cap_{n=1}^{\infty} S_n = G_0$. It is clear that S contains G_0 since S'_n contains every point of G_0 for all n. We have to show that S has no other points. Proceeding towards a contradiction, let us assume that there exists a point $x \in I$ and $y \neq f_0(x)$ such that $(x, y) \in S$. We distinguish two cases.

(1) The case $y > f_0(x)$. Since $(x, y) \in S_n$ for every n, the set S'_n has a point (x, z_n) above (x, y). The sequence (z_n) is obviously bounded, hence it has a limit point $z \geq y$. But S'_n is formed by open balls whose centers are the points of $G_0 \subseteq T$ and whose radii are not larger than $\frac{1}{n}$. Thus $(x, z) \in T$ as T is closed. So T has a point whose first coordinate is x and whose second coordinate is larger than $f_0(x) = \max(T(x))$, a contradiction.

(2) The case $y < f_0(x)$. By a similar argument to the previous one, we might notice that S'_n has a point (x, z_n) below (x, y) for every n. Let $k \in \mathbb{N}$ satisfy $y < f_0(x) - \frac{1}{k}$. Then if $n \geq 2k$, amongst the open balls forming S'_n we might find a ball that intersects v_x and for its center $(x_n, f_0(x_n))$ the inequality $f_0(x_n) < f_0(x) - \frac{1}{2k}$ holds. But by definition, it is impossible: this neighborhood must satisfy the overlapping condition, thus it cannot intersect v_x, a contradiction. Hence f_0 is a function of Baire class 1.

Using Lemma 3.1, we modify f_0 on a countable set A, so that the accumulation set of the new points above A is T. We denote this altered function by f. Then it is a bounded Baire-2 function. Nevertheless, if we consider now the whole graph, $L_f = T$ remains true, since every point of the graph above $I \setminus A$ is in T. Therefore other accumulation points cannot occur. $\square$

In the following, we turn our attention to the not necessarily bounded Baire-2 functions. In this case the conditions are more complicated and the proof is a bit more difficult. However, we give a similar characterization.

We approach the problem by finding out some necessary conditions. During that process, we use only that $f : I \to \mathbb{R}$, as we did earlier in our previous theorem. It is easy to see that T must be closed in this case, too. But it is not true at all that $L_f(x) = T(x)$ must be nonempty for every $x \in I$. For instance, let f be the function that vanishes in 0, and elsewhere its value is $\frac{1}{x}$. Then $L_f(0)$ is empty. Nevertheless, we may suspect that $T(x)$ cannot be empty in any set C. Our lemma is the following:

Lemma 4.1. *If $f : I \to \mathbb{R}$ and $C = \{x \in I : L_f(x) = \emptyset\}$, then C is countable.*

Proof. Proceeding towards a contradiction, let us assume that C is uncountable. Put $C_n = \{x \in C : |f(x)| < n\}$ for every $n \in \mathbb{N}$. Then $C = \cup_{n=1}^{\infty} C_n$, and there exists an uncountable C_n. As a consequence, it contains one of its limit points, c. Thus there exists a sequence (c_i) in C_n $(c_i \neq c)$ that converges to c. Since $(f(c_i))$ is bounded, it has a convergent subsequence, therefore $L_f(c)$ cannot be empty, a contradiction. $\square$

We state that these necessary conditions are also sufficient, namely:

Theorem 4.2. *Suppose $T \subseteq I \times \mathbb{R}$. There is a Baire-2 function $f : I \to \mathbb{R}$ such that $L_f = T$ if and only if*

- *T is closed,*
- *there is a countable $C \subseteq I$ such that $T(x)$ is nonempty for $x \in I \setminus C$.*

Proof. The concept of the proof is similar to our proof given for the bounded case. We begin by the construction of a function f_0 and then we prove that it is a Baire-1 function. The desired function f will be obtained by modifying f_0 on a countable set using Lemma 3.1.

We start by observing that C is a G_δ set. Suppose $c \in C$. Since T is closed, it has a $B_{c,n}$ neighborhood for every $n \in \mathbb{N}$ such that for all $x \in B_{c,n}$ distinct from c, the absolute value of every element of $T(x)$ is larger than n. Otherwise $T(c)$ would not be empty. Then for a given n, the set $B_n = \cup_{c\in C} B_{c,n}$ is an open set containing C. On the other hand, clearly $\cap_{n=1}^{\infty} B_n = C$. Hence the set C is G_δ, as we wanted to show.

Now, we begin the construction of our function. The easier part is its definition on C. We consider an enumeration of the countable set $C = \{c_1, c_2, ...\}$ and we let $f_0(c_n) = n$ for every n. However, the definition of f_0 in $I \setminus C$ cannot be as straightforward as it was in our previous proof. Namely, it is possible that $T(x)$ has no maximum. Therefore we have to be more careful.

For every $n \in \mathbb{N}$, let

$$U_n = \{x \in I : \exists r \in T(x), |r| \leq n\}. \tag{4.2}$$

As T is closed, it is easy to see that each U_n is closed, too. It is also obvious that $U_n \subseteq U_{n+1}$ and $\cup_{n=1}^{\infty} U_n = I \setminus C$. Thus, for every $x \in I \setminus C$ there is a smallest n_x such that $x \in U_{n_x}$. Using this property, we may define $f_0(x)$ as the largest element of $T(x)$, whose absolute value does not exceed n_x. We can do so since $T(x)$ is closed and it has such an element. The inequalities $n_x - 1 < |f_0(x)| \leq n_x$ are also true, as otherwise x would be the element of U_m for some $m < n_x$. (Or, if $n_x = 1$, then $0 = n_x - 1 \leq |f_0(x)| \leq n_x = 1$.)

Now, we have defined f_0 on I. We would like to use Proposition 3.1 to show that f_0 is Baire-1. In order to do this, we construct the open strip S_n for every n. First, we define the open set S'_n constisting of some balls $B((x, f_0(x)), \varepsilon_{x,n})$. We select $\varepsilon_{x,n}$ so that $\varepsilon_{x,n} \leq \frac{1}{n}$ and $\varepsilon_{x,n} \leq \varepsilon_{x,n-1}$ for every $n \geq 2$, as we did earlier. Nevertheless, as we defined f_0 differently in certain sets, our further conditions should be case-specific: we handle separately the case $x \in C$ and the case $x \in I \setminus C$.

(i) The case $x \in C$. It means that $x = c_k$ for some k. Let

$$E_n = \cup_{x\in C} B((x, f_0(x)), \varepsilon_{x,n}),$$

and F_n be its projection onto the x-axis, that is $F_n = \cup_{x\in C} B(x, \varepsilon_{x,n})$. Let us choose these neighborhoods such that $\cap_{n=1}^{\infty} F_n = C$. It is possible since C is a G_δ set. Furthermore, we also demand that $B(c_k, \varepsilon_{c_k,n})$ does not contain the points $c_1, ..., c_n$, with the exception of c_k. We remark that these conditions imply $\cap_{n=1}^{\infty} E_n$ equals the graph of $f_0|C$.

(ii) The case $x \in I \setminus C$. Let us make some remarks concerning this complementary set. Let $V_1 = U_1$, and for $n \geq 2$, let $V_n = U_n \setminus U_{n-1}$. Then the set V_n is F_σ for every n, as the difference of closed sets. Consequently, there exist closed sets $V_{n,i}$ for every n and i such that $V_n = \cup_{i=1}^{\infty} V_{n,i}$. We can take an enumeration $W_1, W_2, ...$ of the sets $V_{n,i}$. Let $x \in V_k$. We can suppose that the $\varepsilon_{x,n}$ are chosen so that $B(x, \varepsilon_{x,n})$ does not contain the points $c_1, c_2, ..., c_n$. Furthermore, we can suppose that $B(x, \varepsilon_{x,n})$ does not intersect the sets $W_1, W_2, ..., W_n$, except for those which contain x. Finally, we have a special overlapping condition, namely that $f_0(r) - f_0(x) < \frac{1}{n}$ for every $r \in B(x, \varepsilon_{x,n}) \cap V_k$. One can prove that this condition can be satisfied as we proved it last time, in the bounded case. It is worth mentioning that if $f_0(x) < 0$, then $(x, -(k-1))$ cannot be a limit point of a sequence of points in G_0 above $I \setminus C$. Since T is closed, if such a sequence would exist, then $(x, -(k-1)) \in T$. But it means that $x \in U_{k-1}$, hence $x \notin V_k$.

Now the open set S'_n is defined for each n. As in the bounded case, our next step is making strips of these open sets: let $S_n(x) = (\inf(S'_n(x)), \sup(S'_n(x)))$ for every $x \in I$. Set $\cap_{n=1}^{\infty} S_n = S$ and similarly $\cap_{n=1}^{\infty} S'_n = S'$. We are going to show that $S = G_0$. Since $G_0 \subseteq S$ is obvious, we can focus on proving $S \subseteq G_0$, or equivalently, proving that S has no point outside of G_0. We examine the relation of these sets independently for every $x \in I$: our goal is $S(x) \subseteq G_0(x)$. We distinguish the same cases which we distinguished during the construction of $S'_n(x)$:

(1) The case $x \in C$, that is, $x = c_k$ for some k. Let us consider the set $S'_n(x)$. If $n \geq k$, amongst the open neighborhoods forming S'_n there can be only one that intersects v_x: the neighborhood of $(x, f_0(x))$. Thus for sufficiently large n the equality $S'_n(x) = S_n(x)$ holds, and $S'_n(x)$ contains only one open interval whose radius is $\frac{1}{n}$. Hence if n converges to infinity, we find that the only element of $S(x)$ is $f_0(x)$. Therefore $S(x) \subseteq G_0(x)$.

(2) The case $x \in I \setminus C$. It means $x \in V_k$ and $x \in W_m$ for some k and m. Let us consider $S'_n(x)$. We would like to find out for which r the open ball $B((r, f_0(r)), \varepsilon_{r,n})$ can intersect v_x. It is clear that for sufficiently large n a neighborhood around a $(c_i, f_0(c_i))$ cannot do so as the intersection of these open balls are exactly the

graph of $f_0|C$. Furthermore, if $n \geq m$, then the neighborhood chosen around $(r, f_0(r))$ can intersect v_x if and only if $r \in W_m$. Indeed, we have chosen these neighborhoods such that they do not intersect $W_1, W_2, ..., W_n$, unless those which are containing r. Thus if n is large enough, v_x can be intersected by a certain $B((r, f_0(r)), \varepsilon_{r,n})$ only if $r \in W_m$. Only these places are relevant if we want to find out what $S(x)$ is. But how did we define W_m? It is a subset of V_k thus the values of f_0 in W_m are between $k-1$ and k. It is important to us that f_0 is bounded here, and $f_0(x) = \max(T_k(x))$ for each element of W_m, where $T_k = (I \times [-k, k]) \cap T$, as in Lemma 3.1. Therefore, in the relevant places we defined f_0 as we would have done in Theorem 4.1, if we had regarded T_k instead of T. Consequently, in this case one can conclude the proof of $S(x) \subseteq G_0(x)$ as it was done there.

After these observations, the conclusion of the proof is clear. We use Lemma 3.1 as we did just before and alter the function on a countable set A, such that $L_f = T$ for the resulting function f. Then f is obviously a Baire-2 function. □

By proving this theorem we finished our characterization of accumulation points of Baire-2 functions. On the other hand, our proofs clarified that for any ordinal number α larger than 2 the Baire-α functions are not interesting concerning our question. Namely, the accumulation set of the graph of a Baire-α function is also the accumulation set of a Baire-2 function. This fact explains why we examine only the Baire-1 and Baire-2 functions.

5. Functions of Baire Class 1

First, we focus again on the bounded case. Since Baire-1 functions are also Baire-2 functions, the conditions we found earlier recur in this case: T should be compact and $T(x)$ should be nonempty, if $x \in I$. Nevertheless, it is clear, that these conditions are not sufficient. Namely, if $L_f = T$ and for a given x the set $T(x)$ has multiple elements, then f is discontinuous at x. But a Baire-1 function cannot have an arbitrary set of discontinuities: it must be a meager F_σ set. Thus if $D = \{x : \#(T(x)) > 1\}$, then D should be a meager F_σ set. As we will see, these conditions suffice. However, before the statement of the actual theorem, let us notice that if we require T to be closed, then it is redundant to require D to be F_σ. Indeed, let $D_n = \{x : \mathrm{diam}(T(x)) \geq \frac{1}{n}\}$ for each $n \in \mathbb{N}$. Then it is easy to see that these sets are closed and their union is D. (Moreover, each D_n

is nowhere dense, otherwise some of them would contain an interval, and D cannot do so.) Consequently, D is an F_σ set. Using this fact, our theorem is simply the following:

Theorem 5.1. *Suppose $T \subseteq I \times \mathbb{R}$. There is a bounded Baire-1 function $f : I \to \mathbb{R}$ such that $L_f = T$ if and only if*

- *T is compact,*
- *$T(x)$ is nonempty, if $x \in I$,*
- *the set $D = \{x : \#(T(x)) > 1\}$ is meager.*

Proof. Let us begin the proof by the construction of f. First, we use Lemma 3.1 to define f on a countable set A such that the accumulation set of the graph of $f|A$ coincides with T. We can suppose that A is disjoint from D. Indeed, in any neighborhood of any point $x \in I$ there are infinitely many points of $I \setminus D$, since D is meager. Thus we have defined f on A. On the other hand, on $I \setminus A$ let us define f as we did it in the bounded Baire-2 case: let $f(x) = \max(T(x))$. For this f, we have $L_f = T$, and obviously f is bounded.

We would like to apply Proposition 3.1 to f. We use the usual method: we define the open set S'_n for each n, which is the union of open balls around points of the graph with $\varepsilon_{x,n}$ radius, and then we extend these sets to open strips. The conditions concerning $\varepsilon_{x,n}$ will be case-specific, except for the usual size conditions.

(i) The case $x \in A = \{a_1, a_2, ...\}$. Then $x = a_k$ for some k. Our first condition on $\varepsilon_{x,n}$ is that $B(x, \varepsilon_{x,n})$ must not contain the points $a_1, a_2, ..., a_n$, except for a_k. The second condition is related to the overlapping of D. Since D is a meager F_σ set, we can choose $D_1, D_2, ...$ nowhere dense closed sets such that $D = \cup_{n=1}^{\infty} D_n$. Moreover, none of these sets contains x since $x \in A$ and the sets A and D are disjoint. Therefore, the condition "$B(x, \varepsilon_{x,n})$ and $\cup_{i=1}^{n} D_i$ are disjoint" can also be satisfied.

(ii) The case $x \in I \setminus A$. First, in order to stay away from the set A, the open ball $B(x, \varepsilon_{x,n})$ must not contain the points $a_1, a_2, ..., a_n$. The second condition is identical to the overlapping condition of the bounded Baire-2 case: if $r \in B(x, \varepsilon_{x,n}) \setminus A$, then $f(r) - f(x) < \frac{1}{n}$.

We have finished the construction of the open set S'_n, and now, we can extend it to obtain the open strip S_n by taking the infimum and the supremum along each v_x. Our goal is to prove that the intersection S of the sets S_n is G. Of course, the challenging part is the verification of $S \subseteq G$. Let us consider $S(x)$ for each x. We separate three cases by the location of x:

(1) The case $x \in A$, that is $x = a_k$. If $n \geq k$, then amongst the neighborhoods forming S'_n there can be only one that intersects v_x, namely, the open ball centered at $(x, f(x))$. Therefore, $S_n(x) = S'_n(x)$, and

$$S_n(x) = (f(x) - \varepsilon_{x,n}, f(x) + \varepsilon_{x,n}) \subseteq \left(f(x) - \frac{1}{n}, f(x) + \frac{1}{n}\right).$$

This fact immediately implies that the only element of $S(x)$ is $f(x)$.

(2) The case $x \in D$. It means that $x \in D_k$ for some k. Thus if $n \geq k$, the neighborhoods $B((a_k, f(a_k)), \varepsilon_{a_k,n})$ cannot intersect v_x. Therefore, if n is sufficiently large, if we want to describe $S_n(x)$, we have to deal only with the points in $I \setminus A$. But above $I \setminus A$ we defined f and the neighborhoods forming S'_n as we defined f_0 and S'_n in the proof of Theorem 4.1. Consequently, the proof given there for $S(x) = G_0(x)$ for any $x \in I$ works.

(3) The case $x \in I \setminus (A \cup D)$. Proceeding towards a contradiction, we assume that $S(x)$ has an element y distinct from $f(x)$. Then $S'_n(x)$ has a point z_n for each n such that $|f(x) - z_n| \geq |f(x) - y|$. By definition, the set G is bounded, thus it is obvious that there exists some $K \in \mathbb{R}$ such that for any n and x, the $S'_n(x)$ has no element larger than K. It implies that the sequence (z_n) is bounded. Therefore, it has a convergent subsequence whose limit is some $z \in \mathbb{R}$. For this limit z the inequality $|f(x) - z| \geq |f(x) - y|$ also holds, thus $f(x) \neq z$. Since there is a point of G whose distance from (x, z_n) does not exceed $\frac{1}{n}$, the point (x, z) is also an accumulation point of G, thus $(x, z) \in L_f$, a contradiction. Namely, for our f the equation $L_f = T$ holds, however, the only element of $T(x)$ is $f(x) \neq z \in L_f(x)$.

Therefore $S = G$, thus we can apply Proposition 3.1. Hence f is a bounded Baire-1 function, such that $L_f = T$. □

As we have characterized the bounded Baire-1 functions, now we might focus on the most challenging problem appearing in this paper: the characterization of the not necessarily bounded Baire-1 functions. However, as we will see, during the proof we will apply the same ideas. Following the usual scheme, we begin by thinking about necessary conditions concerning T.

The conditions we found during the examination of the general Baire-2 case obviously recur: T is a closed set and $T(x) = \emptyset$ can hold only on a countable subset of I. As T is closed, this subset is G_δ. Of course we need more than these simple conditions. We have to pay attention to the fact that a Baire-1 function cannot have an arbitrary set of discontinuities: it must be a meager F_σ set, and at points of continuity, $\#(L_f(x)) = 1$, thus $\#(T(x)) = 1$. However, we must be careful. In the bounded case, the property $\#(L_f(x)) = 1$ already guaranteed that f is continuous at x, or f has a removable discontinuity at x. But in this case, it is not true at all: for instance, if $f(x) = \frac{1}{2x-1}$ for $x > \frac{1}{2}$, and $f(x) = 0$ for $x \leq \frac{1}{2}$, then although $L_f\left(\frac{1}{2}\right) = 0$, it does not imply that f is continuous at $\frac{1}{2}$ or it has a removable discontinuity there. Therefore, we must pay attention to the infinite limits. If we embed T into $I \times \overline{\mathbb{R}}$ and take its closure $\overline{T}$, then we have to demand that this $\overline{T}$ can intersect the extended vertical lines in multiple points only above a meager F_σ set. However, the additional F_σ condition is unnecessary since we supposed that T is closed. Indeed, if $D_n = \{x : \operatorname{diam}(\overline{T}(x)) \geq \frac{1}{n}\}$, then these sets are nowhere dense closed sets and their union is D, hence D is F_σ.

If we collect all of these remarks, we gain a more complicated system of conditions than the ones in the previous cases. We show that it is sufficient.

Theorem 5.2. *Suppose $T \subseteq I \times \mathbb{R}$. There is a Baire-1 function $f : I \to \mathbb{R}$ such that $L_f = T$ if and only if*

- *T is closed,*
- *there is a countable $C \subseteq I$, such that $T(x)$ is nonempty for $x \in I \setminus C$,*
- *the set $D = \{x : \#(\overline{T}(x)) > 1\}$ is meager.*

Proof. We define f on a countable set A, such that the accumulation set of the graph of f restricted to A equals T. We do so using the method given in Lemma 3.1. It is easy to see that we can construct such a set A disjoint from C and D.

Now let us focus on $I \setminus A$. We define f on this set as we defined f_0 in the proof of Theorem 4.2. First, if $C = \{c_1, c_2, ...\}$, then $f(c_n) = n$ for each $n \in \mathbb{N}$. Besides that we also define U_n as we did it in (4.2). These are closed sets in this case, too, though not necessarily disjoint from A. At places which are not in A let us define f as we defined f_0 after (4.2): if $x \in U_n$, let $f_0(x)$ be the largest element of $T(x)$ which has absolute value not exceeding n. Now we are ready with the construction of f and $L_f = T$ clearly holds:

if we consider only the points of the graph above A, it is true by definition, furthermore, sequences containing infinitely many points of the graph above C cannot converge, and points of the graph above $I \setminus (A \cup C)$ are in T. Thus every accumulation point of G is also the accumulation point of the graph of $f|A$, and the set of these accumulation points is T. (We note that C might intersect D, a concern that we will address later.)

We would like to apply Proposition 3.1 to f by giving the open sets S'_n formed by neighborhoods of points of G and extending them to open strips. Again, we separate some cases. We also use our familiar notation: $A = \{a_1, a_2, ...\}$, $C = \{c_1, c_2, ...\}$, and $D = \cup_{n=1}^{\infty} D_n$, where D_n is a nowhere dense, closed set for each n.

(i) The case $x \in C$, $x = c_k$. Here, we define our neighborhoods with $\varepsilon_{x,n}$ radius quite comfortably, namely, we can define the sets E_n and F_n as we did it in (i) of the proof of Theorem 4.2 and repeat the conditions used there. Hence we can choose these open balls such that $\cap_{n=1}^{\infty} F_n = C$, and $B(c_k, \varepsilon_{c_k,n})$ does not contain the points $c_1, ..., c_n$, with the exception of c_k. We also require that this neighborhood is disjoint from $\{a_1, a_2, ..., a_n\}$. We remark that these conditions imply $\cap_{n=1}^{\infty} E_n$ equals the graph of $f_0|C$.

(ii) The case $x \in A$. We evoke the conditions of (i) of the proof of Theorem 5.1. Namely, $B(x, \varepsilon_{x,n})$ does not intersect the closed sets $D_1, D_2, ..., D_n$, and it does not contain $a_1, a_2, ..., a_n$, with the exception of x. Furthermore we give the following additional condition: these neighborhoods have to stay away from C, thus they must not contain $c_1, c_2, ..., c_n$.

(iii) The case $x \in I \setminus (A \cup C)$. We evoke the condition system of (ii) of the proof of Theorem 4.2. We define the sets V_n and W_n as we did there: $V_1 = U_1$, and $V_n = U_n \setminus U_{n-1}$ for $n \geq 2$. Then any set V_n is F_σ. Let $W_1, W_2, ...$ be an enumeration of the closed sets forming them. Now if $x \in V_k$, we require $B(x, \varepsilon_{x,n})$ to be disjoint from $c_1, c_2, ..., c_n$, and also disjoint from the sets $W_1, W_2, ..., W_n$, except for those containing x. Furthermore, of course, we give an overlapping condition: $f_0(r) - f_0(x) < \frac{1}{n}$ for each $r \in B(x, \varepsilon_{x,n}) \cap V_k$. These are exactly the conditions we used in (ii) of the proof of Theorem 4.2. The only additional condition is the following: $B(x, \varepsilon_{x,n})$ must not contain the points $a_1, a_2, ..., a_n$.

Thus we have constructed the open set S'_n for each n. We extend it in the usual way to form the open strip S_n. Our goal is to verify that their intersection S equals G. The challenging part is to show that S contains no points distinct from G. Let us consider $S(x)$ and $S'(x)$ for each x. We separate four cases by the location of x:

(1) The case $x \in C$, $x = c_k$. This is obvious: if $n \geq k$, the only chosen neighborhood that intersects v_x amongst the ones forming $S'_n(x)$ is the neighborhood of $(x, f(x))$, and thus $S'_n(x) = S_n(x)$. Therefore, $S_n(x)$ is an interval whose diameter does not exceed $\frac{2}{n}$ and contains $f(x)$. Thus the only element of $S(x)$ is $f(x)$, as we wanted to show.

(2) The case $x \in A$, $x = a_k$. We can simply repeat our previous argument: for sufficiently large n, there is only one chosen neighborhood that intersects v_x, and since the diameters of these neighborhoods converge to 0, the only element of $S(x)$ is $f(x)$.

(3) The case $x \in D \setminus C$. It means $x \in D_k$ for some $k \in \mathbb{N}$. Now, if $n \geq k$, the neighborhood $B((x', f(x')), \varepsilon_{x',n})$ for $x' \in A$ cannot intersect v_x. It is also true that for sufficiently large n, the neighborhood $B((x', f(x')), \varepsilon_{x',n})$ for $x' \in C$ cannot intersect v_x, since these neighborhoods are nested and their intersection is the graph of $f|C$. Hence it is enough to consider the graph of f above $I \setminus (A \cup C)$. At these places we defined f and the open balls forming S'_n as we defined f_0 and the open balls forming S'_n during the proof of Theorem 4.2. Consequently, case (2) of the proof of Theorem 4.2 can be used to prove $S(x) = G(x)$.

(4) The case $x \in I \setminus (A \cup C \cup D)$. Proceeding towards a contradiction, let us suppose that $S(x)$ contains some $y \in \mathbb{R}$, where $y \neq f(x)$. It means that for every n we can choose a point z_n in $S'_n(x)$, such that $|f(x) - z_n| \geq |f(x) - y|$. Since $z_n \in S'_n(x)$, the point (x, z_n) is in one of the open balls forming S'_n. Here, if n is sufficiently large, then this ball is centered at a point of the graph above $I \setminus C$. Indeed, if n is large enough, the neighborhoods around points of the graph above C cannot intersect v_x by definition. Now, the sequence (z_n) has a limit point z in $\overline{\mathbb{R}}$. Obviously, for this z the inequality $|f(x) - z| \geq |f(x) - y|$ also holds, thus $f(x) \neq z$. However, if n is sufficiently large, there is a point of the graph not above C whose distance from (x, z_n) does not exceed $\frac{1}{n}$. Consequently, there is a sequence (p_n) of points of the graph above $I \setminus C$ such that (p_n) converges to (x, z). Without loss of generality, we might assume that the elements of this sequence are all distinct. Since these points are not above C, they are above A or they are also elements of T. Nevertheless,

if n is sufficiently large, for any given $\varepsilon > 0$, a point p_n that is above A cannot be farther than ε from a point of T, as we noted in Remark 3.1. This fact immediately implies $(x, z) \in \overline{T}$, a contradiction, since the only element of $\overline{T}(x)$ is $f(x)$ by our assumptions.

Hence $S = G$, therefore we might apply Proposition 3.1. Thus f is a Baire-1 function satisfying $L_f = T$. □

6. Concluding Remark

Before the end of this paper, we would like to point out something in connection with our theorems about the not necessarily bounded functions. Namely, amongst the conditions of the last theorem there was one condition about $\overline{T}$. However, $\overline{T} = \overline{L_f}$ does not necessarily hold for the function we constructed.

For instance let T be the following closed set: let $C = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, $c_1 = 0$, and for $n \geq 2$, let $c_n = \frac{1}{n-1}$. For each point x in $I \setminus C$ let $T(x) = \{-\frac{1}{d(x,C)}\}$, where $d(x, C)$ is the distance of x from C. Then it is easy to see that this set T satisfies the conditions of Theorem 5.2 with regards to the not necessarily bounded Baire-1 functions. It is also true, that $\overline{T}(0) = \{-\infty\}$. Now, let us consider f, specifically $\overline{L_f}(0)$. We recall that in our construction $f(c_n) = n$. It implies $\overline{L_f}(0) = \{-\infty, +\infty\}$. It means that although $L_f(0) = T(0) = \emptyset$, $\overline{T}(0) \neq \overline{L_f}(0)$. Thus the sets we examined earlier are equal, but these extended sets are not.

This example raises two new questions: if we regard our theorems about the not necessarily bounded Baire-1 and Baire-2 functions and we do not change the conditions, is it possible to construct a function f in each of these cases that satisfies $L_f = T$ and $\overline{L_f} = \overline{T}$ simultaneously? However, we might answer these questions easily:

Proposition 6.1. *Suppose $T \subseteq I \times \mathbb{R}$.*

- *If there exists a Baire-2 function satisfying $L_f = T$, then it can be chosen such that $\overline{L_f} = \overline{T}$ also holds.*
- *If there exists a Baire-1 function satisfying $L_f = T$, then it can be chosen such that $\overline{L_f} = \overline{T}$ also holds.*

Proof. We will appropriately modify the functions we have constructed in the proofs of Theorem 4.2 and Theorem 5.2. It is clear that for those functions $\overline{T} \subseteq \overline{L_f}$ holds. Indeed, for any point $t \in T$ there are points of G arbitrarily close to t. Thus if we consider a point (x, ∞) of $\overline{T}$, then it is also an accumulation point of G. Hence if $\overline{L_f} \neq \overline{T}$, then $\overline{T}$ is a proper subset of $\overline{L_f}$.

For those functions it is also clear that if $\overline{L_f}$ has a point p which is not in $\overline{T}$, then it is an accumulation point of the graph of $f|C$. Namely, if we take a sequence (p_n) in G which converges in $I \times \overline{\mathbb{R}}$ and contains only finitely many points of G above C, then after a while every term of this sequence is above A or in T. The terms above A will get arbitrarily close to a point of T if n is sufficiently large. Thus if we have a point in $\overline{L_f}$ which is a limit point of such a sequence, then it is also a point of $\overline{T}$. Hence if $\overline{L_f}$ has a point outside $\overline{T}$, then there exists a sequence in the graph of $f|C$ converging to this point.

It is a problem we can easily handle in both cases by modifying f on C: if $C = \{c_1, c_2, ...\}$, then let $|f(c_n)| = n$. The sign is determined by whether $\overline{T}$ contains $(c_n, +\infty)$ or $(c_n, -\infty)$. If both of them occurs, then let $f(c_n) = n$. If we define the function f on C this way, then L_f clearly does not change, the equality $L_f = T$ still holds. Indeed, if a sequence of points of G above C converges to a point in $I \times \overline{\mathbb{R}}$, then the second coordinate of this point is $+\infty$ or $-\infty$. By symmetry, we can consider the $+\infty$ case. For a subsequence (c_{n_k}) the sequence $(c_{n_k}, f(c_{n_k}))$ converges to some $(x, +\infty) \in I \times \overline{\mathbb{R}}$. We can suppose that all the numbers $f(c_{n_k})$ are positive. Then by definition, in the $\frac{1}{n_k}$ neighborhood of c_{n_k} we might choose a point a_k such that $T(a_k)$ has an element larger than n_k. We denote this element of T by t_k. Now it is clear that the sequence (t_k) is in T and it also converges to $(x, +\infty)$. Hence all the elements of $\overline{L_f}$ are in $\overline{T}$, too. Thus we constructed a function of the corresponding Baire class satisfying $L_f = T$ and $\overline{L_f} = \overline{T}$ simultaneously. $\square$

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