Big and little Lipschitz one sets

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Abstract

We denote the local "little" and "big" Lipschitz constants of a function $f : \mathbb{R} \to \mathbb{R}$ by $\operatorname{lip} f$ and $\operatorname{Lip} f$. In this paper we are interested in the following question. Given a set $E \subset \mathbb{R}$ is it possible to find a continuous function f such that $\operatorname{lip} f = \mathbf{1}_E$ or $\operatorname{Lip} f = \mathbf{1}_E$?

For monotone continuous functions we provide the rather straightforward answer.

For arbitrary continuous functions the answer is much more difficult, and a complete answer is not known. We introduce the concept of uniform density type (UDT) and show that if E is G_δ and UDT then there exists a continuous function f satisfying $\operatorname{Lip} f = \mathbf{1}_E$, that is, E is a $\operatorname{Lip} 1$ set.

We also verify that there exist weakly dense G_δ sets which are not $\operatorname{Lip} 1$.

1 Introduction

Throughout this note we assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then the so-called "big Lip" and "little lip" functions are defined as follows:

$$\operatorname{Lip} f(x) = \limsup_{r\to 0^+} M_f(x,r), \qquad \operatorname{lip} f(x) = \liminf_{r\to 0^+} M_f(x,r), \tag{1.1}$$

where

$$M_f(x,r) = \frac{\sup\{|f(x)-f(y)| : |x-y| \le r\}}{r}.$$

We also define

$$L_f = \{x \in \mathbb{R} : \operatorname{Lip} f(x) < \infty\} \text{ and } l_f = \{x \in \mathbb{R} : \operatorname{lip} f(x) < \infty\}.$$

The behaviour of the two functions, $\operatorname{Lip} f$ and $\operatorname{lip} f$, is intimately related to the differentiability of f. For example, the Rademacher-Stepanov Theorem [10] tells us that if $\mathbb{R} \setminus L_f$ has measure zero, then f is differentiable almost everywhere on $\mathbb{R}$. On the other hand, in ([3], 2006) Balogh and Csörnyei construct a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{lip} f = 0$ almost everywhere, but f is nowhere differentiable. However, in the same paper, they also show that if $\mathbb{R} \setminus l_f$ is countable and $\operatorname{lip} f$ is locally integrable, then f is again differentiable almost everywhere on $\mathbb{R}$.

More recently, progress has been made on characterizing the sets L_f and l_f for continuous functions ([5], 2018) and characterizing the sets of non-differentiability for continuous functions with either $L_f = \mathbb{R}$ or $l_f = \mathbb{R}$ ([8], 2016). There are still a number of open problems concerning the relationship between L_f (l_f) and the differentiability properties of f.

We also mention the very recent result ([11], 2019) about little Lipschitz maps of analytic metric spaces with sufficiently high packing dimension onto cubes in $\mathbb{R}^n$.

It is an interesting problem to characterize the functions $\operatorname{Lip} f$ and $\operatorname{lip} f$ for continuous functions f. In this note, we take a first step in this direction by investigating when it is possible for $\operatorname{Lip} f$ (or $\operatorname{lip} f$) to be a characteristic function. Given a set $E \subset \mathbb{R}$ we say that E is Lip 1 (lip 1) if there is a continuous function defined on $\mathbb{R}$ such that $\operatorname{Lip} f = \mathbf{1}_E$, ($\operatorname{lip} f = \mathbf{1}_E$). So we are interested in determining which sets E are Lip 1 or lip 1.

It turns out that it is straightforward to decide this in the special case where f is monotone. We say that E is *monotone* Lip 1 (lip 1) if there is a continuous, monotone function f such that $\operatorname{Lip} f = \mathbf{1}_E$ ($\operatorname{lip} f = \mathbf{1}_E$). In Theorems 3.1 and 3.6 we show that monotone Lip 1 and lip 1 sets can be characterized using simple density conditions. The details for this are laid out in Section 3.

In Section 4 we see that Lip 1 sets are weakly dense G_δ sets (Theorem 4.1) and lip 1 sets are strongly one-sided dense F_σ sets (Theorem 4.7). In Theorem 4.3 we show that a certain ternary decomposition is necessary and sufficient for Lip 1. In Theorem 4.8 it is proved that countable disjoint unions of closed and strongly one-sided dense sets are lip 1.

In Section 5 we consider the more difficult problem of characterizing general Lip 1 sets. Given a measurable set, we introduce a two-parameter family of sets describing its levels of density and use this to define uniform density type (UDT) sets.

Definition 1.1. Suppose that $E \subseteq \mathbb{R}$ is measurable and $\gamma, \delta > 0$. Let

$$E^{\gamma,\delta} = \left\{ x \in \mathbb{R} : \max\left\{ \frac{|(x-r,x) \cap E|}{r}, \frac{|(x,x+r) \cap E|}{r} \right\} \geq \gamma \ \ \forall r : 0 < r \leq \delta \right\},$$

where $|E|$ denotes the Lebesgue measure of the set E.

We say that E has uniform density type (UDT) if there exist sequences $\gamma_n \nearrow 1$ and $\delta_n \searrow 0$ such that $E \subseteq \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E^{\gamma_n,\delta_n}$.

Our main result from Section 5, Theorem 5.5, states that G_δ sets which are UDT are Lip 1.

Finally, in Section 6 we show that the UDT condition in Theorem 5.5 cannot be replaced with one of the weaker density conditions from Section 3.

2 Preliminary definitions and results

The union of disjoint sets A and B is denoted by $A \sqcup B$. For any $S, T \subset \mathbb{R}$ and $x \in \mathbb{R}$ we define $d(S,T)$ to be the distance from S to T, that is $\inf\{|x-y| : x \in S,\ y \in T\}$. Let $d(x,S) = d(\{x\}, S)$.

In the space of continuous functions defined on an interval I we use the supremum norm $||f|| = \sup\{|f(x)| : x \in I\}$ and the metric and topology generated by this norm. In some of our arguments we will need a finer notion, which is not a neighborhood in the usual sense. We will call it vicinity:

Definition 2.1. By a vicinity U of a function $f\colon \mathbb{R} \to \mathbb{R}$ we mean a set of functions of the following form:

$$U = \{g : \forall x \ |f(x) - g(x)| \leq r(x)\},$$

where $r(x)$ is a fixed, continuous, nonnegative function, called the radius of U.

Definition 2.2. Given a sequence of non-degenerate closed intervals $\{I_n\}$, we write $I_n \to x$ if $x \in I_n$ for all $n \in \mathbb{N}$ and $|I_n| \to 0$.

Definition 2.3. The measurable set E is *weakly dense* at x if there exists $I_n \to x$ such that $\frac{|E \cap I_n|}{|I_n|} \to 1$. The set E is *weakly dense* if E is weakly dense at x for each $x \in E$.

The set E is *strongly dense* at x if for every sequence $\{I_n\}$ such that $I_n \to x$ we have $\frac{|E \cap I_n|}{|I_n|} \to 1$. We say that E is *strongly dense* if E is strongly dense at x for each $x \in E$.

Note: E being strongly dense at x, just means that x is a point of density of E.

In this paper a.e. always means Lebesgue almost everywhere.

Lemma 2.4. *Suppose that $E \subset \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ are such that $\operatorname{Lip} f = \mathbf{1}_E$. Then f is a Lipschitz function and $|f(x) - f(y)| \leq |[x, y] \cap E|$ for every $x, y \in \mathbb{R}$ (where $x < y$).*

Proof. Suppose that there exist $x_0 < y_0$ such that $|f(y_0) - f(x_0)| = \gamma |y_0 - x_0|$ with $\gamma > 1$. Then one can find a nested sequence of intervals $[x_n, y_n]$, $n = 0, 1, ...$ such that $|f(y_n) - f(x_n)| \geq \gamma |y_n - x_n|$ and $y_n - x_n = 2^{-n}(y_0 - x_0)$ hold for any n. Let $x^* = \cap_{n=0}^{\infty} [x_n, y_n]$. Clearly, $\operatorname{Lip} f(x^*) \geq \gamma > 1$, a contradiction.

Since f is Lipschitz it is absolutely continuous. Therefore f' exists almost everywhere and $f(y) - f(x) = \int_x^y f'(t)\,dt$. Since $|f'|$ equals $\mathbf{1}_E$ a.e. we obtain that $|f(y) - f(x)| \leq \int_x^y \mathbf{1}_E(t)\,dt = |[x, y] \cap E|$. □

3 Necessary and/or sufficient conditions for monotone Lip 1 and lip 1 sets

For monotone Lip 1 and lip 1 sets it is rather easy to obtain necessary and sufficient conditions.

Theorem 3.1. *The set E is monotone* Lip 1 *if and only if E is weakly dense and E^c is strongly dense.*

Proof. Assume that E is monotone Lip 1. Then we can choose a continuous, monotone increasing function f such that $\mathrm{Lip} f = \mathbf{1}_E$. By Lemma 2.4 f is absolutely continuous. Since $\mathrm{Lip} f = \mathbf{1}_E$ and f is increasing, we conclude that $f'(x) = \mathbf{1}_E(x)$ a.e. and we have

$$f(y) - f(x) = \int_x^y \mathbf{1}_E(t)\, dt = |E \cap [x, y]| \text{ for all } x < y. \tag{3.1}$$

From (3.1) and the definition of $\mathrm{Lip} f$ it is straightforward to show that E is weakly dense and E^c is strongly dense.

Now assume that E is weakly dense and E^c is strongly dense. Then let $f(x) = \int_{x_0}^x \mathbf{1}_E(t)\, dt$ by selecting an arbitrary x_0. It is straightforward to show that $\mathrm{Lip} f = \mathbf{1}_E$ and therefore E is monotone Lip 1. ☐

For the characterization of monotone lip 1 sets we need a few new definitions:

Definition 3.2. Suppose that $I_n \to x$. If each I_n is centered at x we say that $\{I_n\}$ *center converges to* x and we write $I_n \xrightarrow{c} x$.

Definition 3.3. The set E is *weakly center dense* at x if there exists a sequence $\{I_n\}$ such that $I_n \xrightarrow{c} x$, and $\frac{|E \cap I_n|}{|I_n|} \to 1$. The set E is *weakly center dense* if E is weakly center dense at every point $x \in E$.

Definition 3.4. The set E is *strongly one-sided dense* at x if for any sequence $\{I_n\} = \{[x - r_n, x + r_n]\}$ such that $I_n \to x$, we have $\max\{\frac{|E \cap [x - r_n, x]|}{r_n}, \frac{|E \cap [x, x + r_n]|}{r_n}\} \to 1$. The set E is *strongly one-sided dense* if E is strongly one-sided dense at every point $x \in E$.

Remark 3.5. The observant reader will note that we have not defined *strongly center dense* or *weakly one-sided dense*. The reason for this is that defining these terms in the obvious way would be redundant since strongly center dense sets would be equivalent to strongly dense sets and weakly one-sided dense sets would be equivalent to weakly dense sets. We also observe that the following implications hold:

$$\text{strongly dense} \;\Rightarrow\; \text{strongly one-sided dense,}$$
$$\text{weakly center dense} \;\Rightarrow\; \text{weakly dense .}$$

Note that neither of the above implications is reversible: a closed interval is strongly one-sided dense and weakly dense, but not strongly dense or weakly center dense.

Theorem 3.6. *The set E is monotone* lip 1 *if and only if E is strongly one-sided dense and E^c is weakly center dense.*

Proof. The proof of Theorem 3.6 is similar to the proof of Theorem 3.1. ☐

4 Necessary and/or sufficient conditions for general Lip 1 and lip 1 sets

In Theorem 4.1 we give a necessary condition for a set to be Lip 1. We will see in Section 6 (Theorem 6.3) that this condition is not sufficient.

Theorem 4.1. *If* $E \subset \mathbb{R}$ *is* Lip 1 *then* E *is a weakly dense* G_δ *set.*

Proof. Suppose that E is Lip 1. Lemma 2.4 implies that E is weakly dense. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\operatorname{Lip} f = \mathbf{1}_E$. As

$$E = \bigcap_{n=1}^{\infty} \left\{ x \in \mathbb{R} : \text{ there exists } r \in \left(0, \frac{1}{n}\right) \text{ such that } M_f(x, r) > 1 - \frac{1}{n} \right\}$$

and the sets on the right are open, we obtain that E is G_δ. □

The next definition will be used to obtain a necessary and sufficient condition for Lip 1 sets in Theorem 4.3.

Definition 4.2. Let E be a measurable subset of $\mathbb{R}$ and suppose that E_1, E_0, E_{-1} are pairwise disjoint sets whose union is $\mathbb{R}$. Then we say that E_1, E_0, E_{-1} is a *ternary decomposition of* $\mathbb{R}$ with respect to E if the following conditions hold:

$$\forall x \in E \text{ either } E_1 \text{ or } E_{-1} \text{ is weakly dense at } x, \tag{4.1}$$

$$\forall x \in E^c \text{ and } \forall I_n \to x \text{ we have } \frac{||E_1 \cap I_n| - |E_{-1} \cap I_n||}{|I_n|} \to 0. \tag{4.2}$$

If E_1, E_0, E_{-1} is a ternary decomposition of $\mathbb{R}$ with respect to E we write $E \sim (E_1, E_0, E_{-1})$.

Theorem 4.3. *A set* E *is* Lip 1 *if and only if there is a ternary decomposition of* $\mathbb{R}$ *with respect to* E*.*

Proof. Suppose that $E \sim (E_1, E_0, E_{-1})$. Define

$$f(x) = \int_0^x \mathbf{1}_{E_1}(t) - \mathbf{1}_{E_{-1}}(t)\, dt.$$

Then straightforward calculations show that $\operatorname{Lip} f = \mathbf{1}_E$.

Working in the opposite direction, now assume that $\operatorname{Lip} f = \mathbf{1}_E$. Then by Lemma 2.4, f is absolutely continuous and hence f is differentiable almost everywhere and wherever $f'(x)$ is defined $f'(x)$ is equal to either $1, 0$ or -1. For

$i = 1, -1$ define $E_i = \{x : f'(x) = i\}$ and let $E_0 = \mathbb{R} \backslash (E_1 \cup E_{-1})$. By absolute continuity of f we have that

$$f(x) = f(0) + \int_0^x f'(t)\,dt = f(0) + \int_0^x \mathbf{1}_{E_1}(t) - \mathbf{1}_{E_{-1}}(t)\,dt$$

and it is straightforward to show that $E \sim (E_1, E_0, E_{-1})$. □

Remark 4.4. Suppose that $E \sim (E_1, E_0, E_{-1})$. Then we can find F_1, F_0, F_{-1} such that $E \sim (F_1, F_0, F_{-1})$ and $E = F_1 \cup F_{-1}$.

To see the truth of the Remark 4.4 assume that $E \sim (E_1, E_0, E_{-1})$. Recalling that almost every element in a set is a density point of the set, it follows from the definition of a ternary decomposition that $|E_i \backslash E| = 0$ for $i = 1, -1$ and $|E_0 \cap E| = 0$. Thus, if we define $F_1 = (E_1 \cup E_0) \cap E$, $F_{-1} = E_{-1} \cap E$, and $F_0 = \mathbb{R} \setminus E$, then we have $E \sim (F_1, F_0, F_{-1})$ and $E = F_1 \cup F_{-1}$.

Remark 4.5. Although Theorem 4.3 gives a characterization of Lip 1 sets, it is not always easy to verify whether or not a given set E has a ternary decomposition. One simple example is $E = (0, \infty)$. In this case, one can verify that $E_0 = (-\infty, 0]$, $E_{-1} = \cup_{n=1}^{\infty} (\frac{1}{2n+1}, \frac{1}{2n}]$, $E_1 = (\cup_{n=1}^{\infty} (\frac{1}{2n}, \frac{1}{2n-1}]) \cup (1, \infty)$ gives a ternary decomposition of $\mathbb{R}$ with respect to E and therefore E is Lip 1.

Next we want to find some necessary and some sufficient conditions for lip 1 sets. First we state and prove the lip 1 version of Lemma 2.4.

Lemma 4.6. *If $E \subset \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{lip} f = \mathbf{1}_E$ then $|f(a) - f(b)| \le |[a, b] \cap E|$ for every $a, b \in \mathbb{R}$ (where $a < b$).*

Proof. Let $\varepsilon > 0$. For every $x \in \mathbb{R}$ we fix $r_x \in (0, \varepsilon)$ such that $M_f(x, r_x) < 1 + \varepsilon$. We select a finite set $H \subset \mathbb{R}$ for which $\{(x - r_x, x + r_x) : x \in H\}$ is a minimal cover of $[a, b]$ that is every $y \in \mathbb{R}$ is contained by at most two of these open intervals. If $x \in \mathbb{R}$, $r > 0$ and $y \in (x - r, x + r)$, we have $|f(x) - f(y)| \le r M_f(x, r)$. Thus

$$|f(a) - f(b)| \le \sum_{x \in H} 2 r_x M_f(x, r_x) \le \sum_{x \in H} 2 r_x (1 + \varepsilon) \le 2(b + \varepsilon - (a - \varepsilon))(1 + \varepsilon).$$

Hence f is Lipschitz as a, b and ε were chosen arbitrarily.

Since f is Lipschitz it is absolutely continuous. Therefore f' exists almost everywhere and $f(b) - f(a) = \int_a^b f'(t)\,dt$. Since $|f'|$ equals $\mathbf{1}_E$ a.e. we obtain that $|f(b) - f(a)| \le \int_a^b \mathbf{1}_E(t)\,dt = |[a, b] \cap E|$. □

Theorem 4.7. *If $E \subset \mathbb{R}$ is* lip 1 *then E is a strongly one-sided dense F_σ set.*

Proof. Suppose that E is lip 1. Lemma 4.6 implies that E is strongly one-sided dense. Let $f : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{lip} f = \mathbf{1}_E$. As

$$E^c = \bigcap_{n=1}^{\infty} \left\{ x \in \mathbb{R} : \text{ there exists } r \in \left(0, \frac{1}{n}\right) \text{ such that } M_f(x,r) < \frac{1}{2} \right\}$$

and the sets on the right are open, we obtain that E^c is G_δ hence E is F_σ. □

After the above necessary condition here is a sufficient condition for lip 1.

Theorem 4.8. *Suppose that $E = \sqcup_{n=1}^{\infty} E_n$ where for each $n \in \mathbb{N}$, E_n is closed and strongly one-sided dense. Then E is* lip 1.

We should note that simple examples show that Theorem 4.8 does not provide a characterization of lip 1 sets. For example, non-empty, open sets are lip 1, but no non-empty, open set can be expressed as a disjoint, countable union of closed sets.

Remark 4.9. Note that if E is dense in $\mathbb{R}$ and each E_n is nowhere dense, then E is not Lip 1. Indeed, proceeding towards a contradiction suppose that E is Lip 1 and f is a continuous function verifying this property. For every $x \in E^c$ there exists $\delta_x > 0$ such that for any h with $|h| < \delta_x$ we have

$$|f(x+h) - f(x)| < \frac{1}{4}|h|.$$

By Baire's category theorem there exists $n \in \mathbb{N}$ and an interval (a,b), $a < b$ such that

$$\Delta_n = \{x \in E^c : \delta_x > \frac{1}{n}\}$$

is dense in (a,b). Then $\operatorname{Lip} f(x) \leq \frac{1}{2}$ for any $x \in (a,b)$ and hence $E \cap (a,b) = \emptyset$, which contradicts the density of E.

The proof of Theorem 4.8 will depend on the following:

Lemma 4.10. *Suppose that E is closed and strongly one-sided dense. Let $\varepsilon > 0$. Then there exists a continuous function f such that*

(i) $\operatorname{lip} f = \mathbf{1}_E$,

(ii) $0 \leq f(x) \leq \varepsilon$ *for all* $x \in \mathbb{R}$.

Proof of Lemma 4.10. For every $i \in \mathbb{Z}$ define $E_i = [(i-1)\varepsilon, i\varepsilon] = [a_{i-1}, a_i]$ and choose $x_i \in E_i$ such that $|E \cap [a_{i-1}, x_i]| = |E \cap [x_i, a_i]|$. For each $i = 1, 2, \ldots, n$ define $E_i^+ = E \cap [a_{i-1}, x_i]$ and $E_i^- = E \cap [x_i, a_i]$ and let $E^+ = \cup_{i=-\infty}^{\infty} E_i^+$ and $E^- = \cup_{i=-\infty}^{\infty} E_i^-$. Define $f(x) = \int_0^x \mathbf{1}_{E^+}(t) - \mathbf{1}_{E^-}(t)\, dt$. Then it is straightforward to verify that (i) and (ii) hold. □

Proof of Theorem 4.8. Assume that $E = \sqcup_{n=1}^{\infty} E_n$, where each E_n is closed and strongly one-sided dense. By redefining $E_1, E_2, \dots$ we can suppose that if $n \geq 2$ then E_n is bounded.

Using Lemma 4.10 we choose f_1 such that (i) and (ii) of Lemma 4.10 hold with f replaced by f_1, E replaced by E_1 and $\varepsilon = 1$. Again using Lemma 4.10 for each $n \in \mathbb{N} \cap [2, \infty)$ we choose $f_n \geq 0$ such that (i) holds with f replaced by f_n and E replaced with E_n such that

$$0 \leq f_n(x) \leq 2^{-n} \min \left\{ 1, d\left(E_n, \bigcup_{k=1}^{n-1} E_k \right) \right\} \tag{4.3}$$

(we note that the right-hand side is positive, as E_n is compact and $\cup_{k=1}^{n-1} E_k$ is closed). Obviously, for every $n \in \mathbb{N}$

$$f_n \text{ is constant on each interval contiguous to } E_n. \tag{4.4}$$

Let $f(x) = \sum_{n=1}^{\infty} f_n$.

Suppose that $n_0 \in \mathbb{N}$, $\varepsilon > 0$ and $x \in E_{n_0}$. Let

$$n_1 := \max\{n_0 + 1, -\lfloor \log_2(\varepsilon) \rfloor\} \text{ and } r \in \big(0, d\big(\{x\}, \cup_{n \in \mathbb{N} \cap [1, n_1] \setminus \{n_0\}} E_n\big)\big). \tag{4.5}$$

For every $y \in [x - r, x + r]$ we have

$$\begin{aligned}
\left| \frac{|f(x) - f(y)|}{r} - \frac{|f_{n_0}(x) - f_{n_0}(y)|}{r} \right| &\underset{\text{using (4.4)}}{\leq} r^{-1} \sum_{\substack{n \in \mathbb{N} \setminus \{n_0\} \\ E_n \cap (x-r, x+r) \neq \emptyset}} \sup_{t \in \mathbb{R}} f_n(t) \\
\underset{\text{using (4.3)}}{\leq} r^{-1} \sum_{\substack{n \in \mathbb{N} \setminus \{n_0\} \\ E_n \cap (x-r, x+r) \neq \emptyset}} 2^{-n} d(E_n, E_{n_0}) &\leq r^{-1} \sum_{\substack{n \in \mathbb{N} \setminus \{n_0\} \\ E_n \cap (x-r, x+r) \neq \emptyset}} 2^{-n} d(E_n, \{x\}) \\
\leq r^{-1} \sum_{\substack{n \in \mathbb{N} \setminus \{n_0\} \\ E_n \cap (x-r, x+r) \neq \emptyset}} 2^{-n} r &\underset{\text{using (4.5)}}{\leq} \sum_{n = n_1 + 1}^{\infty} 2^{-n} = 2^{-n_1} \leq \varepsilon.
\end{aligned}$$

Thus $\operatorname{lip} f(x) = \operatorname{lip} f_{n_0}(x) = 1$.

Now let $x \notin E$. If x is not an accumulation point of E then obviously $\operatorname{lip} f(x) = 0$. Otherwise, set $n_1 := \max\{1, -\lfloor \log_2(\varepsilon) \rfloor + 1\}$ and $r := d\big(\{x\}, \cup_{n \in \mathbb{N} \cap [1, n_1]} E_n\big)$.

We have that for every $y \in [x-r, x+r]$

$$\frac{|f(x)-f(y)|}{r} \underset{\text{using (4.4)}}{\leq} r^{-1} \sum_{\substack{n\in\mathbb{N} \\ E_n\cap(x-r,x+r)\neq\emptyset}} \sup_{t\in\mathbb{R}} f_n(t)$$

$$\underset{\text{using (4.3)}}{\leq} r^{-1} \sum_{\substack{n\in\mathbb{N} \\ E_n\cap(x-r,x+r)\neq\emptyset}} 2^{-n} d(E_n, \cup_{k\in\mathbb{N}\cap[1,n_1]} E_k)$$

$$\leq r^{-1} \sum_{\substack{n\in\mathbb{N} \\ E_n\cap(x-r,x+r)\neq\emptyset}} 2^{-n}\cdot 2r \leq \sum_{n=n_1+1}^{\infty} 2\cdot 2^{-n} = 2\cdot 2^{-n_1} \leq \varepsilon.$$

Since $r \to 0$ as $n_1 \to \infty$ (and $n_1 \to \infty$ as $\varepsilon \to 0$), we obtain that $\operatorname{lip} f(x) = 0$. □

5 G_δ uniform density type sets are Lip 1

Recall that the sets $E^{\gamma,\delta}$ were defined in Definition 1.1.

Lemma 5.1. *For any $\gamma, \delta > 0$ the set $E^{\gamma,\delta}$ is closed.*

Proof. For $r > 0$ we introduce the notation

$$E_r^\gamma = \left\{ x \in \mathbb{R} : \max\left\{ \frac{|(x-r,x)\cap E|}{r}, \frac{|(x,x+r)\cap E|}{r} \right\} \geq \gamma \right\}.$$

Then we obviously have

$$E^{\gamma,\delta} = \bigcap_{r:0<r\leq\delta} E_r^\gamma.$$

However, the functions $x \mapsto \frac{|(x-r,x)\cap E|}{r}$ and $x \mapsto \frac{|(x,x+r)\cap E|}{r}$ are obviously continuous for any r, hence

$$x \mapsto \max\left\{ \frac{|(x-r,x)\cap E|}{r}, \frac{|(x,x+r)\cap E|}{r} \right\}$$

is also continuous, which immediately yields that each upper level set E_r^γ is closed. Consequently, their intersection $E^{\gamma,\delta}$ is also closed. □

Proposition 5.2. *UDT sets are strongly one-sided dense.*

Proof. Suppose $x \in E$ and $\gamma < 1$. Choose k such that $\gamma_n > \gamma$ when $n \geq k$. Then there exists $n(\gamma, x) \geq k$ such that $x \in E^{\gamma_{n(\gamma,x)},\delta_{n(\gamma,x)}}$ and

$$\max\left\{\frac{|(x-r,x)\cap E|}{r}, \frac{|(x,x+r)\cap E|}{r}\right\} > \gamma_{n(\gamma,x)} > \gamma \text{ holds for } 0 < r < \gamma_{n(\gamma,x)}.$$

Since this is true for any $0 < \gamma < 1$ we see that E is strongly one-sided dense at x. ◻

In Definition 1.1 we considered $\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E^{\gamma_n,\delta_n}$, which is the lim sup of the sequence E^{γ_n,δ_n}. By taking the lim inf we arrive at the following definition:

Definition 5.3. We say that E has strong uniform density type (SUDT) if there exist sequences $\gamma_n \nearrow 1$ and $\delta_n \searrow 0$ such that $E \subseteq \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} E^{\gamma_n,\delta_n}$.

Proposition 5.4. *Let any arising set be a measurable subset of* $\mathbb{R}$.

(i) If a set E has SUDT then it also has UDT.

(ii) Any interval has SUDT (and hence UDT).

(iii) If $E_1, E_2, ...$ have UDT (resp. SUDT) then $E = \bigcup_{m=1}^{\infty} E_m$ also has UDT (resp. SUDT).

(iv) There exists E which has SUDT but its closure $\overline{E}$ is not strongly one-sided dense and hence does not have UDT.

Proof. The proofs of (i) and (ii) are obvious.

In (iii) we will examine the UDT case, the proof of the SUDT case is basically the same. Let us choose sequences existing by definition for each set E_m. Denote such a pair of sequences by $(\gamma_{m,n})_{n=1}^{\infty}, (\delta_{m,n})_{n=1}^{\infty}$. As we explain it below we can take sequences $(\gamma_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$ such that $\gamma_n \nearrow 1$ and $\delta_n \searrow 0$ and for every $m \in \mathbb{N}$ there is an $n_m \in \mathbb{N}$ for which

$$\text{for all } n > n_m \text{ we have } 0 < \delta_n < \delta_{m,n} \text{ and } \gamma_n < \gamma_{m,n} < 1. \tag{5.1}$$

Indeed, one possible way to choose δ_n and γ_n so that (5.1) is satisfied is the following. Choose a strictly increasing sequence $1 = n_1 < n_2 < ...$ such that for every $m \in \mathbb{N}$ we have

$$\gamma_{j,n_m} > 1 - \frac{1}{m} \text{ for } j = 1, ..., m.$$

For $n_m \leq n < n_{m+1}$ let $\gamma_n = 1 - \frac{1}{m}$ and $\delta_n = \min\{\delta_{1,n}, ..., \delta_{m,n}\}$. Then for $n > n_m$ there is $m' \geq m$ such that $n_{m'} \leq n < n_{m'+1}$ and $\gamma_{m,n} \geq \gamma_{m,n_{m'}} > 1 - \frac{1}{m'} = \gamma_n$. Moreover, $\delta_n = \min\{\delta_{1,n}, ..., \delta_{m',n}\} \leq \delta_{m,n}$.

Thus for every $m \in \mathbb{N}$ and $n > n_m$ we have that $E^{\gamma_{m,n},\delta_{m,n}} \subseteq E^{\gamma_n,\delta_n}$, hence

$$E_m \subseteq \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E^{\gamma_{m,n},\delta_{m,n}} = \bigcap_{k=n_m}^{\infty} \bigcup_{n=k}^{\infty} E^{\gamma_{m,n},\delta_{m,n}} \subseteq \bigcap_{k=n_m}^{\infty} \bigcup_{n=k}^{\infty} E^{\gamma_n,\delta_n} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E^{\gamma_n,\delta_n}.$$

This implies

$$E \subseteq \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E^{\gamma_n,\delta_n},$$

that is E has UDT.

Finally, for (iv) consider

$$E = \bigcup_{n=-\infty}^{\infty} \left[2^n - 2^{n-2}, 2^n\right].$$

By (iii), E has SUDT as it is a countable union of intervals. Meanwhile its closure is $\overline{E} = E \cup \{0\}$. But in intervals of the form $(0, 2^n - 2^{n-2})$, $n \in \mathbb{Z}$ the set $\overline{E}$ has density

$$\frac{1}{2^n - 2^{n-2}} \sum_{k=-\infty}^{n-1} 2^{k-2} = \frac{2^{n-2}}{2^n - 2^{n-2}} = \frac{1}{3}.$$

Meanwhile for any interval of the form $(-r, 0)$ for $r > 0$ the set $\overline{E}$ has density 0. Consequently, $0 \in \overline{E}$ witnesses that $\overline{E}$ does not have UDT.

□

Theorem 5.5. *Assume that E is G_δ and E has UDT. Then there exists a continuous function f satisfying* $\operatorname{Lip} f = \mathbf{1}_E$*, that is, the set E is* Lip 1.

In order to prove the theorem we will need a definition and a couple of technical lemmas:

Definition 5.6. Suppose that f is continuous on the interval $[a, b]$ and f_l, f_u are continuous on $[a, b]$ with $f_l < f < f_u$ on (a, b) and $f_l(a) = f_u(a) = f(a)$ and $f_l(b) = f_u(b) = f(b)$. Then we say that (f_l, f_u) is an *envelope* for f on $[a, b]$ and we write $f \in (f_l, f_u)$ on $[a, b]$.

For each of the following lemmas we assume that E is as in the statement of Theorem 5.5 and that $\phi(x) = \int_0^x \mathbf{1}_E(t)\, dt$.

Lemma 5.7. *Assume that f is continuous and monotone on $[a, b]$ and $f \in (f_l, f_u)$ on $[a, b]$. Furthermore, let $0 < \delta < \epsilon \leq 1$ and assume that*

$$|f(x) - f(y)| \leq (1 - \epsilon)|\phi(x) - \phi(y)| \text{ for all } x, y \in [a, b]. \tag{5.2}$$

Then, there exists a continuous function g on $[a, b]$ such that

- *$g \in (f_l, f_u)$ on $[a, b]$,*
- *g is locally piecewise monotone on $[a, b]$,*
- *On any interval of monotonicity of g there exists a constant K depending only on the interval such that $g = K \pm (1-\delta)\phi$.*

Proof. We first note the following useful fact, which follows from the inequalities $0 < \delta < \epsilon$ and inequality (5.2):

Given any interval $[r, s] \subset (a, b)$ we can choose $t \in (r, s)$ such that

$$(1-\delta)(|E \cap [r, t]| - |E \cap [t, s]|) = f(s) - f(r). \tag{5.3}$$

Next, we note that in order to prove the lemma it suffices to prove that for any subinterval $[c, d] \subset (a, b)$ we can construct a continuous function g such that

(i) $f_l(x) < g(x) < f_u(x)$ on $[c, d]$,

(ii) $g(c) = f(c)$ and $g(d) = f(d)$,

(iii) g is piecewise monotone on $[c, d]$,

(iv) $g = K \pm (1-\delta)\phi$ on each interval of monotonicity of g, recall that we use constants K which depend on the interval considered.

Assume that $[c, d] \subset (a, b)$. Let

$$\gamma = \inf_{c \le x \le d} \min\{(f_u(x) - f(x)), (f(x) - f_l(x))\} > 0.$$

Using the uniform continuity of f_u and f_l on $[c, d]$ choose a positive integer n such that

$$\frac{d-c}{n} < \frac{\gamma}{3} \text{ and for } x, y \in [c, d],\ |x - y| < \frac{d-c}{n} \text{ we have} \tag{5.4}$$

$$\max\{|f_u(x) - f_u(y)|, |f_l(x) - f_l(y)|\} < \frac{\gamma}{3}.$$

For $i = 0, 1, 2 \ldots n$ let $c_{2i} = c + i(\frac{d-c}{n})$ so we have $c = c_0 < c_2 < c_4 < \ldots < c_{2n} = d$. Using (5.3) for each $i = 1, 2, \ldots n$ we choose $c_{2i-1} \in (c_{2i-2}, c_{2i})$ such that

$$(1-\delta)(|E \cap [c_{2i-2}, c_{2i-1}]| - |E \cap [c_{2i-1}, c_{2i}]|) = f(c_{2i}) - f(c_{2i-2}). \tag{5.5}$$

Next, for each $j = 0, 1, 2, \ldots, 2n-1$ we define g in $[c_j, c_{j+1}]$ by

$$g(x) = (1-\delta)\int_{c_j}^{x} \mathbf{1}_E(t)\,dt + f(c_j) \text{ if } j \text{ is even}$$

and

$$g(x) = -(1-\delta)\int_x^{c_{j+1}} \mathbf{1}_E(t)\,dt + f(c_{j+1}) \text{ if } j \text{ is odd .}$$

We see that for $j = 0, 1, 2, \ldots, n-1$ we have g is monotone increasing on $[c_{2j}, c_{2j+1}]$ and monotone decreasing on $[c_{2j+1}, c_{2j+2}]$. Furthermore, $g = K \pm (1-\delta)\phi$ on each interval $[c_i, c_{i+1}]$ for $i = 0, 1, 2, \ldots 2n-1$. We also see that (ii) holds and (i) follows from inequality 5.4. This concludes the proof of the lemma. □

Lemma 5.8. *Suppose that f is continuous on $[a, b]$, $f \in (f_l, f_u)$ on $[a, b]$, and H is a closed set such that $H \subset (a, b)\backslash E$. Furthermore, assume that $0 < \delta < \epsilon \le 1$ and (5.2) holds. Then there exists a function g continuous on $[a, b]$ with*

(i) $g \in (f_l, f_u)$ *on* $[a, b]$,

(ii) $g(a) = f(a)$, $g(b) = f(b)$,

(iii) $g' = 0$ *on* H,

(iv) $|g(x) - g(y)| \le (1-\delta)|\phi(x) - \phi(y)|$ *for all* $x, y \in [a, b]$.

Proof. Write (a, b) as a countable union of non-overlapping closed intervals $[c, d]$ which satisfy

$$f_l(x) < \min\{f(c), f(d)\} \le \max\{f(c), f(d)\} < f_u(x) \text{ for all } x \in [c, d]. \tag{5.6}$$

Assume that $[c, d]$ is a closed subinterval of (a, b) satisfying (5.6). We treat the case where $c, d \in H$. The case where either c or d is not in H is handled similarly and is left to the reader. It suffices to show that we can define g on $[c, d]$ such that

$$g(c) = f(c) \text{ and } g(d) = f(d), \tag{5.7}$$

$$g' = 0 \text{ on } H \cap [c, d], \tag{5.8}$$

$$\text{(iv) holds with } [a, b] \text{ replaced by } [c, d], \tag{5.9}$$

and

$$f_l(x) < g(x) < f_u(x) \text{ for all } x \in [c, d]. \tag{5.10}$$

We note that if $E \cap (c, d) = \emptyset$, then from inequality (5.2) it follows that f is constant on $[c, d]$ and we simply define $g = f$ on $[c, d]$. Thus we may as well assume that $E \cap (c, d) \ne \emptyset$ and therefore $|E \cap (c, d)| > 0$. We also assume without loss of generality that $f(d) \ge f(c)$. Next we choose finitely many intervals $I_i = (c_{2i-1}, c_{2i})$, $i = 1, 2, \ldots, n$ which are contiguous to $H \cap [c, d]$ and such that $c \le c_1 < c_2 \le c_3 < c_4 \le \ldots \le c_{2n-1} < c_{2n} \le d$ and

$$(1-\delta)|E \cap (\cup_{i=1}^n [c_{2i-1}, c_{2i}])| > (1-\epsilon)|E \cap [c, d]| \ge f(d) - f(c). \tag{5.11}$$

Furthermore, we choose γ such that

$$\gamma|E \cap \cup_{i=1}^{n}[c_{2i-1}, c_{2i}]| = f(d) - f(c),$$

and note that $\gamma < 1-\delta$. Using this fact, on each interval $[c_{2i-1}, c_{2i}]$ one can define a monotone function g_i so that

$$g_i'(c_{2i-1}) = g_i'(c_{2i}) = 0$$

$$g_i(c_{2i-1}) = 0 \text{ and } g_i(c_{2i}) = \gamma|E \cap [c_{2i-1}, c_{2i}]|,$$

and

$$|g_i(x) - g_i(y)| \leq (1-\delta)|\phi(x) - \phi(y)| \text{ for all } x, y \in [c_{2i-1}, c_{2i}].$$

We also extend g_i to the entire interval $[c, d]$ by defining $g_i = 0$ on $[c, c_{2i-1}]$ and $g_i = g(c_{2i})$ on $[c_{2i}, d]$. Finally, we define $g = f(c) + \sum_{i=1}^{n} g_i$ on $[c, d]$. Then using (5.6) it is straightforward to verify that (5.7)-(5.10) hold and we are done with the proof.

□

Proof of Theorem 5.5. In this proof again we will use a constant K which will depend on the interval considered. Fix sequences γ_n and δ_n witnessing the UDT property of E. Let $E = \bigcap_{n=1}^{\infty} G_n$, where each set G_n is open and $G_{n+1} \subset G_n$ for all $n \in \mathbb{N}$. We also assume, as we may, that each component of G_n intersects E. We also denote the complement of G_n by F_n. Thus $(F_n)_{n=1}^{\infty}$ is an increasing sequence of closed sets. Let $\phi(x) = \int_0^x \mathbf{1}_E(t)dt$ be the integral function of the characteristic function of E. We will construct a sequence of functions $(f_n)_{n=1}^{\infty}$ together with a sequence of vicinities $(U_n)_{n=1}^{\infty}$ with the following properties: (Recall that the vicinity was defined in Definition 2.1.)

(i) f_n is differentiable on F_n and its derivative vanishes there.

(ii) For any $m \geq n$ we have $f_m \restriction_{F_n} = f_n \restriction_{F_n}$.

(iii) For any $x, y \in \mathbb{R}$ we have $|f_n(x) - f_n(y)| \leq (1 - 2^{-3n})|\phi(x) - \phi(y)|$.

(iv) If (a, b) is an interval contiguous to F_n, then for any $x \in E^{\gamma_n, \delta_n} \cap (a, b)$ there exists $y_n(x) \in (a, b)$ such that $|x - y_n(x)| \leq \delta_n$ and $|f_n(x) - f_n(y_n(x))| > (1 - 2^{-2n})\gamma_n|x - y_n(x)|$. Moreover, $y_n(x)$ may be chosen so that $|y_n(x) - x|$ is bounded away from 0 on compact subsets of (a, b). Additionally, f_n is locally monotone on (a, b) and on each interval of monotonicity we have $f_n = K \pm (1 - 2^{-3n})\phi$.

(v) Each U_n has a continuous radius r_n satisfying $r_n(x) \le \min\{2^{-n}, d(x,F_n)^2\}$ for all $x \in \mathbb{R}$ and $r_n(x) > 0$ for all $x \in G_n$ and $r_n(x) \le 2^{-2n}\gamma_n|x - y_n(x)|$ for all $x \in E^{\gamma_n,\delta_n}$. Moreover, $f_m \in U_n$ for all $m \ge n$ and $U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$.

(vi) For any $m \ge n$ and $x \in G_n \cap E^{\gamma_n,\delta_n}$ we have

$$|f_m(x) - f_m(y_n(x))| > (1 - 2^{-n})\gamma_n|x - y_n(x)|.$$

(vii) For any $g \in U_n$ we have $g' = 0$ on F_n.

Assume for the moment that we have established (i)-(vii). From (v) it follows that the f_ns converge uniformly to some function f. Then, $f \in U_n$ for all $n \in \mathbb{N}$ so by (vii) we may conclude that $f' = 0$ on $\cup_{n=1}^{\infty} F_n = \mathbb{R}\backslash E$ and therefore $\mathrm{Lip}\, f = 0$ on $\mathbb{R}\backslash E$. On the other hand, if $x \in E$, we have $x \in E^{\gamma_n,\delta_n}$ for infinitely many choices of n as E has UDT. Hence by (iv) and (vi) there exists $y_n(x)$ satisfying $|x - y_n(x)| < \delta_n$ and $|f_m(x) - f_m(y_n(x))| > (1 - 2^{-n})\gamma_n|x - y_n(x)|$ for all $m \ge n$, which yields $|f(x) - f(y_n(x))| \ge (1 - 2^{-n})\gamma_n|x - y_n(x)|$. As we have $(1 - 2^{-n})\gamma_n \to 1$, we deduce that $\mathrm{Lip}\, f(x) \ge 1$. On the other hand, by (iii), $\mathrm{Lip}\, f \le 1$ everywhere and we have $\mathrm{Lip}\, f(x) = 1$ for $x \in E$ which concludes the proof.

In the following we construct the f_ns and U_ns and verify that (i)-(vii) are valid. We begin by constructing f_1 and then define the other functions recursively.

To begin we set $f_0 = f_0^* \equiv 0$ and we also define $f_1 = 0$ on F_1. Set

$$\mathcal{E}^1(x) = d(x,F_1)^2 \text{ and } \mathcal{E}_1(x) = -d(x,F_1)^2.$$

Now, consider an interval (a,b) contiguous to F_1 see Figure 1. We need to ensure that f_1 has derivative 0 at a and b. Note that $f_0 \in (\mathcal{E}_1, \mathcal{E}^1)$ on $[a,b]$. Now applying Lemma 5.7 with $f = f_0$, $\delta = 2^{-3}$, $\epsilon = 1$ and $(f_l, f_u) = (\mathcal{E}_1, \mathcal{E}^1)$ we can define f_1 on $[a,b]$ so that

$$f_1 \in (\mathcal{E}_1, \mathcal{E}^1) \text{ on } [a,b] \tag{5.12}$$

$$f_1 \text{ is locally monotonic on } (a,b), \tag{5.13}$$

and on any interval of monotonicity $[c,d]$ of f_1 we have

$$f_1 = K \pm (1 - 2^{-3})\phi. \tag{5.14}$$

Note that by defining f_1 in this fashion on each contiguous interval of F_1 we ensure that f_1 is differentiable on F_1 with $f_1' = 0$ on F_1. It follows that (i) is satisfied for $n = 1$ and it is easy to see that (iii) holds as well. (In the upcoming steps of the construction we will require that $f_n \in (\mathcal{E}_1, \mathcal{E}^1)$ on $[a,b]$ as well. This will make sure that the limit function f is differentiable on F_1 and its derivative vanishes there as desired.)

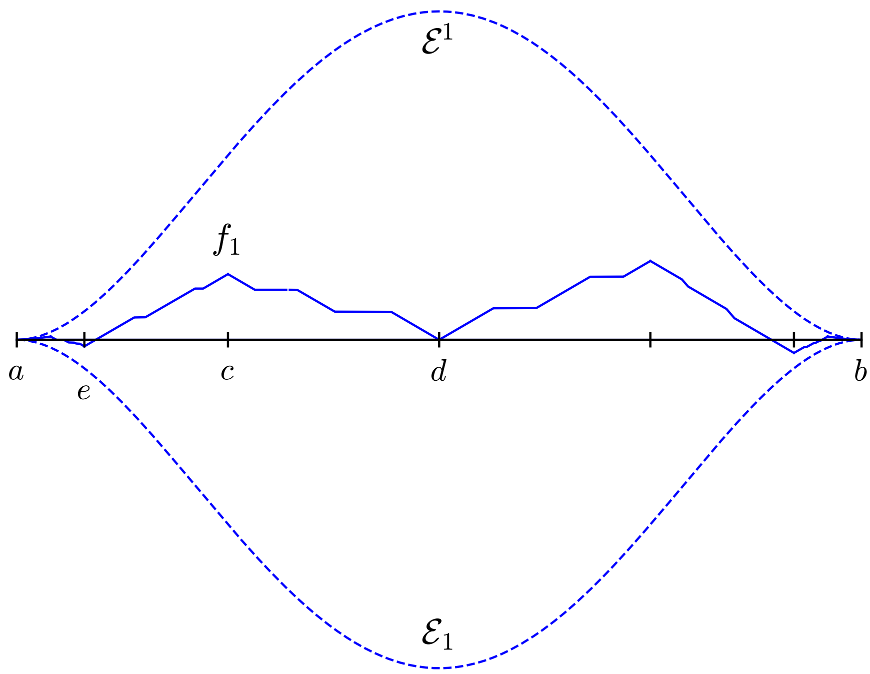

Figure 1: definition of f_1 on (a,b)

We next demonstrate that (iv) holds. We assume without loss of generality that $\gamma_1 \geq \frac{1}{2}$ and let $x \in E^{\gamma_1,\delta_1} \cap (a,b)$. Then choose an interval of monotonicity $[c,d]$ of f_1. We can assume without loss of generality that x is in the left half of $[c,d]$ so that $x \in [c, \frac{c+d}{2}]$. Let $[e,c]$ be the adjacent interval of monotonicity of f_1. Choose

$$\delta = \frac{1}{100}\min\{c-e, d-c, \delta_1\}. \tag{5.15}$$

Suppose first of all that $x \in [c+\delta, \frac{c+d}{2}]$. In this case f_1 is monotone on $[x-\delta, x+\delta]$ and using the definition of f_1, the fact that $x \in E^{\gamma_1,\delta_1}$ and the fact that $\delta \leq \delta_1$, we see that $|f_1(x) - f_1(y)| \geq (1-2^{-3})\gamma_1|x-y| > (1-2^{-2})\gamma_1|x-y|$ must hold for either $y = x-\delta$ or $y = x+\delta$.

Now suppose that $x \in [c, c+\delta]$. Since $x \in E^{\gamma_1,\delta_1}$ and $100\delta \leq \delta_1$ we have that $\max\{|E \cap [x, x+100\delta]|, |E \cap [x-100\delta, x]|\} \geq 100\gamma_1\delta$. Suppose first of all that $|E \cap [x, x+100\delta]| \geq 100\gamma_1\delta$ and let $y = x+100\delta$. In this case, since $[x,y] \subset [c,d]$, by the definition of f_1, we obtain $|f_1(x)-f_1(y)| \geq (1-2^{-3})\gamma_1|x-y| > (1-2^{-2})\gamma_1|x-y|$. Now suppose that $|E \cap [x-100\delta, x]| \geq 100\gamma_1\delta$. Note that $[x-100\delta, c] \subset [e,c]$. Setting $y = x-100\delta$, $S_1 = \int_y^c (1-2^{-3})1_E(t)\,dt$ and $S_2 = \int_c^x (1-2^{-3})1_E(t)\,dt$, we get $|f_1(x)-f_1(y)| \geq S_1 - S_2$. On the other hand, we know that $S_2 \leq (1-2^{-3})(x-c) \leq (1-2^{-3})\delta$ and $S_1 + S_2 = (1-2^{-3})|E \cap [y,x]| \geq 100\gamma_1\delta$. Using the fact that $\gamma_1 \geq \frac{1}{2}$,

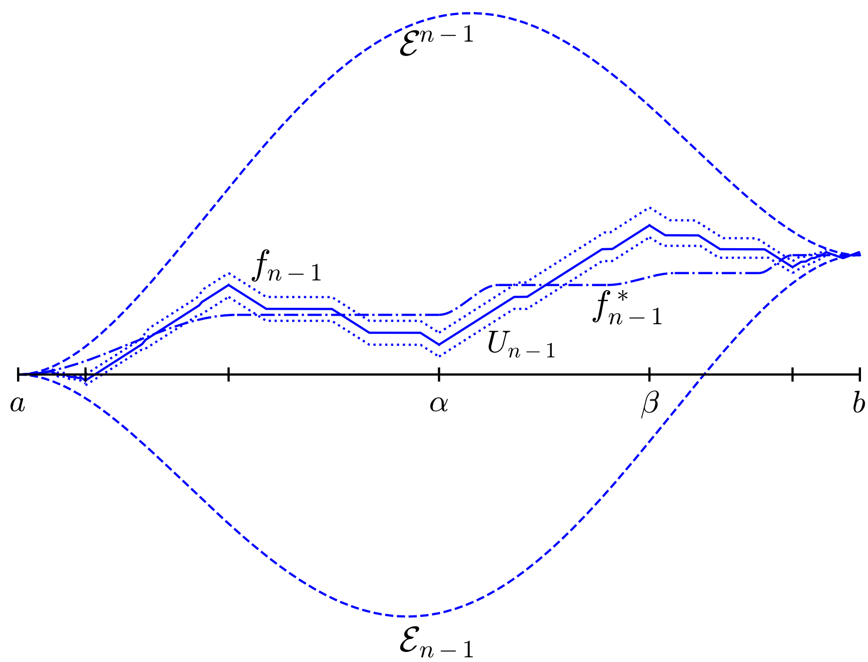

Figure 2: f_{n-1} on (a, b)

we see that

$$|f_1(x)-f_1(y)| \geq S_1-S_2 \geq (1-2^{-3})(100\gamma_1-2)\delta > (1-2^{-2})100\gamma_1\delta = (1-2^{-2})\gamma_1|x-y|.$$

Summing up, we see that in each of the two cases considered: $x \in [c, c+\delta]$ or $x \in [c+\delta, \frac{c+d}{2}]$, we can choose $y = y_1(x)$ such that $\delta \leq |x-y| \leq \delta_1$ and $|f_1(x) - f_1(y)| \geq (1-2^{-2})\gamma_1|x-y|$. Note that the definition of δ in (5.15) ensures that $|x - y_1(x)|$ is bounded away from 0 on compact subsets of (a, b). This establishes (iv).

Using the fact that $|x - y_1(x)|$ is locally bounded away from 0, we see that we can define a continuous, non-negative function $r_1 \leq \mathcal{E}^1$ so that $r_1 = 0$ on F_1, $r_1 > 0$ on G_1 and $r_1(x) \leq 2^{-2}|x - y_1(x)|$ for all $x \in E^{\gamma_1,\delta_1} \cap G_1$ and $||r_1||_\infty \leq 1/2$. Letting U_1, be the vicinity of f_1 with radius r_1 we see that for any $g \in U_1$ we have $g \in (-\mathcal{E}_1, \mathcal{E}_1)$ on any interval $[a, b]$ contiguous to F_1. It follows that (v)-(vii) have been established provided that we assume that at later steps $f_m \in U_m \subset U_1$ for $m > 1$.

Now assume that we have already defined the functions $f_1, f_2, ..., f_{n-1}$ and the decreasing sequence of vicinities $U_1, U_2, ..., U_{n-1}$ with radii $r_1, r_2, \ldots, r_{n-1}$ for some $n \geq 2$ so that they have the prescribed properties. Since r_{n-1} is continuous and positive on G_{n-1}, it follows that r_{n-1} is bounded away from 0 on all compact subsets of G_{n-1}. Now we would like to define f_n and U_n. First we define an auxiliary function f_n^*. Roughly f_n^* will be defined so that it has the same increase as f_{n-1} in any interval of monotonicity of f_{n-1}, but has vanishing derivative on F_n.

To this end consider an interval (a,b) contiguous to F_{n-1}. See Figure 2. On this figure the function f_{n-1} is drawn with a continuous line, the boundaries of the vicinity U_{n-1} are marked with dotted lines, the envelope boundaries $\mathcal{E}_{n-1}$ and $\mathcal{E}^{n-1}$ used in step $n-1$ are marked with dashed lines, finally the auxiliary function f_{n-1}^* used at the previous step is marked with dash-dot line.

By assumption we have

$$|f_{n-1}(x) - f_{n-1}(y)| \le (1 - 2^{-3(n-1)})|\phi(x) - \phi(y)| \text{ in } [a,b] \tag{5.16}$$

and clearly $f_{n-1} \in (f_{n-1} - \frac{r_{n-1}}{2}, f_{n-1} + \frac{r_{n-1}}{2})$ on $[a,b]$. Let δ' satisfy $2^{-3n} < \delta' < 2^{-3(n-1)}$. Then by Lemma 5.8 used with $\varepsilon = 2^{-3(n-1)}$ and $\delta = \delta'$ we can define f_n^* on $[a,b]$ so that

$$f_n^* \in (f_{n-1} - \frac{r_{n-1}}{2}, f_{n-1} + \frac{r_{n-1}}{2}) \text{ on } [a,b] \tag{5.17}$$

$$f_n^*(a) = f_{n-1}(a) \text{ and } f_n^*(b) = f_{n-1}(b) \tag{5.18}$$

$$(f_n^*)' = 0 \text{ on } F_n \cap (a,b) \tag{5.19}$$

$$|f_n^*(x) - f_n^*(y)| \le (1 - \delta')|\phi(x) - \phi(y)| \text{ for all } x, y \in [a,b]. \tag{5.20}$$

Now define $\epsilon_n(x) = \min\{d(x,F_n)^2, \frac{r_{n-1}(x)}{2}\}$ and let $\mathcal{E}_n = f_n^* - \epsilon_n$ and $\mathcal{E}^n = f_n^* + \epsilon_n$. Let (a',b') be contiguous to F_n in (a,b) so we have $f_n^* \in (\mathcal{E}_n, \mathcal{E}^n)$ on $[a',b']$. See Figure 3. Noting that (5.20) holds, we can apply Lemma 5.7 with $\varepsilon = \delta'$ and $\delta = 2^{-3n}$ to define a function f_n such that on each $[a',b']$ we have that $f_n \in (\mathcal{E}_n, \mathcal{E}^n)$, that f_n is locally monotone and that on each interval of monotonicity we have $f_n = K \pm (1 - 2^{-3n})\phi$.

From our construction we see that (i)-(iii) hold. On the other hand, (iv) is verified in a similar way to its verification in the case $n = 1$. Finally, we consider (v)-(vii). As f_n is piecewise monotonic, for given $x \in E^{\gamma_n,\delta_n}$ we can choose $y_n(x)$ in (iv) so that $|x - y_n(x)|$ is locally bounded away from 0 on (a',b'), where (a',b') is contiguous to F_n. Consequently, we can define $r_n \in C[a,b]$ such that $r_n \le \min\{2^{-n}, \epsilon_n\}$ and $r_n > 0$ on each interval (a',b') contiguous to F_n. Moreover we can suppose that if the vicinity U_n has radius r_n we have for any function $g \in U_n$ that $|g(x) - g(y_n(x))| > (1-2^{-n})\gamma_n|x - y_n(x)|$. Note that $r_n \le \epsilon_n$ guarantees $U_n \subseteq U_{n-1}$. Thus, if we select a sufficiently small r_n such that condition $r_n \le 2^{-2n}\gamma_n|x - y_n(x)|$ is also satisfied for all $x \in E^{\gamma_n,\delta_n}$ property (v) is verified. Moreover (vi) and (vii) follow easily as well.

By our earlier observations this concludes the proof: (v) guarantees that the sequence (f_n) has a uniform limit function f, for which $\operatorname{Lip} f(x) = 1$ in E by (iv) and (v). On the other hand, $\operatorname{Lip} f(x) = 0$ in the complement of E by (ii) and (v), as f has a vanishing derivative there by the choice of the vicinities. □

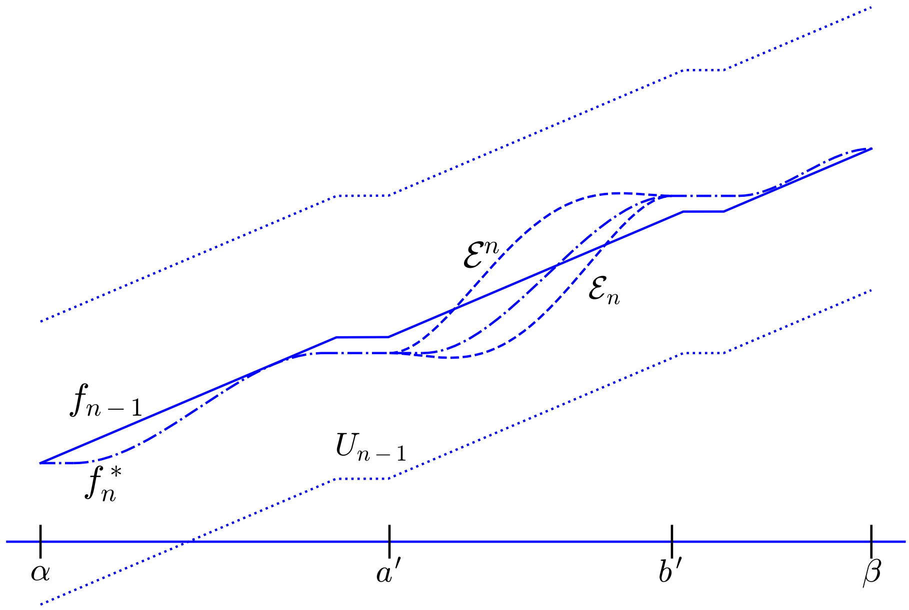

Figure 3: f_{n-1}, f_n^* on (α, β)

Note that sets of full measure are trivially UDT sets so an interesting consequence of Theorem 5.5 is that G_δ sets of full measure are Lip 1. This tells us, for example, that the set of irrationals is a Lip 1 set!

6 A weakly dense G_δ set which is not Lip 1

Recall that in Section 4 we proved that Lip 1 sets are weakly dense, G_δ sets (Theorem 4.1). In this section (Theorem 6.3) we show that weakly dense, G_δ sets need not be Lip 1. For the proof of the theorem we will need the following:

Lemma 6.1. *Suppose that* $E \subset \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$ *and* $\operatorname{Lip} f = \mathbf{1}_E$. *Then for every* $x \in E$ *and* $\varepsilon > 0$ *there is a* $y \in E \cap (x - \varepsilon, x + \varepsilon)$ *for which* $|f(x) - f(y)| > (1 - \varepsilon)|x - y|$.

Proof. Take $y' \in \mathbb{R}$ such that $|f(x) - f(y')| > \left(1 - \frac{\varepsilon}{2}\right)|x - y'|$ and $|x - y'| < \varepsilon$. We can assume that $\varepsilon < 1$ and $y' < x$. By Lemma 2.4 there is a $y \in E \cap (y', x)$ for which $|E\cap(y',y)|<\frac{\varepsilon}{2}|f(x)-f(y')|$ and

$$\frac{|f(x)-f(y)|}{|x-y|}\geq\frac{|f(x)-f(y')|-|E\cap(y',y)|}{|x-y|}>\frac{|f(x)-f(y')|-\frac{\varepsilon}{2}|f(x)-f(y')|}{|x-y|}$$
$$\geq\frac{\left(1-\frac{\varepsilon}{2}\right)^2|x-y'|}{|x-y'|}\geq 1-\varepsilon.$$

□

Remark 6.2. Recall Definition 2.3. It is easy to see that the following two statements are equivalent:

- E is weakly dense at x,
- for every $\varepsilon>0$ there is an $r\in(0,\varepsilon)$ such that

$$\max\left\{\frac{|E\cap(x-r,x)|}{r},\frac{|E\cap(x,x+r)|}{r}\right\}>1-\varepsilon. \tag{6.1}$$

Theorem 6.3. *There exists a weakly dense, G_δ set $E\subset\mathbb{R}$ which is not* Lip 1.

Proof. We use recursion to define E. Set $F_1:=[0,1]$. Suppose that n is a non-negative integer and for some $(i_0,\dots,i_n)\in\{1\}\times\dots\times\{1,\dots,4^n\}$ we have already defined a non-degenerate closed interval $F_{i_0,\dots,i_n}$. Let $U_{i_0,\dots,i_n}$ be the left half of $F_{i_0,\dots,i_n}$, that is

$$U_{i_0,\dots,i_n}:=\left[\min F_{i_0,\dots,i_n},\frac{\min F_{i_0,\dots,i_n}+\max F_{i_0,\dots,i_n}}{2}\right].$$

For every $i_{n+1}\in\{1,\dots,4^{n+1}\}$ let

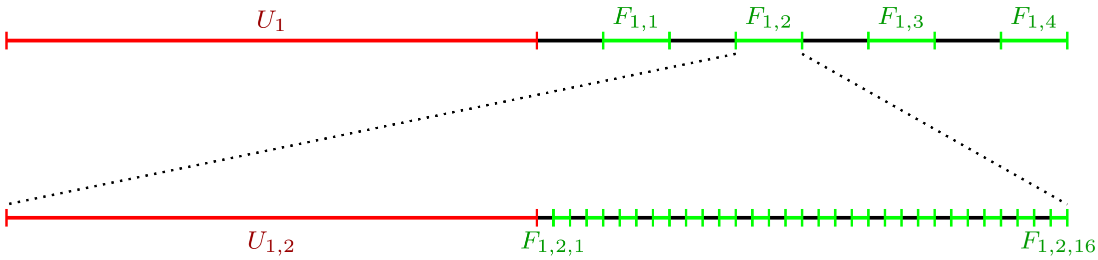

Figure 4: The first two steps of the recursion

$$F_{i_0,\dots,i_n,i_{n+1}}:=\left[\frac{(2\cdot4^{n+1}-2i_{n+1}+1)\max U_{i_0,\dots,i_n}+(2i_{n+1}-1)\max F_{i_0,\dots,i_n}}{2\cdot4^{n+1}},\right.$$
$$\left.\frac{(2\cdot4^{n+1}-2i_{n+1})\max U_{i_0,\dots,i_n}+(2i_{n+1})\max F_{i_0,\dots,i_n}}{2\cdot4^{n+1}}\right].$$

We define $U_{i_0,\dots,i_n}$ and $F_{i_0,\dots,i_n}$ recursively in this way for every $n \in \mathbb{N}$ and $(i_0,\dots,i_n) \in \{1\} \times \{1,\dots,4\} \times \dots \times \{1,\dots,4^n\}$. We are now ready to define E. First define

$$\mathcal{I} = \{1\} \times \{1,2,3,4\} \times ... \times \{1,2,...,4^n\} \times ...$$

and let $\mathcal{I}_1 = \{(i_j) \in \mathcal{I} \mid i_j = 1 \text{ for infinitely many } j \in \mathbb{N}\}$. Set

$$F := \cup_{(i_j)\in\mathcal{I}_1} \cap_{n=1}^{\infty} F_{i_1,i_2,\dots,i_n} \tag{6.2}$$

and

$$U := \cup_{(i_j)\in\mathcal{I}} \cup_{j=0}^{\infty} U_{i_0,i_1,\dots,i_j}.$$

The set F is a Cantor set minus countably many Cantor sets, hence it is G_δ. For every $n \in \mathbb{N}$ and $(i_0,\dots,i_n) \in \{1\} \times \dots \times \{1,\dots,4^n\}$ there is an open set $U'_{i_0,\dots,i_n}$ such that $U_{i_0,\dots,i_n} \subset U'_{i_0,\dots,i_n} \subset (\mathbb{R} \setminus U) \cup U_{i_0,\dots,i_n}$. Thus U is also G_δ. This implies that

$$E := U \cup F$$

is also G_δ.

If $x \in U$ then E is clearly weakly dense at x. If $x \in F$ and $\varepsilon > 0$ then using (6.2) take $n \in \mathbb{N}$ and $(i_0,\dots,i_n) \in \{1\} \times \dots \times \{1,\dots,4^n\}$ such that $x \in F_{i_0,\dots,i_n,1}$ and $\varepsilon > \min\{|F_{i_0,\dots,i_n}|, 4^{-n-1}\}$. By the definition of $F_{i_0,\dots,i_n,1}$ we have $4 \cdot 4^{n+1} |F_{i_0,\dots,i_n,1}| = |F_{i_0,\dots,i_n}|$, hence

$$\begin{aligned}\frac{|(\min F_{i_0,\dots,i_n}, x) \cap E|}{x - \min F_{i_0,\dots,i_n}} &\geq \frac{|U_{i_0,\dots,i_n}|}{\max F_{i_0,\dots,i_n,1} - \min F_{i_0,\dots,i_n}} = \frac{\frac{1}{2}|F_{i_0,\dots,i_n}|}{\frac{1}{2}|F_{i_0,\dots,i_n}| + 2|F_{i_0,\dots,i_n,1}|} \\ &= \frac{\frac{1}{2}|F_{i_0,\dots,i_n}|}{\left(\frac{1}{2} + \frac{2}{4\cdot 4^{n+1}}\right)|F_{i_0,\dots,i_n}|} = \frac{4^{n+1}}{4^{n+1}+1} \\ &= 1 - \frac{1}{4^{n+1}+1} > 1 - \varepsilon.\end{aligned} \tag{6.3}$$

By $(\min F_{i_0,\dots,i_n}, x) \subset (x - \varepsilon, x)$ and (6.3) we obtain that E is weakly dense at x.

Proceeding towards a contradiction assume the existence of a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\operatorname{Lip} f = \mathbf{1}_E$. We will show that there is a point $x^* \in \mathbb{R}$ for which $0.1 \leq \operatorname{Lip} f(x^*) \leq 0.9$. We will define $(i_0, i_1, \dots) \in \{1\} \times \{1,\dots,4\} \times \dots$ recursively such that $\{x^*\} = \cap_{n=0}^{\infty} F_{i_0,\dots,i_n}$. Set $a_0 := 0$ and $i_0 := 1$. Suppose that $n \in \mathbb{N}$ and we have already defined a non-negative integer a_{n-1} and $i_m \in \{1,\dots,4^m\}$ for every $m \in \{0,\dots,a_{n-1}\}$. Let $\{y_n\} = \cap_{l=1}^{\infty} F_{i_0,\dots,i_{a_{n-1}},1^l}$, where $1^l = \underbrace{1,...,1}_{l \text{ many times}}$.

Observe that $y_n = \min\left(F \cap F_{i_0,\dots,i_{a_{n-1}}}\right) \in E$. By Lemma 6.1 used with $x = y_n$

and $0 < \varepsilon < \min\{|F_{i_0,\dots,i_{a_{n-1}}}|, 1/10\}$ we can find an $x_n \in E$ satisfying $|y_n - x_n| < \varepsilon$ and

$$|f(x_n) - f(y_n)| > 0.9|x_n - y_n|. \tag{6.4}$$

This implies that $x_n \in F_{i_0,\dots,i_{a_{n-1}}}$. Since $x_n \neq y_n$ there exists $a_n > a_{n-1}$ such that $x_n \in F_{i_0,\dots,i_{a_n-1}} \backslash F_{i_0,\dots,i_{a_n-1},1}$ while $y_n \in F_{i_0,\dots,i_{a_n-1},1}$. The property $x_n \in F_{i_0,\dots,i_{a_n-1}}$ defines i_m for $m \in \{a_{n-1}+1, ..., a_n - 1\}$. We might be able to find many x_ns satisfying the above property but we select an x_n for which a_n is minimal among the possible choices. Then $i_m = 1$ for every $m \in \{a_{n-1}+1, ..., a_n - 1\}$.

If $k \in \mathbb{N}$, $(j_0, \dots, j_k) \in \{1\} \times \dots \times \{1, \dots, 4^k\}$, $j_{k+1}, j'_{k+1} \in \{1, \dots, 4^{k+1}\}$, $j_{k+1} < j'_{k+1}$, $z \in F_{j_0,\dots,j_k,j_{k+1}}$ and $z' \in F_{j_0,\dots,j_k,j'_{k+1}}$, then by Lemma 2.4 and the elementary fact

$$0 \leq a < b \text{ and } 0 \leq c \text{ implies } \frac{a}{b} \leq \frac{a+c}{b+c} \tag{6.5}$$

we obtain

$$\begin{aligned} \frac{|f(z) - f(z')|}{|z - z'|} &\leq \frac{|E \cap [z, z']|}{|z - z'|} \\ &\leq \frac{z - \min F_{j_0,\dots,j_k,j_{k+1}} + |E \cap [z, z']| + \max F_{j_0,\dots,j_k,j'_{k+1}} - z'}{\max F_{j_0,\dots,j_k,j'_{k+1}} - \min F_{j_0,\dots,j_k,j_{k+1}}} \\ &\leq \frac{(|j'_{k+1} - j_{k+1}| + 1)\left|F_{j_0,\dots,j_k,j_{k+1}}\right|}{(2|j'_{k+1} - j_{k+1}| + 1)\left|F_{j_0,\dots,j_k,j_{k+1}}\right|} \leq \frac{2}{3}. \end{aligned} \tag{6.6}$$

This applied with $z = x_n$, $z' = y_n$ and $k = a_n - 1$ would imply that for $x_n \notin U_{i_0,\dots,i_{a_n-1}}$ we would have $|f(x_n) - f(y_n)| \leq \frac{2}{3}|x_n - y_n|$, contradicting (6.4). Hence $x_n \in U_{i_0,\dots,i_{a_n-1}}$.

For every $x \in U_{i_0,\dots,i_{a_n-1}}$ and $y \in F_{i_0,\dots,i_{a_n-1},4^{a_n}}$ again Lemma 2.4 and (6.5) imply that

$$\begin{aligned} \frac{|f(y) - f(x)|}{y - x} &\leq \frac{|E \cap [x, y]|}{y - x} \\ &\leq \frac{|E \cap [x, y]| + \left(x - \min F_{i_0,\dots,i_{a_n-1}}\right) + \left(\max F_{i_0,\dots,i_{a_n-1}} - y\right)}{y - x + \left(x - \min F_{i_0,\dots,i_{a_n-1}}\right) + \left(\max F_{i_0,\dots,i_{a_n-1}} - y\right)} \\ &\leq \frac{\left|U_{i_0,\dots,i_{a_n-1}}\right| + \sum_{m=1}^{4^{a_n}} \left|F_{i_0,\dots,i_{a_n-1},m}\right|}{\left|F_{i_0,\dots,i_{a_n-1}}\right|} = \frac{3}{4}. \end{aligned}$$

Next we define i_{a_n}. We select an integer $i_{a_n} \in \{1, \dots, 4^{a_n}\}$ (let it be the least one) such that for every $\widetilde{x} \in U_{i_0,\dots,i_{a_n-1}}$ and $\widetilde{y} \in F_{i_0,\dots,i_{a_n}}$ we have

$$\frac{|f(\widetilde{y}) - f(\widetilde{x})|}{\widetilde{y} - \widetilde{x}} \leq 0.9. \tag{6.7}$$

Since $x_n \in U_{i_0,\dots,i_{a_n-1}}$ and $y_n \in E \cap F_{i_0,\dots,i_{a_n-1},1}$ by (6.4) we have that i_{a_n} is larger than one.

As there are $w \in F_{i_0,\dots,i_{a_n-1},i_{a_n}-1}$ and $v \in U_{i_0,\dots,i_{a_n-1}}$ for which

$$\frac{|f(w)-f(v)|}{w-v} > 0.9 \text{ and hence}$$

$$|[v,w]\setminus E| = w-v-|[v,w]\cap E| \leq w-v-|f(w)-f(v)| \leq \frac{w-v}{10}, \tag{6.8}$$

for every $\widetilde{y} \in F_{i_0,\dots,i_{a_n}}$ we obtain

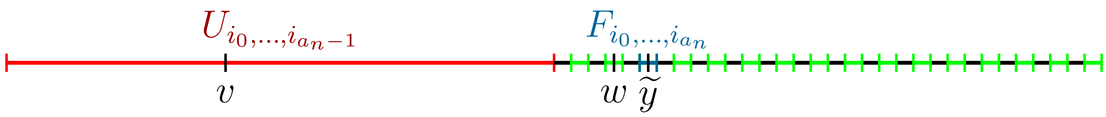

Figure 5: The position of v, w and $\widetilde{y}$

$$\begin{aligned}
\frac{|f(\widetilde{y})-f(v)|}{\widetilde{y}-v} &\geq \frac{|f(w)-f(v)|-|E\cap[w,\widetilde{y}]|}{\widetilde{y}-w+w-v} \\
&\geq \frac{|f(w)-f(v)|-|E\cap[w,\widetilde{y}]|}{\max F_{i_0,\dots,i_{a_n}} - \min F_{i_0,\dots,i_{a_n}-1} + w - v} \\
&\geq \frac{|f(w)-f(v)|-|E\cap[w,\widetilde{y}]|}{3\left|F_{i_0,\dots,i_{a_n}}\right| + (w-v)} \\
&\geq \frac{|f(w)-f(v)|-2\left|F_{i_0,\dots,i_{a_n}}\right|}{3\left|F_{i_0,\dots,i_{a_n}}\right| + (w-v)} \geq \frac{0.9(w-v)-2\left|F_{i_0,\dots,i_{a_n}}\right|}{3\left|F_{i_0,\dots,i_{a_n}}\right| + (w-v)} \qquad (6.9) \\
&\geq \frac{0.9(w-v)-2\left|F_{i_0,\dots,i_{a_n}}\right|}{4(w-v)} \underset{\text{using (6.8)}}{\geq} \frac{0.9\cdot 10|[v,w]\setminus E| - 2\left|F_{i_0,\dots,i_{a_n}}\right|}{4\cdot 10|[v,w]\setminus E|} \\
&\geq \frac{0.9\cdot 10\left(\min F_{i_0,\dots,i_{a_n-1},1} - \max U_{i_0,\dots,i_{a_n-1}}\right) - 2\left|F_{i_0,\dots,i_{a_n}}\right|}{4\cdot 10\left(\min F_{i_0,\dots,i_{a_n-1},1} - \max U_{i_0,\dots,i_{a_n-1}}\right)} \\
&= \frac{0.9\cdot 10\left|F_{i_0,\dots,i_{a_n}}\right| - 2\left|F_{i_0,\dots,i_{a_n}}\right|}{4\cdot 10\left|F_{i_0,\dots,i_{a_n}}\right|} = \frac{7}{40} > 0.1.
\end{aligned}$$

We define a_n and $i_0,\dots,i_{a_n}$ recursively for every $n \in \mathbb{N}$.

Set $\{x^*\} := \cap_{n=1}^{\infty} F_{i_0,\dots,i_n}$. From (6.9) we have $\operatorname{Lip} f(x^*) > 0.1$. We claim that

$$\frac{|f(\widehat{x})-f(x^*)|}{|\widehat{x}-x^*|} \leq 0.9 \tag{6.10}$$

for every $\widehat{x} \in \mathbb{R}\setminus\{x^*\}$. Suppose that an $\widehat{x}$ does not satisfy (6.10). Since f is continuous and it is constant on every complementary interval of the closure of

E, we can assume that $\widehat{x} \in E$. By (6.6) there is a $k \in \mathbb{N}$ such that $\widehat{x} \in U_{i_0,\dots,i_k-1}$ and $x^* \in F_{i_0,\dots,i_k-1,i_k}$. Since $i_{a_n} > 1$ for $n \in \mathbb{N}$ we have $x^* \neq \min\left(F \cap F_{i_0,\dots,i_k-1}\right) = \cap_{l=1}^{\infty} F_{i_0,\dots,i_k-1,1^l}$. This implies

$$\frac{|f(\widehat{x}) - f(x^*)|}{|\widehat{x} - x^*|} \leq$$
$$\leq \max\left\{\frac{|f(\widehat{x}) - f\left(\min\left(F \cap F_{i_0,\dots,i_k-1}\right)\right)|}{|\widehat{x} - \min\left(F \cap F_{i_0,\dots,i_k-1}\right)|}, \frac{|f(x^*) - f\left(\min\left(F \cap F_{i_0,\dots,i_k-1}\right)\right)|}{|x^* - \min\left(F \cap F_{i_0,\dots,i_k-1}\right)|}\right\}. \tag{6.11}$$

Since (6.7) shows that $k \neq a_n$ for any $n \in \mathbb{N}$, from the definition of $(a_n)_{n=0}^{\infty}$ we obtain

$$\frac{|f(\widehat{x}) - f\left(\min\left(F \cap F_{i_0,\dots,i_k-1}\right)\right)|}{|\widehat{x} - \min\left(F \cap F_{i_0,\dots,i_k-1}\right)|} \leq 0.9. \tag{6.12}$$

Moreover (6.6) implies that

$$\frac{|f(x^*) - f\left(\min\left(F \cap F_{i_0,\dots,i_k-1}\right)\right)|}{|x^* - \min\left(F \cap F_{i_0,\dots,i_k-1}\right)|} \leq \frac{2}{3}. \tag{6.13}$$

Hence by (6.11), (6.12) and (6.13) we have

$$\frac{|f(\widehat{x}) - f(x^*)|}{|\widehat{x} - x^*|} \leq 0.9,$$

which is impossible.

Thus $\operatorname{Lip} f(x^*) \neq \mathbf{1}_E(x^*)$, which is a contradiction. □

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