Lipschitz one sets modulo sets of measure zero

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Abstract

We denote the local "little" and "big" Lipschitz functions of a function $f : \mathbb{R} \to \mathbb{R}$ by $\operatorname{lip} f$ and $\operatorname{Lip} f$. In this paper we continue our research concerning the following question. Given a set $E \subset \mathbb{R}$ is it possible to find a continuous function f such that $\operatorname{lip} f = \mathbf{1}_E$ or $\operatorname{Lip} f = \mathbf{1}_E$?

To give some partial answers to this question uniform density type, UDT and strong uniform density type, SUDT sets play an important role.

In this paper we show that modulo sets of zero Lebesgue measure any measurable set coincides with a Lip 1 set.

On the other hand, we see that one can find a measurable SUDT set E such that for any G_δ set $\widetilde{E}$ satisfying $|E \Delta \widetilde{E}| = 0$ the set $\widetilde{E}$ does not have UDT. Combining these two results shows that there exist Lip 1 sets not having UDT, that is, the converse of one of our earlier results does not hold.

1 Introduction

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the so-called "big Lip" and "little lip" functions are defined as follows:

$$\operatorname{Lip} f(x) = \limsup_{r \to 0^+} M_f(x,r), \qquad \operatorname{lip} f(x) = \liminf_{r \to 0^+} M_f(x,r), \tag{1.1}$$

where

$$M_f(x,r) = \frac{\sup\{|f(x) - f(y)| \colon |x-y| \le r\}}{r}.$$

By the Rademacher-Stepanov Theorem [8] if $\operatorname{Lip} f(x) < \infty$ for Lebesgue almost every x, then f is differentiable almost everywhere. On the other hand, in ([4], 2006) Balogh and Csörnyei showed that this property is not true if one considers $\operatorname{lip} f$. This line of research was continued in ([6], 2016) and ([3], 2018).

As other activity concerning lip exponents it also worth to mention the very recent result ([9], 2019).

The current paper is a continuation of [2].

Following [2], we say that $E \subset \mathbb{R}$ is Lip 1 (lip 1) if there exists a continuous function f defined on $\mathbb{R}$ so that $\operatorname{Lip} f = \mathbf{1}_E$ ($\operatorname{lip} f = \mathbf{1}_E$). In [2] we considered the challenging problem of characterizing these sets, focusing primarily on the Lip 1 case. Simple arguments show that being a G_δ set is a necessary, but not sufficient condition for being a Lip 1 set.

To obtain sufficient conditions for Lip 1 sets we need some assumptions about uniform density properties of sets. To state these definitions first we need to define the sets $E^{\gamma,\delta}$.

Definition 1.1. Let $E \subseteq \mathbb{R}$ be measurable and $\gamma, \delta > 0$. Then

$$E^{\gamma,\delta} = \left\{ x \in \mathbb{R} : \max\left\{ \frac{|(x-r,x) \cap E|}{r}, \frac{|(x,x+r) \cap E|}{r} \right\} \geq \gamma \ \ \forall r : 0 < r \leq \delta \right\}.$$

(Note that we use $|A|$ to indicate the Lebesgue measure of a set A.)

In [2] the following density conditions were introduced:

Definition 1.2. We say that E has uniform density type (UDT) if there exist sequences $\gamma_n \nearrow 1$ and $\delta_n \searrow 0$ such that $E \subseteq \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E^{\gamma_n,\delta_n}$.

In the previous definition we considered $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E^{\gamma_n,\delta_n}$, which is the lim sup of the sequence E^{γ_n,δ_n}. By taking the lim inf we arrive at the following definiton:

Definition 1.3. We say that E has strong uniform density type (SUDT) if there exist sequences $\gamma_n \nearrow 1$ and $\delta_n \searrow 0$ such that $E \subseteq \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E^{\gamma_n,\delta_n}$.

One of the main results from [2] (Theorem 5.5), states that if a set E is G_δ and UDT, then E is Lip 1.

In the present paper we show that every measurable subset of $\mathbb{R}$ is "close" to being a Lip 1 set. More precisely, we prove

Theorem 1.4. *For every measurable set $\widetilde{E}$ there exists a G_δ,* Lip 1 *set E such that* $|E \triangle \widetilde{E}| = 0$.

Quite often, in measure theory such theorems are not too difficult to prove, but in our case the proof of this theorem is not that easy.

On the other hand, we also prove the following:

Theorem 1.5. *There exists a measurable set $E \subseteq \mathbb{R}$ having SUDT such that for any G_δ set $\widetilde{E}$ satisfying $|E \Delta \widetilde{E}| = 0$ the set $\widetilde{E}$ does not have UDT.*

Combining these two theorems reveals that there exist Lip 1 sets which fail to be UDT so the converse of Theorem 5.5 in [2] is false.

The layout of this paper is as follows: In Section 2 we introduce our notation and recall some of the results from [2]. In Section 3, we introduce a class of Cantor sets which have SUDT and use them to construct the set E given in Theorem 1.5. Finally, Section 4 is devoted to the proof of Theorem 1.4.

2 Notation and preliminaries

From [2], we recall several definitions and results.

Definition 2.1. We write that $I_n \xrightarrow{l} x$ if $\{I_n\}$ is a sequence of closed intervals with $I_n = [x - r_n, x]$ and $r_n \searrow 0$.

Definition 2.2. We also have the right-sided version of the previous notation $I_n \xrightarrow{r} x$ if $\{I_n\}$ is a sequence of closed intervals with $I_n = [x, x + r_n]$ and $r_n \searrow 0$.

Definition 2.3. The set E is *right* (*left*) *dense* at x if for any sequence $\{I_n\}$ such that $I_n \xrightarrow{r} x$ ($I_n \xrightarrow{l} x$) we have $\frac{|E \cap I_n|}{|I_n|} \to 1$. The set E is *one-sided dense* if E is either right or left dense at every point $x \in E$.

From Definition 1.1 it is straightforward to check the following lemma from [2]:

Lemma 2.4. *For any $\gamma, \delta > 0$ the set $E^{\gamma,\delta}$ is closed.*

The following proposition and theorem were also proved in [2]:

Proposition 2.5. *Let any arising set be a measurable subset of $\mathbb{R}$.*

(i) If a set E has SUDT then it also has UDT.

(ii) Any interval has SUDT (and hence UDT).

(iii) If $E_1, E_2, \ldots$ have UDT (resp. SUDT) then $E = \bigcup_{n=1}^{\infty} E_n$ also has UDT (resp. SUDT).

(iv) There exists E which has SUDT but its closure $\overline{E}$ does not have UDT.

Theorem 2.6. *Assume that E is G_δ and E has UDT. Then there exists a continuous function f satisfying $\operatorname{Lip} f = \mathbf{1}_E$, that is the set E is* Lip 1.

While the proof of the proposition is rather standard and we encourage the reader to come up with one, the proof of the theorem is quite elaborate as one of the main results of that paper.

3 An SUDT set which is not approximable by a G_δ UDT set

Notation 3.1. Suppose that $\{\alpha_n\}$ satisfies $0 < \alpha_n < 1$ for all $n \in \mathbb{N}$ and E is a Cantor set constructed by starting with $[0, 1]$ and then removing the open interval of length α_1 centered at $1/2$ from $[0, 1]$. Then continuing with a standard "middle interval" construction after the nth step there will be 2^n closed intervals remaining, each of the same length. If I is one of these intervals at the next stage of the construction we remove an open interval centered at the midpoint of I and of length $\alpha_{n+1}|I|$ from I. We let $\mathcal{I}_n$ be the collection of closed intervals remaining after the nth step of the construction and let d_n be the length of each of these intervals. Finally, we define $E = \cap E_n$ where $E_n = \cup_{I\in\mathcal{I}_n} I$. In this case we use the notation $E \sim \{\alpha_n\}$.

Theorem 3.2. *Using Notation 3.1 suppose that $E \sim \{\alpha_n\}$ where $\sum \alpha_n < \infty$. Then E has SUDT.*

Proof. Suppose that $I = [a, b] \in \mathcal{I}_n$, so $|I| = d_n$. Note that

$$\frac{|E \cap I|}{|I|} = \prod_{k=n+1}^{\infty} (1 - \alpha_k) = \beta_n,$$

where $\beta_n \nearrow 1$ since $\sum \alpha_n < \infty$. Choose $\gamma_n = 1 - 12(1 - \beta_n)$ and $\delta_n = \frac{1}{2}d_n$.

We claim that

$$E \subset \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} E^{\gamma_n,\delta_n}, \tag{3.1}$$

and therefore E has SUDT.

To verify the claim let $x \in E$ and for each $n \in \mathbb{N}$ choose $I_n = [a_n, b_n] \in \mathcal{I}_n$ such that $x \in I_n$ so $\{x\} = \cap I_n$. Now let $r_n = \max\{x - a_n, b_n - x\}$. We assume without loss of generality that $\alpha_n < 1/3$ for each n. Then it follows easily that

$$d_{n+1} > \frac{1}{3}d_n, \qquad \frac{1}{2}d_n \le r_n \le d_n, \qquad \frac{1}{6}r_n < r_{n+1} < r_n.$$

For each $n \in \mathbb{N}$ we let $J_n = [x - r_n, x] = [a_n, x]$ if $x - a_n > b_n - x$ and $J_n = [x, x + r_n] = [x, b_n]$ otherwise. Then it follows from the fact that $r_n \ge \frac{1}{2}d_n$ that we have $\frac{|E\cap J_n|}{|J_n|} \ge 1 - 2(1 - \beta_n)$. Similarly, for every δ satisfying $r_{n+1} \le \delta \le r_n$ we can take $J = J_{n,\delta}$ to be a closed interval of length δ with x as one endpoint and contained in J_n and we have $\frac{|E\cap J|}{|J|} \ge 1 - 12(1 - \beta_n) = \gamma_n$. It now follows easily that $x \in \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} E^{\gamma_n,\delta_n}$ and therefore (3.1) holds. □

At first glance, one might believe that if K is an SUDT set, then each of its points is a left or right density point. The following theorem and its corollary refutes this belief, as they can be used to show that the SUDT set provided by Theorem 3.2 does not have this property, being a nowhere dense closed set.

Theorem 3.3. *Assume that K is a nowhere dense closed set. Then the set $D_L(K)$ of left density points (resp. the set $D_R(K)$ of right density points) is meager in K.*

Proof. We use an argument similar to the one used in [1]. Proceeding towards a contradiction, assume that $D_L(K)$ is of second category. Set

$$H_n = \left\{ x \in K : \frac{|(x - h, x) \cap K|}{h} \ge 0.9, \forall h \in \left(0, \frac{1}{n}\right) \right\}. \tag{3.2}$$

Then $D_L(K) \subseteq \bigcup_{n=1}^{\infty} H_n$ clearly holds, hence there exists n such that H_n is of second category in K. Consequently, there exists an open interval J such that $J \cap K \neq \emptyset$ and H_n is dense in $J \cap K$. As K is nowhere dense, we can choose an interval $I = (a, b)$ contiguous to K such that $[a, b] \subseteq J$ and $b - a < \frac{1}{n}$. By the observation about the density of H_n we can choose a point $x \in H_n$ such that $0 < x - b < \frac{b-a}{2}$. However, for this x and $h = b - a < \frac{1}{n}$ we have

$$\frac{|(x-h, x) \cap K|}{h} \leq 0.5,$$

contradicting $x \in H_n$. This concludes the proof. □

Corollary 3.4. *If K is a non-empty nowhere dense closed set, then it has points which are not one-sided density points.*

Proof. The set of one-sided density points is the union of $D_L(K)$ and $D_R(K)$, hence it is meager by the previous theorem. However, as $K \subseteq \mathbb{R}$ is closed, it is a Baire space, and thus we can apply the Baire category theorem to obtain the statement of the corollary. □

Using the above ideas, one can easily generalize these results with a bit of extra care. First we introduce a definition:

Definition 3.5. The set $E \subseteq \mathbb{R}$ is *weakly nowhere dense* if for any interval J we have that $E \cap J$ does not have full measure in J.

Let us notice that if E is weakly nowhere dense and $\alpha \in (0, 1)$ is a fixed positive real number then by Lebesgue's density theorem applied to the complement of E we obtain that for any interval J there exists a subinterval $I \subseteq J$ such that $|I \cap E| < \alpha |I|$.

Theorem 3.6. *Assume that E is a weakly nowhere dense set. Then the set $D_L(E)$ of left density points (resp. the set $D_R(E)$ of right density points) is meager in E.*

We note that Theorem 3.3 is implied by Theorem 3.6 as the former one is a special case of the latter.

Proof of Theorem 3.6. We can repeat the proof of Theorem 3.3 almost word for word. Proceeding towards a contradiction, assume that $D_L(E)$ is of second category. Recall the definition of H_n from (3.2). Then $D_L(E) \subseteq \bigcup_{n=1}^{\infty} H_n$ clearly holds, hence there exists n such that H_n is of second category in E. Consequently, there exists an open interval J such that $J \cap E \neq \emptyset$ and H_n is dense in $J \cap E$. As E is weakly nowhere dense, by the previous remark we can choose an interval $I = (a, b)$ such that $[a, b] \subseteq J$, we have $b - a < \frac{1}{n}$, and $\frac{|(a,b) \cap E|}{b-a} \leq 0.1$. Moreover, we may assume $b \in E$ as otherwise we can translate the interval I to the right until we arrive at such a point. Consequently, by the observation about the density of H_n we can choose a point $x \in H_n$ such that $0 < x - b < (b-a)/4$. However, for this x and $h = b - a < \frac{1}{n}$ we have

$$\frac{|(x-h,x) \cap E|}{h} \leq \frac{0.1 \cdot (b-a) + \frac{1}{4}(b-a)}{(b-a)} = 0.35,$$

contradicting $x \in H_n$. This concludes the proof. □

Corollary 3.7. *If E is a non-empty weakly nowhere dense G_δ set, then it has points which are not one-sided density points.*

Proof. The set of one-sided density points is the union of $D_L(E)$ and $D_R(E)$, hence it is meager by the previous theorem. However, as $E \subseteq \mathbb{R}$ is G_δ, it is a Baire space by Alexandrov's Theorem (see [7] for example), thus we can apply the Baire Category Theorem to obtain the statement of the corollary. □

Now we will prove Theorem 1.5 with the help of Theorem 3.2.

Proof of Theorem 1.5. Using Notation 3.1 let $E_n^* \sim \{\alpha_{k,n}\}_{k=1}^\infty$ such that $|E_n^*| = \frac{1}{2^n}$. It is easy to check that there exist such sequences $\{\alpha_{k,n}\}_{k=1}^\infty$. Then the set of intervals which are contiguous to any of these sets is countable. Now set $E_1 = E_1^*$. Next we let E_2 be a homothetic image of E_2^* centered in a contiguous interval to E_1 in $[0,1]$. Next we define E_3 as a homothetic image of E_3^* centered in a contiguous interval to $E_1 \cup E_2$, etc. We proceed recursively so that none of the occuring complementary intervals remain empty by the end of the process. By countability we can do so. Consequently the set $E = \bigcup_{n=1}^\infty E_n$ is a dense, F_σ set. By Theorem 3.2 and (iii) of Proposition 2.5 it has SUDT. We claim that it is a good example for the statement of the theorem.

To verify that take any G_δ set $\widetilde{E}$ satisfying $|E \Delta \widetilde{E}| = 0$. By construction, the set E has positive measure in any nontrivial subinterval of $[0,1]$. Consequently $\widetilde{E}$ must be dense in $[0,1]$. As $\widetilde{E}$ is also G_δ, we have that $\widetilde{E}$ is residual. Proceeding towards a contradiction, assume that $\widetilde{E}$ has UDT, that is

$$\widetilde{E} \subseteq \bigcap_{i=1}^\infty \bigcup_{j=i}^\infty \widetilde{E}^{\gamma_j,\delta_j} \tag{3.3}$$

for suitable sequences (γ_j), (δ_j). As $\widetilde{E}$ equals E modulo null-sets, we obviously have that $E^{\gamma,\delta} = \widetilde{E}^{\gamma,\delta}$ for any choice of γ, δ. Hence (3.3) can be rewritten as

$$\widetilde{E} \subseteq \bigcap_{i=1}^\infty \bigcup_{j=i}^\infty E^{\gamma_j,\delta_j}. \tag{3.4}$$

For example we have

$$\widetilde{E} \subseteq \bigcup_{j=1}^{\infty} E^{\gamma_j,\delta_j}. \tag{3.5}$$

By Lemma 2.4 each of the sets E^{γ_j,δ_j} is closed and their union contains the residual set $\widetilde{E}$. Consequently, for suitable i the set E^{γ_i,δ_i} contains an open interval I. Since E is dense in $[0,1]$, E must contain a point in I, let it be $x \in E_n$. Clearly we can choose n sufficiently large to have $\sum_{j=n}^{\infty} \frac{1}{2^j} < \gamma_i$. As $\bigcup_{l=1}^{n} E_l$ is perfect we can choose $y \in I \cap \bigcup_{l=1}^{n} E_l$ such that $x \neq y$ and $|x-y| < \delta_i$. We assume without loss of generality that $y > x$. However, $\bigcup_{l=1}^{n} E_l$ is nowhere dense and thus the existence of x and y guarantees that there exists an interval $J = (a,b) \subseteq (x,y)$ contiguous to $\bigcup_{l=1}^{n} E_l$. By the definition of E it is clear that

$$\left| E \cap \left(a, \frac{a+b}{2}\right) \right| \leq \sum_{j=n}^{\infty} \frac{1}{2^j} \frac{b-a}{2} \tag{3.6}$$

and

$$\left| E \cap \left(\frac{a+b}{2}, b\right) \right| \leq \sum_{j=n}^{\infty} \frac{1}{2^j} \frac{b-a}{2}, \tag{3.7}$$

as we fill J with some of the sets E_j for $j > n$, they are symmetric to the midpoint of J and they cannot have measure larger than $\frac{1}{2^j}|J| = \frac{b-a}{2^j}$. However, this means that E has smaller density than γ_i both in $\left(a, \frac{a+b}{2}\right)$ and $\left(\frac{a+b}{2}, b\right)$. These are the left and right neighborhoods of $\frac{a+b}{2}$ with radius $\frac{b-a}{2}$, which is smaller than δ_i. This yields that $\frac{a+b}{2} \notin E^{\gamma_i,\delta_i}$, a contradiction. □

4 Approximating measurable sets with Lip 1 sets

Lemma 4.1. *If $U \subset \mathbb{R}$ is open, $\widetilde{H} \subset U$ is measurable and $\varepsilon > 0$, then there is an open set $H \subset U$ such that $|\widetilde{H} \setminus H| = 0$, and if $I = (a,b)$ is a bounded component of H, then $\widetilde{H}$ is right dense at a and left dense at b and for every $r \in (0, b-a)$ we have*

$$\max\left\{ \frac{|(a,a+r) \setminus \widetilde{H}|}{r}, \frac{|(b-r,b) \setminus \widetilde{H}|}{r} \right\} < \varepsilon. \tag{4.1}$$

Proof. If $|\widetilde{H}| = 0$ then $H := \emptyset$ is the suitable choice, hence we can assume that $|\widetilde{H}| > 0$.

First we prove that if x is a density point of $\widetilde{H}$ and $\varepsilon_x > 0$, then there is an interval $I_x = (a_x, b_x) \subset U$ which contains x, its endpoints are density points of $\widetilde{H}$ and for every $r_x \in (0, b_x - a_x)$ we have

$$\max\left\{ \frac{|(a_x, a_x + r_x) \setminus \widetilde{H}|}{r_x}, \frac{|(b_x - r_x, b_x) \setminus \widetilde{H}|}{r_x} \right\} < \varepsilon_x. \tag{4.2}$$

Since x is a density point of $\widetilde{H}$ we can take an open interval $I'_x = (a'_x, b'_x) \subset U$ centered at x for which

$$\frac{|I'_x \setminus \widetilde{H}|}{|I'_x|} < \frac{\varepsilon_x}{16}. \tag{4.3}$$

Let

$$H_x := \Big\{ x' \in \Big[a'_x, \frac{a'_x + b'_x}{2}\Big] : \exists\, r'_{x'} \in \Big[0, \frac{b'_x - a'_x}{2}\Big] \text{ such that} \tag{4.4}$$
$$\frac{|[x', x' + r'_{x'}] \setminus \widetilde{H}|}{r'_{x'}} \geq \varepsilon_x \Big\}.$$

For every $x' \in H_x$ fix such an $r'_{x'}$. Choose a finite subset X' of H_x such that

$$\text{for every } z \in \mathbb{R} \text{ we have that } \#\Big\{ x' \in X' : z \in [x', x' + r'_{x'}] \Big\} \leq 2 \tag{4.5}$$

and

$$\Big| \bigcup_{x' \in H_x} [x', x' + r'_{x'}] \setminus \bigcup_{x' \in X'} [x', x' + r'_{x'}] \Big| < \frac{|I'_x|}{16}. \tag{4.6}$$

By (4.6) we obtain

$$\begin{aligned}
|H_x| &\leq \frac{|I'_x|}{16} + \Big| \bigcup_{x' \in X'} [x', x' + r'_{x'}] \Big| \leq \frac{|I'_x|}{16} + \sum_{x' \in X'} r'_{x'} \\
&\underset{\text{by (4.4)}}{\leq} \frac{|I'_x|}{16} + \sum_{x' \in X'} \frac{1}{\varepsilon_x} |[x', x' + r'_{x'}] \setminus \widetilde{H}| \\
&\underset{\text{by (4.5)}}{\leq} \frac{|I'_x|}{16} + \frac{1}{\varepsilon_x} \cdot 2 |I'_x \setminus \widetilde{H}| \\
&\underset{\text{by (4.3)}}{\leq} \frac{|I'_x|}{16} + \frac{1}{\varepsilon_x} \cdot 2 \cdot \frac{\varepsilon_x}{16} |I'_x| = \frac{3|I'_x|}{16} = \frac{3(b'_x - a'_x)}{16}.
\end{aligned}$$

Thus, by Lebesgue's density theorem, there exists a density point a_x of $\widetilde{H}$ in $((3a'_x + b'_x)/4, (a'_x + b'_x)/2)$ such that $a_x \notin H_x$ and hence $r_x \in \Big(0, \frac{b'_x - a'_x}{2}\Big)$ implies

$$\frac{|(a_x, a_x + r_x) \setminus \widetilde{H}|}{r_x} < \varepsilon_x.$$

Similarly, there exists a density point b_x of $\widetilde{H}$ in $((a'_x + b'_x)/2, (a'_x + 3b'_x)/4)$ such that $r_x \in (0, \frac{b'_x - a'_x}{2})$ implies

$$\frac{|(b_x - r_x, b_x) \setminus \widetilde{H}|}{r_x} < \varepsilon_x.$$

As $b_x - a_x < \frac{b'_x - a'_x}{2}$ and $x = \frac{a'_x + b'_x}{2} \in (a_x, b_x)$, the points a_x and b_x satisfy (4.2).

We choose a subset $X = \{x_1, x_2, \ldots\}$ of the density points of $\widetilde{H}$ with their corresponding neighbourhoods $\{I_{x_1}, I_{x_2}, \ldots\}$, a sequence of positive numbers $\{\varepsilon_{x_1}, \varepsilon_{x_2}, \ldots\}$ and a strictly increasing sequence of natural numbers $(m_i)_{i=1}^{\infty}$ such that they satisfy

- (4.2) holds with x replaced with x_n

- $$\text{each real number is contained by at most two of } \{I_{x_1}, I_{x_2}, \ldots\}, \tag{4.7}$$

- $$\varepsilon_{x_k} < \frac{\varepsilon}{4} \text{ for every } k \in \mathbb{N}, \tag{4.8}$$

- $$\varepsilon_{x_k} < \frac{\varepsilon}{4i} \text{ for every } i \in \mathbb{N} \text{ and } k \geq m_i, \tag{4.9}$$

- and
$$\left| \left(\widetilde{H} \cap [-i, i] \right) \setminus \bigcup_{j=1}^{m_i} I_{x_j} \right| < \frac{1}{i}. \tag{4.10}$$

Denote
$$H := \bigcup_{j=1}^{\infty} I_{x_j}. \tag{4.11}$$

By (4.10) we have that $|\widetilde{H} \setminus H| = 0$. Let $I = (a, b)$ be a component of H and $r \in (0, b - a)$. By (4.11), there is a $j_r \in \mathbb{N}$ such that
$$\left| (a, a + r) \setminus \bigcup_{j=1}^{j_r} I_{x_j} \right| < r \frac{\varepsilon}{2}. \tag{4.12}$$

Hence we have

$$\begin{aligned}
\frac{|(a,a+r)\setminus \widetilde{H}|}{r} &\le \frac{\left|(a,a+r)\setminus \bigcup_{j=1}^{j_r} I_{x_j}\right| + \left|\left(\bigcup_{j=1}^{j_r} I_{x_j}\cap(a,a+r)\right)\setminus \widetilde{H}\right|}{r}\\
&\underset{\text{by (4.12)}}{<} \frac{\frac{\varepsilon}{2}\cdot r + \sum_{j=1}^{j_r} |\left(I_{x_j}\cap(a,a+r)\right)\setminus \widetilde{H}|}{r}\\
&\underset{\text{by (4.2)}}{\le} \frac{\frac{\varepsilon}{2}\cdot r + \sum_{j=1}^{j_r} \varepsilon_{x_j}|I_{x_j}\cap(a,a+r)|}{r}\\
&\underset{\text{by (4.8)}}{\le} \frac{\frac{\varepsilon}{2}\cdot r + \sum_{j=1}^{j_r} \frac{\varepsilon}{4}|I_{x_j}\cap(a,a+r)|}{r}\\
&\underset{\text{by (4.7)}}{\le} \frac{\frac{\varepsilon}{2}\cdot r + \frac{\varepsilon}{4}\cdot 2r}{r} = \varepsilon.
\end{aligned}$$

Similarly, we obtain

$$\frac{|(b-r,b)\setminus \widetilde{H}|}{r} < \varepsilon,$$

hence H satisfies (4.1).

To show that a is a right density point of $\widetilde{H}$ take an arbitrary $\varepsilon^* > 0$. If a is a left endpoint of I_{x_k} for some $k \in \mathbb{N}$, we are done. Otherwise, take an $i^* \in \mathbb{N}$ such that

$$\frac{\varepsilon}{i^*} < \varepsilon^*, \tag{4.13}$$

and a $\delta^* \in (0, b-a)$ such that

$$(a, a+\delta^*) \cap \bigcup_{j=1}^{m_{i^*}} I_j = \emptyset. \tag{4.14}$$

According to (4.11) we have that $(a,a+\delta^*) \subset \bigcup_{j=1}^{\infty} I_{x_j}$, hence (4.14) implies that $(a,a+\delta^*) \subset \bigcup_{j=m_{i^*}}^{\infty} I_{x_j}$. Consequently, there is a $j_{\delta^*} \in \mathbb{N}$ for which

$$\left|(a,a+\delta^*)\setminus \bigcup_{j=m_{i^*}}^{j_\delta^*} I_{x_j}\right| < \frac{\varepsilon^*}{2}\cdot\delta^*. \tag{4.15}$$

Thus

$$\frac{|(a,a+\delta^*)\setminus \widetilde{H}|}{\delta^*} < \frac{\left|(a,a+\delta^*)\setminus \bigcup_{j=m_{i^*}}^{j_{\delta^*}} I_{x_j}\right| + \left|\left(\bigcup_{j=m_{i^*}}^{j_{\delta^*}} I_{x_j}\cap (a,a+\delta^*)\right)\setminus \widetilde{H}\right|}{\delta^*}$$

$$\underset{\text{by (4.15)}}{\leq} \frac{\frac{\varepsilon^*}{2}\cdot\delta^* + \sum_{j=m_{i^*}}^{j_{\delta^*}} |(I_{x_j}\cap(a,a+\delta^*))\setminus \widetilde{H}|}{\delta^*}$$

$$\underset{\text{by (4.2)}}{\leq} \frac{\frac{\varepsilon^*}{2}\cdot\delta^* + \sum_{j=m_{i^*}}^{j_{\delta^*}} \varepsilon_{x_j}|I_{x_j}\cap(a,a+\delta^*)|}{\delta^*}$$

$$\underset{\text{by (4.9)}}{\leq} \frac{\frac{\varepsilon^*}{2}\cdot\delta^* + \sum_{j=m_{i^*}}^{j_{\delta^*}} \frac{\varepsilon}{4i^*}|I_{x_j}\cap(a,a+\delta^*)|}{\delta^*}$$

$$\underset{\text{by (4.13)}}{\leq} \frac{\frac{\varepsilon^*}{2}\cdot\delta^* + \sum_{j=m_{i^*}}^{j_{\delta^*}} \frac{\varepsilon^*}{4}|I_{x_j}\cap(a,a+\delta^*)|}{\delta^*}$$

$$\underset{\text{by (4.7)}}{\leq} \frac{\frac{\varepsilon^*}{2}\cdot\delta^* + \frac{\varepsilon^*}{4}\cdot 2\delta^*}{\delta^*} = \varepsilon^*.$$

Hence a is a right density point of $\widetilde{H}$, and we obtain in the same way that b is a left density point of $\widetilde{H}$. This concludes the proof. □

The next lemma is Lemma 2.4 in [2].

Lemma 4.2. *Suppose that $E \subset \mathbb{R}$ and $f\colon \mathbb{R}\to\mathbb{R}$ such that $\operatorname{Lip} f = \mathbf{1}_E$. Then f is a Lipschitz function and $|f(x)-f(y)| \leq |[x,y]\cap E|$ for every $x,y\in\mathbb{R}$ (where $x<y$).*

We turn now to the proof of Theorem 1.4

We first note here that if there exists a G_δ set $\widetilde{E}$ having UDT and satisfying $|E\triangle\widetilde{E}| = 0$, then Theorem 1.4 trivially follows from Theorem 2.6. However, as Theorem 1.5 highlights, it is not always possible to find such a set, even if E has nice density behaviour.

Proof of Theorem 1.4. The construction is analogous to, but more complicated than the proof of Theorem 2.6, which is presented in [2] with the numbering Theorem 5.5.

To avoid some technical difficulties we observe that we can suppose that we work with essentially unbounded sets, that is for all $\alpha,\beta\in\mathbb{R}$ we have $|\widetilde{E}\cap$

$(-\infty,\alpha)| > 0$ and $|\widetilde{E}\cap(\beta,+\infty)| > 0$. Indeed, suppose that we proved our theorem for such cases and, for example, there exists $\alpha\in\mathbb{R}$ such that $|\widetilde{E}\cap(-\infty,\alpha)| = 0$ but for all $\beta\in\mathbb{R}$ $|\widetilde{E}\cap(\beta,+\infty)| > 0$. Then one can use $\widetilde{E}\cup(-\infty,\alpha-2]$ to obtain a Lip 1 set E' such that $|E'\Delta(\widetilde{E}\cup(-\infty,\alpha-2))| = 0$. Suppose that h is a continuous function such that $\mathrm{Lip}\, h(x) = \mathbf{1}_{E'}(x)$. Then $h'(x) = 0$ on $(\alpha-2,\alpha)$. Set $E = (\alpha-1,+\infty)\cap E'$. Letting $f(x) = h(x)$ for $x\ge\alpha-1$ and $f(x) = h(\alpha-1)$ otherwise we obtain a continuous function for which $\mathrm{Lip}\, f(x) = \mathbf{1}_E(x)$ and $|\widetilde{E}\Delta E| = 0$. The reduction of the other essentially bounded cases to the unbounded case is similar.

Given an open set G we say that a set D is locally discrete in G if $D\subset G$ and for any $x\in G$ there is a $\delta>0$ such that $D\cap(x-\delta,x+\delta)$ is finite.

We will define a nested sequence of open sets $(G_n)_{n=0}^\infty$ and uniformly convergent sequences of continuous functions $(f_n)_{n=0}^\infty$, $(\mathcal{E}_n)_{n=0}^\infty$ and $(\mathcal{E}^n)_{n=0}^\infty$ such that for every $m,n\in\mathbb{N}\cup\{0\}$, $m\le n$ we have

(A) $|\widetilde{E}\setminus G_n| = 0$,

(B) $|\widetilde{E}\triangle\bigcap_{n=1}^\infty G_n| = 0$,

(C) $1-\frac{1}{n}\le \mathrm{Lip}(f_n)$ on G_n for $n\ge1$, and $\mathrm{Lip}(f_n)\le1$ on $\mathbb{R}$ for $n\ge0$ and hence f_n is continuous,

(D) $\mathcal{E}_n$ and $\mathcal{E}^n$ have vanishing derivative on $F_n := \mathbb{R}\setminus G_n$, and $\mathcal{E}_n|_{F_n} = \mathcal{E}^n|_{F_n} = f_n|_{F_n}$,

(E) for $n\ge1$ there is a locally discrete set D_n in G_n such that for every $x\in G_n$ there are $d_1,d_2\in D_n$ for which

$$x\in[d_1,d_2],\ 0<|d_1-d_2|\le\frac{1}{n}\ \text{and}\ \left|\frac{f_n(d_1)-f_n(d_2)}{d_1-d_2}\right|\ge1-\frac{1}{n},$$

(F) $f_n|_{F_m\cup D_m} = f_m|_{F_m\cup D_m}$,

(G) $\mathcal{E}_m\le\mathcal{E}_n\le f_n\le\mathcal{E}^n\le\mathcal{E}^m$.

Next we see that the above assumptions imply Theorem 1.4. By (E) for every $x\in\mathbb{R}$ there is an $x'\in F_m\cup D_m$ such that $|x-x'|\le\frac{1}{2m}$. Thus Lemma 4.2, (C) and (F) imply that

$$\begin{aligned}|f_n(x)-f_m(x)| &\le |f_n(x)-f_n(x')|+|f_n(x')-f_m(x')|+|f_m(x')-f_m(x)|\\ &\le |x-x'|+0+|x-x'|\le 2|x-x'|\le\frac{1}{m},\end{aligned}$$

that is $\|f_n - f_m\| \le \frac{1}{m}$, i.e. $f := \lim_{n\to\infty} f_n$ exists. Moreover, (F) and (E) imply that $\mathrm{Lip}(f) \ge 1$ on $\bigcap_{n=1}^{\infty} G_n$, and $\mathrm{Lip}(f) \le 1$ by (C). According to (D), (F) and (G) if $x \in \mathbb{R} \setminus \bigcap_{n=1}^{\infty} G_n$ then $\mathrm{Lip}\, f(x) = 0$. Thus $E := \bigcap_{n=1}^{\infty} G_n$ will be a suitable choice by (B) and this proves Theorem 1.4.

Now we turn to the proof of the fact that conditions (A-G) can be satisfied (the places where the individual conditions are verified are marked by $\circledast$). Let

- $G_0 := \mathbb{R}$,
- $f_0 :\equiv 0$,
- $D_0 := \{z \in \mathbb{Z} : |[z, z+1] \cap \widetilde{E}| > 0\}$.

By these definitions we can define continuous functions $\mathcal{E}_0$ and $\mathcal{E}^0$ for which $\mathcal{E}_0 \le f_0 \le \mathcal{E}^0$ and

$$\begin{gathered}\text{if } d_1, d_2 \in D_0 \text{ are adjacent in } D_0 \text{ and } x \in [d_1, d_2] \text{ then}\\ \min\big\{f_0(d_1) - \mathcal{E}_0(x), f_0(d_2) - \mathcal{E}_0(x),\\ \mathcal{E}^0(x) - f_0(d_1), \mathcal{E}^0(x) - f_0(d_2)\big\} > |d_1 - d_2|.\end{gathered}$$

Now we assume that $n \in \mathbb{N}$, and we have already defined G_k, f_k, $\mathcal{E}_k$, $\mathcal{E}^k$ and D_k for every $k \in \{0, \ldots, n-1\}$ so that they satisfy (A), (D), (F), (G) and the following assumptions:

$$\begin{gathered}\text{if } d_1, d_2 \in D_{n-1} \text{ are adjacent in } D_{n-1} \text{ and } x \in [d_1, d_2] \subset G_{n-1} \text{ then}\\ \min\big\{f_{n-1}(d_1) - \mathcal{E}_{n-1}(x), f_{n-1}(d_2) - \mathcal{E}_{n-1}(x),\\ \mathcal{E}^{n-1}(x) - f_{n-1}(d_1), \mathcal{E}^{n-1}(x) - f_{n-1}(d_2)\big\} > |d_1 - d_2|,\end{gathered} \tag{4.16}$$

if $d_1, d_2 \in D_{n-1}$ are adjacent, then

$$|\widetilde{E} \cap (d_1, d_2)| > |f_{n-1}(d_1) - f_{n-1}(d_2)|, \tag{4.17}$$

if $n > 1$, and (a, b) is a component of G_{n-1}, then

$$\big\{\text{accumulation points of } \big(D_{n-1} \cap (a, b)\big)\big\} = \{a, b\}. \tag{4.18}$$

Observe that G_0, D_0, f_0, $\mathcal{E}_0$ and $\mathcal{E}^0$ indeed satisfy (A), (G), (4.16) and (4.17). As (D) and (F) say nothing when $n = 0$, they also hold.

We continue by defining G_n. First we define the sets $\widetilde{G}_n^l \supset G_n^l \supset \widetilde{G}_n^{l+1}$... by mathematical induction. Let

$$\widetilde{G}_n^1 := G_{n-1} \setminus \Big(D_{n-1} \cup \Big\{\frac{z}{n} : z \in \mathbb{Z}\Big\} \cup \{\text{midpoints of the} \tag{4.19}$$

$$\text{components of } G_{n-1} \setminus D_{n-1}\}\Big).$$

Let $l > 0$ and suppose that we have already defined an open set $\widetilde{G}^l_n$. According to Lemma 4.1 there is an open set $G^l_n \subset \widetilde{G}^l_n$ such that

$$|(\widetilde{E} \cap \widetilde{G}^l_n) \setminus G^l_n| = 0, \tag{4.20}$$

and it also satisfies the property that if $I = (a, b)$ is a component of G^l_n, then a is a right density point of $\widetilde{E} \cap \widetilde{G}^l_n$, b is a left density point of $\widetilde{E} \cap \widetilde{G}^l_n$ and for every $r \in (0, b - a)$ we have

$$\begin{aligned} &\max\left\{\frac{|(a, a + r) \setminus \widetilde{E}|}{r}, \frac{|(b - r, b) \setminus \widetilde{E}|}{r}\right\} \\ &\leq \max\left\{\frac{|(a, a + r) \setminus (\widetilde{E} \cap \widetilde{G}^l_n)|}{r}, \frac{|(b - r, b) \setminus (\widetilde{E} \cap \widetilde{G}^l_n)|}{r}\right\} < \frac{1}{4(n + l)^2}. \end{aligned} \tag{4.21}$$

If $G^l_n = \emptyset$, let $I^l_j := \emptyset$ for every $j \in \mathbb{N}$. Otherwise, we take some components $I^l_1, I^l_2, I^l_3, \ldots$ of G^l_n such that every bounded interval contains finitely many of them and

$$\left|G^l_n \setminus \bigcup_{k=1}^{\infty} I^l_k\right| < 2^{-l}. \tag{4.22}$$

Define $\widetilde{G}^{l+1}_n := G^l_n \setminus \bigcup_{k=1}^{\infty} I^l_k$ and continue the induction.

Set

$$G_n := \bigcup_{l=1}^{\infty}\bigcup_{k=1}^{\infty} I^l_k.$$

By mathematical induction for every $l^* \in \mathbb{N}$ we will prove

$$\left|\widetilde{E} \setminus \left(G^{l^*}_n \cup \bigcup_{l=1}^{l^*-1}\bigcup_{k=1}^{\infty} I^l_k\right)\right| = 0. \tag{4.23}$$

As

$$\left|\widetilde{E} \setminus G^1_n\right| \underset{\text{by (4.20)}}{=} \left|\widetilde{E} \setminus \widetilde{G}^1_n\right| \underset{\text{by (4.19)}}{=} \left|\widetilde{E} \setminus G_{n-1}\right|,$$

(4.23) is true for $l^* = 1$. Suppose that it holds for some $l^* \in \mathbb{N}$, then

$$\begin{aligned} \left|\widetilde{E} \setminus \left(G^{l^*+1}_n \cup \bigcup_{l=1}^{l^*}\bigcup_{k=1}^{\infty} I^l_k\right)\right| &\underset{\text{by (4.20)}}{=} \left|\widetilde{E} \setminus \left(\widetilde{G}^{l^*+1}_n \cup \bigcup_{l=1}^{l^*}\bigcup_{k=1}^{\infty} I^l_k\right)\right| \\ &\leq \left|\widetilde{E} \setminus \left(G^{l^*}_n \cup \bigcup_{l=1}^{l^*-1}\bigcup_{k=1}^{\infty} I^l_k\right)\right| = 0. \end{aligned}$$

Hence, by (4.22) we obtain $|\widetilde{E} \setminus G_n| = 0$. ⊛ Thus (A) holds at step n.

Moreover, according to (4.21), if $I \subset G_n$ is a bounded interval such that at least one of its endpoints is an endpoint of a component of G_n, we have that $|\big(G_n \cap I\big) \setminus \widetilde{E}| \le \frac{1}{4n^2}|I|$, ⊛ which implies (B).

Now we construct f_n. We set $f_n := \mathcal{E}_n := \mathcal{E}^n := f_{n-1}$ on $F_{n-1} \cup D_{n-1}$. As (F) held in the previous steps of the induction, ⊛ (F) holds at step n as well.

Take an arbitrary interval

$$I = (a, b) \text{ contiguous to } F_{n-1} \cup D_{n-1}. \tag{4.24}$$

Then $F_0 = \emptyset$ and (4.18) for $n > 1$ imply that $a, b \in D_{n-1}$. According to (4.17), for some $k^* \in \mathbb{N}$ there are finitely many different components $I_1, \dots, I_{k^*}$ of $G_n \cap I$ such that $|f_{n-1}(a) - f_{n-1}(b)| < \sum_{i=1}^{k^*} |I_i \cap \widetilde{E}|$. We index these components in an increasing order on the real line. We can assume without loss of generality that $f_{n-1}(a) \le f_{n-1}(b)$. Denote by a_i and b_i the endpoints of I_i for every $i \in \{1, \dots, k^*\}$, and let

$$f_n(a_i) := f_n(a) + \frac{\sum_{j=1}^{i-1} |I_j \cap \widetilde{E}|}{\sum_{j=1}^{k^*} |I_j \cap \widetilde{E}|} \left(f_n(b) - f_n(a)\right)$$

and

$$f_n(b_i) := f_n(a) + \frac{\sum_{j=1}^{i} |I_j \cap \widetilde{E}|}{\sum_{j=1}^{k^*} |I_j \cap \widetilde{E}|} \left(f_n(b) - f_n(a)\right).$$

On $I \setminus G_n$ set

$$f_n(x) := \mathcal{E}_n(x) := \mathcal{E}^n(x) := \max\big(\{f_n(b_i) : i \in \{1, \dots, k^*\} \text{ and } b_i \le x\} \cup \{f(a)\}\big). \tag{4.25}$$

Let $I' = (a', b')$ be a component of $I \cap G_n$. We separate two cases:

(a) Let $I' \in \{I_1, \dots, I_{k^*}\}$. As a' is a right density point of $\widetilde{E}$, if we choose an $a'_0 \in I'$ close enough to a', then by (4.21) there is a $b'_0 \in (a'_0, b')$ such that

$$\Big(1 - \frac{1}{n}\Big)(b'_0 - a'_0) < f_n(b') - f_n(a') < |(a'_0, b'_0) \cap \widetilde{E}|.$$

Set $f_n(a'_0) := f_n(a')$ and $f_n(b'_0) := f_n(b')$ and let f_n be linear on $[a'_0, b'_0]$. (4.26)

Define D_n on $[a'_0, b'_0]$ such that $D_n \cap [a'_0, b'_0] := \{a'_0, b'_0\}$. We have that

$$\circledast \text{ (C) holds on } (a'_0, b'_0), \text{ and } \circledast \text{ (E) and (4.17) hold on } [a'_0, b'_0]. \tag{4.27}$$

(b) If $I' = (a', b')$ is a component of $I \cap G_n \setminus \{I_1, \dots, I_{k^*}\}$, then set

$$f_n(a') := f_n(b') := f_n(\max(\{a\} \cup \{b_i | b_i \le a'\})).$$

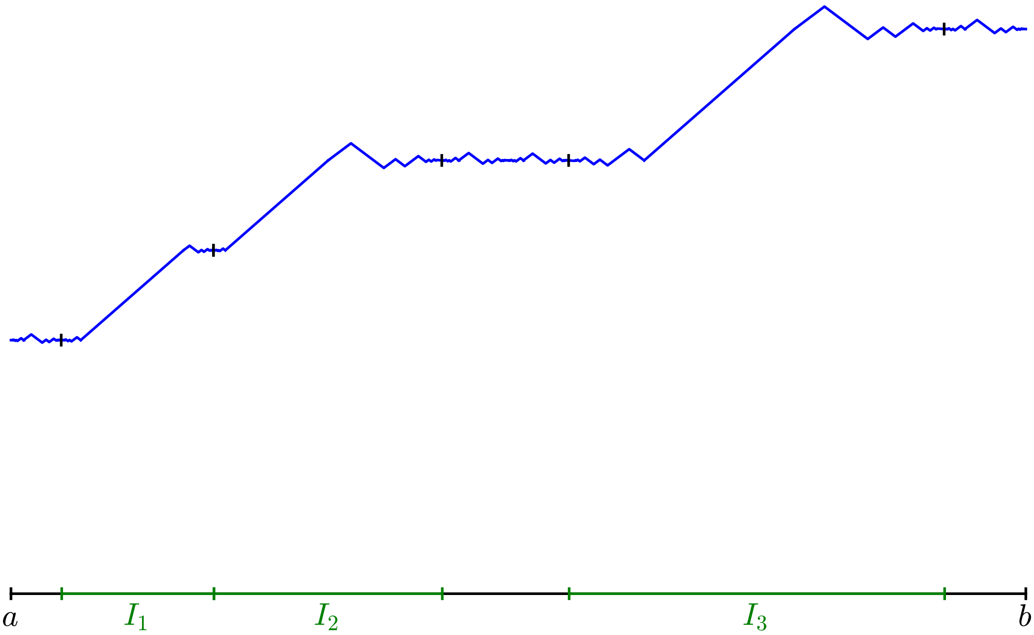

Figure 1: Graph of f_n on $I = [a, b]$

In the following, we will define f_n, $\mathcal{E}_n$, $\mathcal{E}^n$ and D_n in an arbitrary component I' of $I \cap G_n$. If we do not mention which case we investigate, the statements will hold in both cases (a) and (b). However, if $I' \notin \{I_1, \dots, I_{k^*}\}$, then we put $a'_0 := b'_0 := \frac{b'+a'}{2}$ amd $f_n(a'_0) := f_n(b'_0) := f_n(a') = f_n(b')$.

Let $l' := \max\{l | I' \subset \widetilde{G}^l_n\}$. We will define a strictly decreasing sequence $(a'_k)_{k=1}^\infty$ in $(a', a'_0]$ converging to a'. Suppose that we have already defined $a'_0, \dots, a'_{k-1}$ for some $k \in \mathbb{N}$. We choose $a'_k \in (a', a'_{k-1})$ to satisfy

$$|(a'_k, a'_{k-1})| = \min \left\{ \frac{1}{n+l'} |(a', a'_{k-1})|, \frac{1}{k} |(a', a'_{k-1})| + 4(n+l') |(a', a'_{k-1}) \setminus \widetilde{E}| \right\}. \tag{4.28}$$

Next we show that $\lim_{k \to \infty} a'_k = a'$. Since a'_k is monotone decreasing and bounded by a' from below it has a finite limit a''. If $a' = a''$ then we are done. If $a'' > a'$ then for large enough k (4.28) implies that $|(a'_k, a'_{k-1})| \geq |(a', a'_{k-1})|/k \geq |(a', a'')|/k$. Since $\sum \frac{1}{k}$ diverges, this is impossible.

By (4.21) we have that

$$4(n+l') |(a', a'_{k-1}) \setminus \widetilde{E}| < \frac{1}{n+l'} |(a', a'_{k-1})|,$$

hence using that $4(n+l')|(a', a'_{k-1}) \setminus \widetilde{E}|$ is less than the second expression in $\min\{\ ,\ \}$

of (4.28) we obtain that

$$4(n+l')|(a',a'_{k-1})\setminus\widetilde{E}|<|(a'_k,a'_{k-1})|.$$

This implies that

$$|(a'_k,a'_{k-1})\setminus\widetilde{E}|\le|(a',a'_{k-1})\setminus\widetilde{E}|<\frac{1}{4(n+l')}|(a'_k,a'_{k-1})|. \tag{4.29}$$

As a' has been defined to be a right density point of $\widetilde{E}$, by (4.28) we have

$$\lim_{k\to\infty}\frac{|(a'_k,a'_{k-1})|}{|(a',a'_{k-1})|}\le\lim_{k\to\infty}\left(\frac{1}{k}+4(n+l')\frac{|(a',a'_{k-1})\setminus\widetilde{E}|}{|(a',a'_{k-1})|}\right)=0. \tag{4.30}$$

We define a sequence $(b'_k)_{k=1}^{\infty}$ in (b_0,b') similarly.

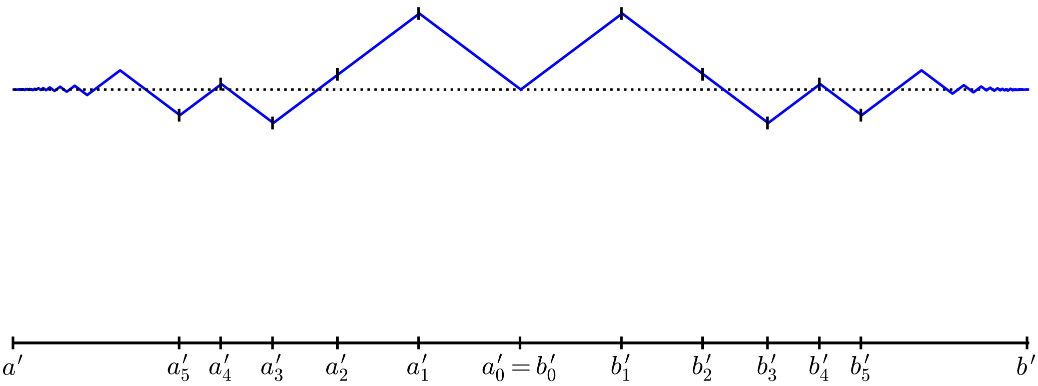

Figure 2: The graph of f_n on $I'=[a',b']$ if $I'\notin\{I_1,\dots,I_{k^*}\}$

For every $k\in\mathbb{N}$ let

$$f_n(a'_k):=\begin{cases}f_n(a'_{k-1})+(1-\frac{1}{n})|(a'_{k-1},a'_k)| \text{ if } f_n(a'_{k-1})\le f_n(a')=f_n(a'_0),\\ f_n(a'_{k-1})-(1-\frac{1}{n})|(a'_{k-1},a'_k)| \text{ if } f_n(a'_{k-1})>f_n(a')=f_n(a'_0),\end{cases} \tag{4.31}$$

and let f_n be linear on $[a'_k,a'_{k-1}]$. We define f_n in an analogous way on (b'_0,b') using $(b'_k)_{k=1}^{\infty}$ in place of $(a'_k)_{k=1}^{\infty}$.

From definition (4.31) and $1-\frac{1}{1}=0$ it follows that

$$\text{if } n=1, \text{ then } f_n|_{[a',a'_0]}\equiv f_n(a')=f_n(a'_0) \text{ and } f_n|_{[b'_0,b']}\equiv f_n(b'_0)=f_n(b'). \tag{4.32}$$

Suppose that $n>1$. By (4.28),

$$(a'_{k-1}-a'_k)\le\frac{1}{n+l'}(a'_{k-1}-a')\le\frac{1}{3}(a'_{k-1}-a'). \tag{4.33}$$

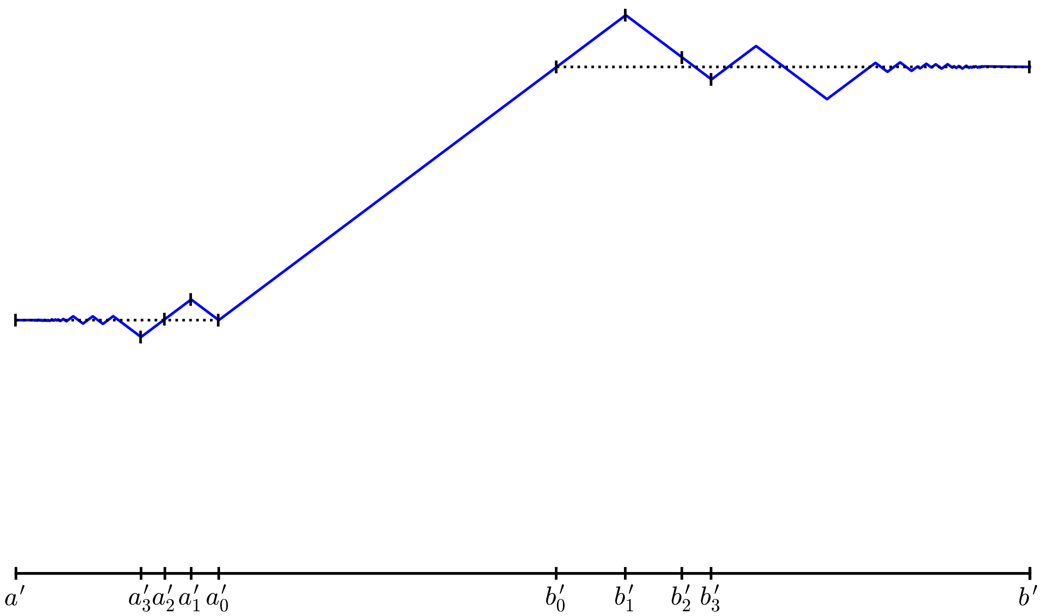

Figure 3: The graph of f_n on $I' = [a', b']$ if $I' \in \{I_1, \dots, I_{k^*}\}$

Next we show that for all $k = 0, 1, ...$

$$|f_n(a'_k) - f_n(a')| < \left(1 - \frac{1}{n}\right)(a'_k - a'). \tag{4.34}$$

Observe that $0 = |f_n(a'_0) - f_n(a')| < (1 - \frac{1}{n})(a'_0 - a')$ and hence (4.34) holds for $k = 0$.

Suppose that for a $k \geq 0$ we have (4.34).

If $(f_n(a'_{k+1}) - f_n(a')) \cdot (f_n(a'_k) - f_n(a')) > 0$ then our definition in (4.31) implies that (4.34) holds for $k + 1$ instead of k.

If $(f_n(a'_{k+1}) - f_n(a')) \cdot (f_n(a'_k) - f_n(a')) \leq 0$ then

$$|f_n(a'_{k+1}) - f_n(a')| \leq |f_n(a'_{k+1}) - f_n(a'_k)| \underset{\text{by (4.31)}}{=} \left(1 - \frac{1}{n}\right)|[a'_{k+1}, a'_k]|$$

$$\underset{\text{by (4.33)}}{\leq} \left(1 - \frac{1}{n}\right)\frac{1}{3}(a'_k - a') \underset{\text{by (4.33)}}{\leq} \left(1 - \frac{1}{n}\right) \cdot \frac{1}{3} \cdot \frac{3}{2}(a'_{k+1} - a').$$

Therefore, by using (4.34) one can see that

$$\begin{aligned} &\text{if } n > 1, \text{ then } \big(f_n(a'_k) - f_n(a')\big)_{k=1}^{\infty} \text{ changes its sign infinitely often} \\ &\text{and similarly } \big(f_n(b'_k) - f_n(b')\big)_{k=1}^{\infty} \text{ changes its sign infinitely often.} \end{aligned} \tag{4.35}$$

It also follows from (4.31) that if x is a local extremum of f_n in (a', a'_0) then there exists $k_x > 0$ such that $x = a_{k_x}$ and

$$(f_n(a'_{k_x-1}) - f_n(a')) \cdot (f_n(a'_{k_x}) - f_n(a')) \leq 0. \tag{4.36}$$

However, it may happen for some $k \in \mathbb{N}$ that $(f_n(a'_{k-1}) - f_n(a'))$ and $(f_n(a'_k) - f_n(a'))$ are of the same sign.

Set

$$D_n \cap I' := \{a'_0, a'_1, \dots\} \cup \{b'_0, b'_1, \dots\}.$$

This definition means that

$$D_n \text{ satisfies (4.18) on } I'. \tag{4.37}$$

By (4.19) we have that $|I'| \leq \frac{1}{n}$, hence $D_n \cap I'$ is a $\frac{1}{n}$-mesh on I'. Thus by (4.27) and (4.31), ⊛ (E) is true on I'.

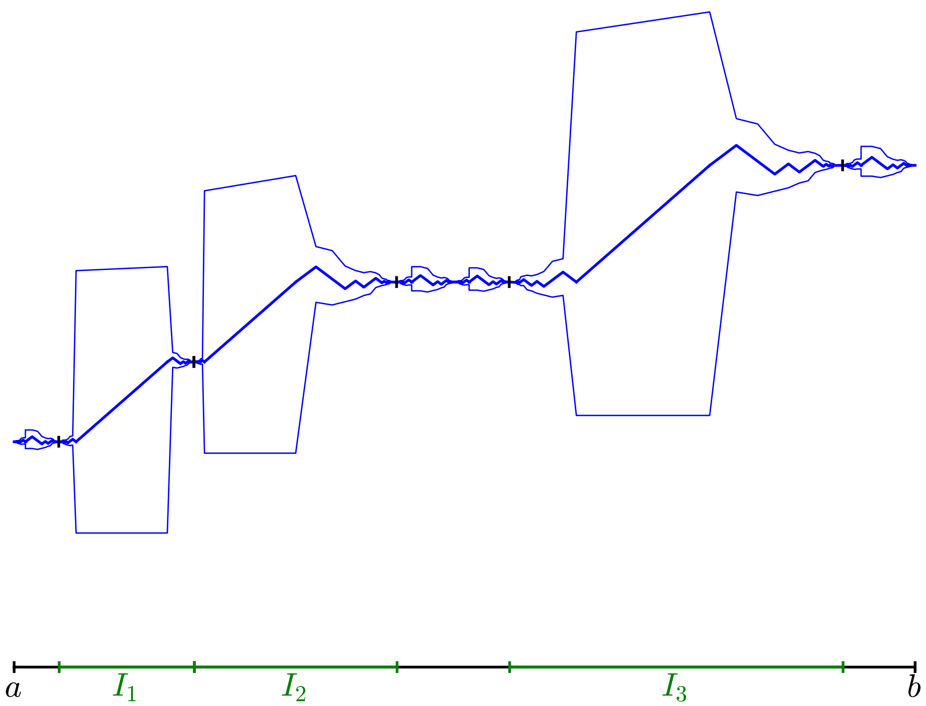

Figure 4: The graphs of $\mathcal{E}_n$, $\mathcal{E}^n$ and f_n on $I = [a, b]$

By (4.27), (4.29) and (4.31) we have that

$$f_n \text{ and } D_n \text{ satisfy (4.17) on } I'. \tag{4.38}$$

Moreover, (4.27) and (4.31) also imply that

$$\circledast \text{ (C) holds on whole of } (a', b'). \tag{4.39}$$

According to (4.32), (4.30), (4.31) and (4.35)

$$\text{the right derivative of } f_n \text{ is 0 in } a' \text{ and the left derivative of } f_n \text{ is 0 in } b'. \tag{4.40}$$

If d_1, d_2 and d_3 are adjacent points of $D_n \cap I'$ and $d_1 < d_2 < d_3$, then we set

$$\mathcal{E}_n(d_2) := \min \{f_n(d_1), f_n(d_2), f_n(d_3)\} - \max \{d_2 - d_1, d_3 - d_2\}$$

and

$$\mathcal{E}^n(d_2) := \max \{f_n(d_1), f_n(d_2), f_n(d_3)\} + \max \{d_2 - d_1, d_3 - d_2\}.$$

Set $\mathcal{E}_n$ and $\mathcal{E}^n$ linear between adjacent points of D_n. This definition immediately implies that

$$f_n, \mathcal{E}_n, \mathcal{E}^n \text{ and } D_n \text{ satisfy (4.16) on } I', \tag{4.41}$$

and

$$\mathcal{E}_n \le f_n \le \mathcal{E}^n \text{ on } I = (a, b). \tag{4.42}$$

By (4.30) and (4.40) we obtain that

$$\begin{gathered}\text{the right derivatives of } \mathcal{E}_n \text{ and } \mathcal{E}^n \text{ are 0 in } a' \text{ and}\\ \text{the left derivatives of } \mathcal{E}_n \text{ and } \mathcal{E}^n \text{ are 0 in } b'.\end{gathered} \tag{4.43}$$

Recall that I was defined at (4.24). For every $x \in I = (a, b)$ we have that

$$\begin{aligned}\mathcal{E}_n(x) \ge& \min\{f_n(x') : x' \in [a, b]\}-\\ &- \max\{|d - d'| : d \text{ and } d' \text{ are adjacent elements of } D_n \cap I\}.\end{aligned}$$

Hence by (4.39), (4.25) and (4.19)

$$\begin{aligned}\mathcal{E}_n(x) \ge& \Big(\min\{f_n(a), f_n(b)\} - \frac{b-a}{2}\Big) - \frac{b-a}{2}\\ =& \min\{f_n(a), f_n(b)\} - (b - a) = \min\{f_{n-1}(a), f_{n-1}(b)\} - (b - a)\end{aligned}$$

thus by (4.16)

$$\mathcal{E}_n(x) \ge \min\{f_{n-1}(a), f_{n-1}(b)\} - (\min\{f_{n-1}(a), f_{n-1}(b)\} - \mathcal{E}_{n-1}(x)) = \mathcal{E}_{n-1}(x),$$

and similarly $\mathcal{E}^n(x) \le \mathcal{E}^{n-1}(x)$. Hence (4.42) implies that

$$\circledast \text{ (G) holds on } I \text{ for } n, \tag{4.44}$$

since it held in the previous steps of the induction.

Take a component $I' = (a', b')$ of $I \cap G_n \setminus \{I_1, \dots, I_{k^*}\}$. Let $x \in (a', a'_0)$. We want to prove that

$$\frac{|f_n(x) - f_n(a')|}{x - a'} \leq \frac{1}{n + l' - 1}.$$

We know that f_n is linear between a'_k and a'_{k-1} for every $k \in \mathbb{N}$, there are infinitely many local extremum points in $\{a'_0, a'_1, \dots\}$ by (4.36), and $\mathrm{Lip}(f_n) = 1 - \frac{1}{n}$ on (a', a'_0) by (4.31). Consequently, we can assume that x is a local extremum point of f_n, i.e. $x = a'_{k_x}$ for some $k_x \in \mathbb{N}$. We can also suppose without loss of generality that $f_n(a'_{k_x-1}) > f_n(a')$, hence by (4.36), $f_n(a'_{k_x}) \leq f_n(a')$. Thus

$$\begin{aligned}
\frac{|f_n(x) - f_n(a')|}{x - a'} &= \frac{|f_n(a'_{k_x}) - f_n(a')|}{a'_{k_x} - a'} \leq \frac{|f_n(a'_{k_x}) - f_n(a'_{k_x-1})|}{a'_{k_x} - a'} \leq \frac{a'_{k_x-1} - a'_{k_x}}{a'_{k_x} - a'} \\
&= \frac{a'_{k_x-1} - a'_{k_x}}{a'_{k_x-1} - a' - (a'_{k_x-1} - a'_{k_x})} \\
&\underset{\text{by (4.28)}}{\leq} \frac{a'_{k_x-1} - a'_{k_x}}{(n + l')(a'_{k_x-1} - a'_{k_x}) - (a'_{k_x-1} - a'_{k_x})} = \frac{1}{n + l' - 1}.
\end{aligned}$$

We can prove similarly for every $x \in (a', b')$ that

$$\max\left\{\frac{|f_n(x) - f_n(a')|}{x - a'}, \frac{|f_n(x) - f_n(b')|}{b' - x}\right\} \leq \frac{1}{n + l' - 1}. \tag{4.45}$$

Let $d_1, d_2, d_3, d_4 \in D_n \cap I'$ be adjacent and increasing in this order. If $x \in [d_2, d_3]$ then by the definition of $\mathcal{E}^n$ we have that

$$\begin{aligned}
&\frac{\mathcal{E}^n(x) - \mathcal{E}^n(a')}{x - a'} \leq \frac{\max\{\mathcal{E}^n(d_2), \mathcal{E}^n(d_3)\} - \mathcal{E}^n(a')}{d_2 - a'} \\
&\leq \frac{\max\{f_n(d_1), f_n(d_2), f_n(d_3), f_n(d_4)\} + \max\{d_2 - d_1, d_3 - d_2, d_4 - d_3\} - \mathcal{E}^n(a')}{d_2 - a'}.
\end{aligned} \tag{4.46}$$

By (4.45)

$$\begin{aligned}
&\max\{f_n(d_1), f_n(d_2), f_n(d_3), f_n(d_4)\} - \mathcal{E}^n(a') \\
&= \max\{f_n(d_1), f_n(d_2), f_n(d_3), f_n(d_4)\} - f_n(a') \leq \frac{1}{n + l' - 1}(d_4 - a'),
\end{aligned}$$

by (4.28)

$$\max\{d_2 - d_1, d_3 - d_2, d_4 - d_3\} \leq \frac{1}{n + l'}(d_4 - a')$$

and

$$d_2 - a' = \frac{d_2 - a'}{d_3 - a'} \cdot \frac{d_3 - a'}{d_4 - a'} \cdot (d_4 - a') \geq \left(1 - \frac{1}{n + l'}\right)^2 (d_4 - a').$$

Writing these inequalities into (4.46) we have

$$\frac{\mathcal{E}^n(x)-\mathcal{E}^n(a')}{x-a'} \le \frac{\frac{1}{n+l'-1}(d_4-a')+\frac{1}{n+l'}(d_4-a')}{(1-\frac{1}{n+l'})^2(d_4-a')} = \frac{\frac{1}{n+l'-1}+\frac{1}{n+l'}}{(1-\frac{1}{n+l'})^2}$$
$$\le \frac{\frac{1}{n+l'-1}+\frac{1}{n+l'-1}}{(1-\frac{1}{n+l'})^2} \le \frac{\frac{1}{n+l'-1}+\frac{1}{n+l'-1}}{(\frac{1}{2})^2} \le \frac{8}{n+l'-1}.$$

We can prove similarly that for every $x \in I'$

$$\max\left\{\frac{|\mathcal{E}_n(x)-\mathcal{E}_n(a')|}{x-a'}, \frac{|\mathcal{E}_n(x)-\mathcal{E}_n(b')|}{b'-x}, \frac{|\mathcal{E}^n(x)-\mathcal{E}^n(a')|}{x-a'}, \frac{|\mathcal{E}^n(x)-\mathcal{E}^n(b')|}{b'-x}\right\} < \frac{8}{n+l'-1}. \tag{4.47}$$

Next we show that the right derivative of $\mathcal{E}^n$ is 0 on $F_n \cap [a,b)$. Let $\varepsilon > 0$ and $x \in I \setminus G_n$. Suppose that x is not the left endpoint of a component of G_n (by (4.43), in such endpoints $\mathcal{E}^n$ has 0 right derivative). Then there is a positive δ such that

$$(x, x+\delta) \cap (I_1 \cup \ldots \cup I_{k^*} \cup \bigcup_{l=1}^{\lceil 8\varepsilon^{-1}\rceil} \bigcup_{k=1}^{\infty} I_n^l) = \emptyset. \tag{4.48}$$

Take an arbitrary $y \in (x, x+\delta)$. If $y \in I \setminus G_n$, then $\mathcal{E}^n(x) = \mathcal{E}^n(y)$ by (4.25) and (4.48). Otherwise, we denote by $J = (a_J, b_J)$ the component of G_n, which contains y. By (4.25) and (4.48), we have that $\mathcal{E}^n(x) = \mathcal{E}^n(a_J)$, and (4.47) and (4.48) implies

$$\frac{|\mathcal{E}^n(y)-\mathcal{E}^n(a_J)|}{y-a_J} \le \frac{8}{n+8\lceil\varepsilon^{-1}\rceil+1-1},$$

hence

$$\frac{|\mathcal{E}^n(y)-\mathcal{E}^n(x)|}{y-x} = \frac{|\mathcal{E}^n(y)-\mathcal{E}^n(a_J)|}{y-x} < \frac{|\mathcal{E}^n(y)-\mathcal{E}^n(a_J)|}{y-a_J}$$
$$\le \frac{8}{n+8\lceil\varepsilon^{-1}\rceil+1-1} \le \varepsilon.$$

It can be verified similarly that the left derivative of $\mathcal{E}^n$ is 0 in $(a,b]$, and the same procedure works for $\mathcal{E}_n$. As I is an arbitrary interval contiguous to $F_{n-1} \cup D_{n-1}$, by (4.18) we have that $\mathcal{E}_n' = (\mathcal{E}^n)' = 0$ on $F_n \setminus F_{n-1}$. Hence (4.44) and the induction hypothesis imply that $\circledast$ we have proved (D) on I.

The places marked by $\circledast$ imply that all (A), (B), (E), (C), (D), (F) and (G) are satisfied for n, and by induction for all ns. Moreover, all the assumptions (4.16), (4.17) and (4.18) of the next induction step are satisfied by (4.41), (4.38) and (4.37). This concludes the proof. □

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