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Generic properties in function spaces

Ph.D. thesis

DOI: 10.15476/ELTE.2023.093

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Acknowledgements

While I was writing my B.Sc. thesis, my brother Péter told me that a long acknowledgements section indicates that the author was unsure if the thesis would ever be finished, and considers it a highly important moment in his or her life. Having academic plans in mind, this prompted me to write minimalistic acknowledgements, only mentioning my supervisor. The next time, in my M.Sc thesis, I barely extended it. Now I am going to break this tradition of mine, not because of the extraordinary importance of achieving this degree compared to the previous ones, but because gratitude is never misplaced, not to mention how happy I am to increase the length of the "to be read" part of this thesis.

There are many people in my life who deserve my gratitude. In fact, due to the highly sensitive dependence on initial conditions in the chaotic system of our world, I daresay each person in the world contributed to the final form of this thesis, for better or for worse. I do not intend to refer to all of them below, regrettably. Instead, I provide an admittedly incomplete list.

Above all, I am thankful to my advisor, Zoltán Buczolich, who has been part of my university life from its first week to date. He has been my supervisor during the course of all four of my scientific student papers (TDK), my B.Sc, my M.Sc, and finally during my PhD. I am grateful for his inviting me to join a high number of his projects, despite my young age, which not only resulted in his being my most frequent coauthor, but vice versa, for that I feel truly privileged. I learned a lot from him both about mathematics and about conducting research. It was a pleasure to work with him from the practical point of view as well, due to our similar approach to deadlines and answering e-mails.

I am also thankful to my other coauthors for their inspiring work Gáspár Vértesy, Bruce Hanson, Ryo Moore, and Péter Maga, already mentioned and also to be mentioned at a later phase. I am thankful to Miklós Abért for convincing me to return to academia.

I am thankful to every further mathematician I have had discussions with or heard giving talks, for shaping the way I think about mathematics. My teachers, from primary school to graduate school deserve special emphasis. I would like to grab the opportunity to extend this highlight to all teachers who do an incredible amount of inexpressibly important work for sometimes incomparably small compensation.

I am thankful to all the students I ever taught for helping me understand my subjects more thoroughly and improve my professional communication. The Analysis 1-2 group of the academic year 2018-2019 deserves honorable mention for their surprising amount of affection.

I am thankful to my colleagues I worked together with during my venture in the industry, especially the co-owners of my businesses (to be mentioned by name below). Attempting entrepreneurship together was a pleasant and highly enlightening journey.

And now, for something completely different...

I would like to thank my fiancée, Gigi, her evergrowing love and care I happened to experience during my graduate years, and the support she did not hesitate to give when I struggled to navigate the maze of my career, convincing me that my happiness has foremost importance. Her interest in my research and willingness to be infected by maths amazes me, while the challenge of explaining what on earth it is that my coauthors and I do is a great source of inspiration to make abstractions more concrete. By the way, our acquaintance dates back to a few months before I started my Ph.D, while our engagement precedes finishing this thesis. I look forward to finding out how these parallels between the love I share with her and the one I share with mathematics are going to unfold in the future. (Disclaimer: only the former one is going to be confirmed by vow.)

I am grateful to my family beyond words. Firstly to my parents for their everlasting support which they exhibited even at times when my choices brought difficulties upon them, such as when I packed up and split for the city, or when it seemed I would go abroad to study. My father, who was the first mathematician in my life, a circumstance which had a great impact on my childhood affection for the subject. My mother, who wanted me to become a musician, and enrolled me in music school, a decision which I reckon shaped my spirit beyond comprehension eventually. My eldest brother, Péter, who was a clear role model in my career and traced the path I followed as well, with substantial ease, thanks to his advice. My other brother, Dávid, who has been the best flatmate one can wish for in the past thirteen years, and thanks to whom I have become a considerably more practical person. And finally, all other members of my family, still with us or already gone, showering me with their love all my life to an undeserved extent.

And all my friends... It would be quite lengthy to provide personal reasoning, though they would certainly deserve it. I opt to give a simplified enumeration of those who have been the closest ones continuously in the past eight to sixteen years: Tamás Tossenberger, Boglárka Gehér[1], Máté Fellner, Tímea Tamási, Fausztin Asztrik Virág, Benedek Vince Soós, Mihály Török, Gusztáv László Gaál, Ágnes Timea Kúsz. I owe them a lot for the immeasurable joy they brought to my life. I am privileged to know them, and also to live with the thought that this list could be extended by considering broader social circles in my life. To those who do not find their name listed and are hurt by being left out: I apologize. Please, contact me, so we can discuss it. I would be happy to meet you.

Finally, I am thankful to those who do not or did not like me, harmed me, or even despised me. Some of them certainly served as a lasting inspiration.

[1] My special gratitude goes to Bogi and her exquisite knowledge of the English language, which improved these acknowledgements significantly.

Chapter 1

Introduction

This thesis synthesizes two research projects I participated in, which have the following common theme: both of them concern with the Hausdorff dimension of level sets and generic properties in function spaces in the Baire category sense. (In what follows, genericity is always understood this way.) While these projects do not have an intimate connection, the utilized techniques certainly enjoy shared features, and they give a fine exposition of my research interests. Indeed, while the papers laying the foundation of this thesis do not exhaust my publication list, the omitted ones also recurringly deal with the question of genericity. An exhaustive enumeration of these papers can be found at the end of this thesis in the Parallel Research section, not to be mistaken with the References.

1.1 Level sets of Hölder functions

1.1.1 Background

In [27], B. Kirchheim proved that for the generic continuous function defined on $[0,1]^p$, almost every level set has Hausdorff dimension $p-1$. (Some people prefer to use the term typical in the Baire category sense instead of generic.) It is a very natural question what happens when the domain is replaced by a more complicated set, for example with one of a fractal structure. This problem was addressed by R. Balka, Z. Buczolich, and

M. Elekes in [7], where they introduced the concept of topological Hausdorff dimension, which is the underlying notion of dimension that determines the Hausdorff dimension of almost every level set of the generic continuous function. (The definition of the topological Hausdorff dimension and the definition of some other concepts used in this introduction can be found in Section 2.1.) The topological Hausdorff dimension is related to some sort of "conductivity" properties of some fractal "networks" and outside of Mathematics, papers in Physics are also dealing with this concept, see for example works of A. Balankin, i.e. [3], [2], [5], [4], and [6]. It is a natural question to ask what happens if the level regions of our functions are not "infinitely compressible" and hence due to thickness of the level regions we cannot use for almost every levels the parts of our fractal domains where they are the "thinnest". The simplest way to impose a bound on compressibility is considering Hölder functions instead of arbitrary continuous functions. Motivated by this, it is interesting to consider level sets of 1-Hölder-α functions defined on fractals. Introducing a bound on the Hölder-constant is a customary practice (see e.g. [1], [31], and [36]) as it significantly tames the function space in question by making it complete and separable.

Level sets of 1-Hölder-α functions can get quite complicated. In some very special cases when either the function is linear, or constant on hyperplanes with a fixed normal vector, we need to consider intersections of these hyperplanes with our fractal. Investigating such intersections is a classical topic (see for example Marstrand's classical slicing theorem, [30]), which even in the case of the Sierpiński triangle, or carpet is still subject of more recent research as well, see [9] and [29].

1.1.2 Our contribution

Chapter 2 summarizes our papers discussing the aforementioned questions, [12] and [11], co-authored by my advisor, Zoltán Buczolich, and my co-graduate, Gáspár Vértesy. The outline of our research and the organization of these chapters are summarized below. In this thesis, we gradually pass from the most general results – theorems being valid for any measurable set – towards highly specific ones, such as estimates for specific fractals, for instance the Sierpiński triangle. We note that the mathematical content of the papers [12] and [11] is slightly reorganized to fit into this logical structure, instead of simply put after one another.

In Section 2.1 among other things we define $D_*(\alpha, F)$ which is the essential supremum of the Hausdorff dimension of the level sets of the generic 1-Hölder-α function. If F is the disjoint union of two fractals F_1 and F_2, with $D_*(\alpha, F_1) < D_*(\alpha, F_2)$ then it is easy to see that it is not necessarily true that for the generic 1-Hölder-α function $D_*(\alpha, F)$ equals the Hausdorff dimension of almost every level set in the range of the function. However, in Subsection 2.4.1 we show that for connected self-similar sets such a result holds if $0 < \alpha < 1$. The Lipschitz case, that is $\alpha = 1$ needs a different approach and can be the subject of some further research.

In Section 2.2, we provide an enumeration of the main results of Chapter 2. This is warmly recommended to the casual reader.

Section 2.3 is dedicated to the qualitative foundations of the theory. In Subsection 2.3.1 we establish some density and approximation results we need for proving results about generic functions.

Next in Subsection 2.3.2 we prove Theorem 2.2.1 according to which $D_*(\alpha, F)$ either equals zero, or it is always less or equal than the upper box dimension of F minus one.

In Subsection 2.3.3 we prove Theorem 2.2.2, that is we show that there is a dense G_δ subset $\mathcal{G}$ of 1-Hölder-α functions such that for every $f \in \mathcal{G}$ the essential supremum of the Hausdorff dimension of the level sets of f equals $D_*(\alpha, F)$. This shows that the complicated looking definition of $D_*(\alpha, F)$ in (2.1.4) can be significantly simplified, and it indeed makes sense to speak about the essential supremum of the Hausdorff dimension of the level sets of the generic 1-Hölder-α function.

In Subsection 2.3.4 we verify Theorem 2.2.3, that is we show that $D_*(\alpha, F)$ is monotone increasing in α for any compact set F.

Section 2.4 contains further robust results about certain families of fractals. In Subsection 2.4.1 we prove Theorem 2.2.4, stating that if F is a connected self-similar set, and $0 < \alpha < 1$, then one can select a dense G_δ set such that for any $f \in \mathcal{G}$ for almost every $r \in f(F)$ the Hausdorff dimension of the level set $f^{-1}(r)$ equals $D_*(\alpha, F)$. It means that the Hausdorff dimension of the Lebesgue-typical level set of the generic

1-Hölder-α function is a well-defined quantity in this case.

In Subsection 2.4.2 we show that if our fractal F is a self-similar set satisfying the strong separation condition then the Hausdorff dimension of almost every level set of a generic 1-Hölder-α function is constant zero for all $\alpha \in (0,1)$, that is the introduction of generic 1-Hölder-α functions is not giving any new information compared to the case of continuous functions.

Section 2.5 contains constructions and explicit calculations of $D_*(\alpha, F)$ for certain fractals. In Subsection 2.5.1 we give the details of the calculation of $D_*(\alpha, F)$ for F defined in Theorem 2.2.6. This is an example fractal $F \subseteq [0,1/2]^2$, which is a big "sponge" of positive Lebesgue measure and its complement is a dense system of very thin "tubes". In a "rough heuristic language" if we put our fractal sponge into $[0,1/2]^2$ then almost every level set of a typical continuous function can "run" in the complement of F, hence these level sets have Hausdorff dimension 0. However, using Hölder level sets one can see that $D_*(\alpha, F) = 1$ for any $\alpha \in (0,1]$, showing that it is criss-crossed by only very "narrow" tubes and these tubes are too thin to "contain" almost every level set of a generic 1-Hölder-α function. For this example the calculation is relatively easy.

A bit more difficult and interesting variant is investigated in Subsection 2.5.2, in which we discuss and illustrate a phenomenon which we call phase transition. We give an example of a fractal F for which the Hausdorff dimension of almost every level set of a generic 1-Hölder-α function for small values α equals the Hausdorff dimension of almost every level set of a generic continuous function defined on F. This means, at a heuristic level, that for such fractals the level sets of generic 1-Hölder-α functions are as flexible/compressible as those of a continuous function. On the other hand, for larger values of α we have $D_*(\alpha, F) > 0$, that is after a critical value of α these level sets are not as flexible/compressible as those of a continuous function and we experience some “traffic” jams as we try to push across the fractal the level sets of generic 1-Hölder-α functions. The fractal F discussed in this subsection will be the Cartesian product of a fat Cantor set with itself, hence it will be of zero topological dimension. Note that due to Subsection 2.4.2, such a construction requires fat Cantor sets. Indeed, a self-similar Cantor set cannot have the above properties. This example is also interesting in view of

Subsection 2.3.4 and Theorem 2.2.3, stating that $D_*(\alpha, F)$ is monotone increasing, as it demonstrates that $D_*(\alpha, F)$ is not necessarily continuous, even restricted to $\alpha > 0$, where $D_*(\alpha, F)$ measures Hölder level sets.

Section 2.6 concerns with estimating $D_*(\alpha, \Delta)$ for the Sierpiński triangle Δ. The fractals in Section 2.5 might give us the false impression that $D_*(\alpha, F)$ is easy to determine. However, they are intentionally constructed with the goal to be able to precisely calculate $D_*(\alpha, F)$. In the case of fractals not fine-tuned for this problem, one encounters significant difficulties, as demonstrated in this section. In Subsection 2.6.1 and Subsection 2.6.2, instead of determining $D_*(\alpha, \Delta)$, we give lower and upper estimates, respectively, displayed in Figure 1.1. It should be noted that both the lower and the upper estimates are positive and tend to 0 as $\alpha \to 0+$, hence the Sierpiński triangle does not admit phase transition.

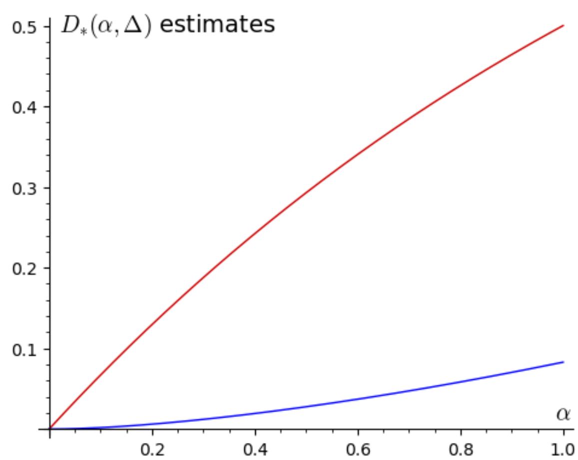

Figure 1.1: Lower and upper estimates of $D_*(\alpha, \Delta)$

1.2 Level sets of Birkhoff averages

1.2.1 Background

If $(X, \mathcal{F}, \mu, T)$ is a measure-preserving system, $x \in X$, and f is a summable function, one might be interested in the limit of the time averages $\frac{1}{N}\sum_{n=1}^{N} f(T^n x)$ as $N \to \infty$. Due to the celebrated ergodic theorem of Birkhoff, if T is ergodic, this time average converges to the space average $\int f\,d\mu$ almost everywhere. In other words, if we introduce the notation $E_f(\alpha) := \{x \in X : \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^n x) = \alpha\}$ then $\mu(E_f(\alpha)) = 1$ if $\alpha = \int f\,d\mu$, and 0 otherwise.

Thus the level sets of the time average behave trivially from the measure theoretic point of view. However, from the geometric point of view, one encounters a highly nontrivial and beautiful behaviour. Notably, we get rather interesting values by considering the Hausdorff dimension of the sets $E_f(\alpha)$ (including the irregular set $E_f' := \{x \in X : \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^n x)$ does not exist.$\}$). The function $S_f(\alpha) := \dim_H(E_f(\alpha))$ is called the *Birkhoff spectrum* of f.

Investigating the Birkoff spectrum belongs to the broader topic of multifractal analysis. Such investigation has been initiated in [35] by Y. Pesin and H. Weiss for Hölder functions in the context of thermodynamic formalism. While we have no inherent reason to believe that this spectrum should be anything else than pathological, quite surprisingly, imposing the Hölder assumption on f yields that S_f is a concave, analytic function, a phenomenon which is deservedly called "multifractal miracle" in the literature. Birkhoff spectrum of continuous functions was studied in [19] by A. Fan, D. Feng, and J. Wu. In their study, they have shown a variational formula between the dimension of the level set and the metric entropy, which we will recall precisely in Theorem 3.1.5. They have also shown that $S_f(\alpha)$ is concave and upper semicontinuous (hence continuous by the nature of concave functions; see [38, §10]) on the interior of the set $\{\alpha \in \mathbb{R}^d : E_f(\alpha) \neq \emptyset\}$. The question regarding the behavior of the spectrum at the boundary of its support remained open. It is mentioned in the introduction of [19] that even for Hölder regular functions discussions of $S_f(\alpha)$ at this boundary are scarce, which is actually a subtle problem.

In case of one-dimensional range the support of the spectrum of $f \in C(\Omega)$ is always a (possibly degenerate) closed interval L_f and concave and upper semiconinuous functions are always continuous on such intervals. However, it may happen that S_f, as a function defined on $\mathbb{R}$ has a jump discontinuity at the endpoints of L_f. Such functions were called degenerate by J. Schmeling in [39], in which the continuity of the spectrum for the generic Hölder function was proved. In fact, this combined with results in [32] and

[19] imply the continuity of the spectrum for the generic continuous function in our setting.

Due to concavity, we know that the Birkhoff spectrum has one-sided derivatives. In [41], F. Takens and E. Verbitskiy determines the Birkhoff spectrum of the Manneville–Pomeau map, and they show that it has a finite one-sided derivative at one of the endpoints.

For other studies of the Birkhoff spectrum, we refer to, for instance, [10], [41], [15], [20], [25], [34], and [23]. For more information on multifractal analysis (especially with its relationship to thermodynamic formalism), we refer to [14], [37] and to the survey paper [16] of V. Climenhaga.

1.2.2 Our contribution

Chapter 3 summarizes our contribution to the topic, presented originally in [13], a joint paper with my advisor, Zoltán Buczolich, and Ryo Moore. We focus our attention to $\Omega = \{0,1\}^{\mathbb{N}}$ with the $\frac{1}{2}$ Bernoulli product measure, the shift map σ being the ergodic transformation. The outline of our research and the organization of these chapters is summarized below.

In Section 3.1 after introducing some notation we give some simple examples and recall one of the main results of [19]. Nevertheless we introduce some basic notation here, to make this subsection more readable. For $f \in C(\Omega)$, that is for f continuous on Ω, we denote by $\alpha_{f,\max}$ (resp. $\alpha_{f,\min}$) the maximum (resp. minimum) value of $f \in C(\Omega)$, We also introduce the notation $\alpha^*_{f,\max} := \sup\{\alpha \in \mathbb{R} : E_f(\alpha) \neq \emptyset\}$, and $\alpha^*_{f,\min} := \inf\{\alpha \in \mathbb{R} : E_f(\alpha) \neq \emptyset\}$, and put $L_f = [\alpha^*_{f,\min}, \alpha^*_{f,\max}]$.

In Section 3.2, we provide an enumeration of the main results of Chapter 3 in a similar manner to the preceding chapter. This is warmly recommended to the casual reader.

Next, in Section 3.3 we discuss some tools used later. First, in Subsection 3.3.1 we show that given a continuous function f, any continuous function that is sufficiently close to f would have its Birkhoff spectrum also close to S_f on L_f except for a neighborhood of the endpoints of the spectrum. This will be proven in Theorem 3.2.1.

In Subsection 3.3.2 we prove some results about piecewise constant continuous (or simply PCC) functions, that is about functions which depend on finitely many coordinates. Among other results we show that for such functions f there is always a periodic ω in $E_f(\alpha^*_{f,\max})$.

Section 3.4 will concern with the continuity of a Birkhoff spectrum. Given $f \in C(\Omega)$, we say that the spectrum S_f is *continuous* if it is continuous on $\mathbb{R}$, and *discontinuous* otherwise. Equivalently, S_f is continuous when $S_f(\alpha^*_{f,\min}) = S_f(\alpha^*_{f,\max}) = 0$. We will first show that continuous, in fact PCC functions with discontinuous spectrum are dense in $C(\Omega)$ (Theorem 3.2.3). On the other hand, we give a direct proof of the fact that generic continuous functions have continuous spectrum (Theorem 3.2.5).

In Subsection 3.4.3 we show that for a dense open subset of $C(\Omega)$ the support of the spectrum is in the interior of $[\alpha_{f,\min}, \alpha_{f,\max}]$.

Section 3.5 concerns with one-sided derivatives of a Birkhoff spectrum at the endpoints/boundary points of the spectrum. Given $\varphi : \mathbb{R} \to \mathbb{R}$, we denote by $\partial^-\varphi(\alpha)$ the left-hand derivative of φ at α (if the value exists). Similarly, $\partial^+\varphi(\alpha)$ denotes the right-hand derivative. We will show that the spectrum of a generic continuous function f has infinite one-sided derivatives at the endpoints of L_f, i.e. $\partial^+ f(\alpha^*_{f,\min}) = \infty$, and $\partial^- f(\alpha^*_{f,\max}) = -\infty$ (Theorem 3.2.8). We construct a continuous function with continuous spectrum for which the one-sided derivatives at the endpoints are finite (Theorem 3.2.9). This function will also have a very small spectrum. By concavity of the spectrum on its support there is always a triangle which should be under the graph of the spectrum. Our example will provide an example when the spectrum is very close to this lower estimate.

It is not that obvious that functions with finite one-sided derivatives at the endpoints of the spectrum exist since for some well-known examples of functions with continuous spectrum, like the one discussed in Example 3.1.1 we have $\partial^+ f(\alpha^*_{f,\min}) = \infty$, and $\partial^- f(\alpha^*_{f,\max}) = -\infty$, however this function does not have a "generic spectrum" since $\alpha^*_{f,\min}$ equals $\alpha_{f,\min}$ and $\alpha^*_{f,\max}$ equals $\alpha_{f,\max}$. As we mentioned earlier for the generic continuous functions we always have $\alpha_{f,\min} < \alpha^*_{f,\min} < \alpha^*_{f,\max} < \alpha_{f,\max}$, see Theorem 3.2.7. In Theorem 3.2.10 we prove that for PCC functions f with continuous spectrum we always have $\partial^+ f(\alpha^*_{f,\min}) = \infty$, and $\partial^- f(\alpha^*_{f,\max}) = -\infty$. This illustrates that for the proof of Theorem 3.2.9 one needs to use a more involved construction than a PCC function.

Chapter 2

Generic Hölder level sets

2.1 Notation and preliminaries

The distance of $x, y \in \mathbb{R}^p$ is denoted by $|x-y|$. If $A \subseteq \mathbb{R}^p$ then the diameter of A is denoted by $|A| = \sup\{|x-y| : x, y \in A\}$. The open ball of radius ϱ centered at x is denoted by $B(x, \varrho)$. For a set $E \subseteq \mathbb{R}^p$ its ϱ-neighborhood $\{x : \inf\{|x-y| : y \in E\} < \varrho\}$ is denoted by $U_\varrho(E)$.

Assume that $F \subseteq \mathbb{R}^p$ for some $p > 0$. In what follows, F will be some fractal set, usually we suppose that it is compact.

We say that a function $f : F \to \mathbb{R}$ is c-Hölder-α for $c > 0$ and $0 < \alpha \leq 1$ if $|f(x) - f(y)| \leq c|x-y|^\alpha$. The space of such functions will be denoted by $C_c^\alpha(F)$, or if F is fixed then by C_c^α. The space of Hölder-α functions will be denoted by C^α, that is $C^\alpha = \bigcup_c C_c^\alpha$. We say that f is c^--Hölder-α if there exists $c' < c$ such that f is c'-Hölder-α. The set of such functions is denoted by $C_{c^-}^\alpha$, that is $C_{c^-}^\alpha = \bigcup_{c'<c} C_{c'}^\alpha$.

In the space of Hölder-α functions often the norm

$$\|f\|_{C^{0,\alpha}} = \|f\|_\infty + \sup_{x,y \in F,\ x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

is considered. This is a Banach space and one can consider typical properties in these spaces as well. However, these spaces are usually non-separable and often it is more convenient to consider Hölder functions as subsets of continuous functions equipped with the supremum norm $\|f\|_\infty = \sup_{x\in F}|f(x)|$. To obtain a closed subset of $C^\alpha(F)$ we will consider 1-Hölder-α functions, $C_1^\alpha(F)$ and use the metric coming from the supremum norm. One could use $C_c^\alpha(F)$ with any fixed positive constant c instead of 1. The results would be the same. In Lipschitz and Hölder spaces it is not unusual to consider these spaces. For example in [36] and [31] the one Lipschitz cases, in our notation $C_1^1([0,1])$ and $C_1^1([0,1]^n)$ were used. In Theorem 2.13 of [1] generic results in the spaces $C_1^\alpha([0,1])$, $0<\alpha<1$ were considered, even our notation is identical to the one used there.

For $\rho>0$ and $f\in C(F)$ we denote by $B(f,\rho)$ the open ball of radius ρ centered at f, the ball taken in the supremum norm. If $f\in C_1^\alpha(F)$ then $B(f,\rho)\cap C_1^\alpha(F)$ will denote the corresponding open ball in the subspace $C_1^\alpha(F)$.

Since similarities are not changing the geometry of a fractal set to avoid some unnecessary technical difficulties we suppose that we work with fractal sets F of diameter not exceeding one, unless stated otherwise in a specific construction. This way

$$C_1^\alpha(F)\subseteq C_1^{\alpha'}(F) \text{ if } \alpha>\alpha'. \tag{2.1.1}$$

Suppose $A\subseteq\mathbb{R}^p$. Given $\delta>0$ we say that the sets U_j form a δ-cover of A if $|U_j|<\delta$ for all j and $A\subseteq\bigcup_j U_j$.

The s-dimensional Hausdorff measure (see its definition for example in [18]) is denoted by $\mathcal{H}^s$. Recall that the Hausdorff dimension of $A\subseteq\mathbb{R}^p$ is given by

$$\dim_H A=\inf\{s:\mathcal{H}^s(A)=0\}= \tag{2.1.2}$$

$$\inf\{s:\exists\mathbf{C}_s>0,\ \forall\delta>0,\ \exists\{U_j\} \text{ a } \delta\text{-cover of } A \text{ s.t. } \sum_j|U_j|^s<\mathbf{C}_s\}.$$

One can observe that in the above definition instead of arbitrary δ-covers of A one can use open δ-covers, that is we can assume that the sets U_j are open.

Since the topological Hausdorff dimension is a less known concept here we quickly mention some definitions and results from [7]. First we recall the definition of the (small inductive) topological dimension.

Definition 2.1.1. Set $\dim_t \emptyset = -1$. The *topological dimension* of a non-empty metric space X is defined by induction as

$$\dim_t X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_t \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\}.$$

The topological Hausdorff dimension is defined analogously to the topological dimension.

In the next definition we adopt the convention that $\dim_H \emptyset = -1$.

Definition 2.1.2. Set $\dim_{tH} \emptyset = -1$. The *topological Hausdorff dimension* of a non-empty metric space X is defined as

$$\dim_{tH} X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\}.$$

Both notions of dimension can attain the value ∞ as well.

If K is a compact metric space and $\dim_t K = 0$ then the generic $f \in C(K)$ is well-known to be one-to-one, so every non-empty level set is a singleton. We do not know where this folklore fact was first proved but its simple proof can be found for example in [8].

Assume $\dim_t K > 0$. The following results from [7] show the connection between the topological Hausdorff dimension and the level sets of the generic $f \in C(K)$.

Theorem 2.1.3. *If K is a compact metric space with $\dim_t K > 0$ then for the generic $f \in C(K)$*

1. $\dim_H f^{-1}(y) \leq \dim_{tH} K - 1$ *for every $y \in \mathbb{R}$,*
2. *for every $\varepsilon > 0$ there exists an interval $I_{f,\varepsilon}$ such that $\dim_H f^{-1}(y) \geq \dim_{tH} K - 1 - \varepsilon$ for every $y \in I_{f,\varepsilon}$.*

Corollary 2.1.4. *If K is a compact metric space with $\dim_t K > 0$ then $\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}\} = \dim_{tH} K - 1$ for the generic $f \in C(K)$.*

There are many equivalent definitions of the box or Minkowski dimension. We will use the following one:

Definition 2.1.5. Given a non-empty set $F \subseteq \mathbb{R}^p$ let $a_N(F)$ denote the number of closed 2^{-N} grid hypercubes intersected by F. The lower and upper box dimensions of F equal $\underline{\dim}_B F = \liminf_{N\to\infty} \frac{\log a_N(F)}{N \log 2}$, $\overline{\dim}_B F = \limsup_{N\to\infty} \frac{\log a_N(F)}{N \log 2}$. If $\underline{\dim}_B F = \overline{\dim}_B F$ then this common value is the box dimension of F, denoted by $\dim_B F$. For an empty set F we put $\underline{\dim}_B F = \overline{\dim}_B F = \dim_B F = 0$.

The above definition makes sense for an arbitrary set of $F \subseteq \mathbb{R}^p$, but in this paper we will mainly work with measurable sets.

We need approximations by smooth functions. We will use the bump function

$$\eta(x) = \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right) & if|\mathrm{x}|<1, \\ 0 & otherwise, \end{cases} \tag{2.1.3}$$

and the corresponding mollifier

$$\eta_r(x) = c_r \eta\left(\frac{x}{r}\right),$$

where c_r is defined such that $\int_{\mathbb{R}^p} \eta_r(x)dx = 1$.

We want to study the Hausdorff dimension of the level sets of arbitrary 1-Hölder-α functions and also of the generic 1-Hölder-α functions.

To make it more precise, we introduce the following notation: let $D^f(r, F) = D^f(r) = \dim_H(f^{-1}(r))$ for any function $f : F \to \mathbb{R}$, that is $D^f(r)$ denotes the Hausdorff dimension of the function f at level r.

We are interested in those values for which the level set is of large Hausdorff dimension for many level sets in the sense of Lebesgue measure. This motivates the following definition.

$$D_*^f(F) = D_*^f = \sup\{d : \lambda\{r : D^f(r, F) \geq d\} > 0\},$$

where λ denotes the one-dimensional Lebesgue measure. Later we will assume that our fractal F is compact, but the above definition makes sense for more general measurable sets as well.

The definition of $D_*^f(F)$ depends on f. In case we want a definition depending only on the fractal F we can first take

$$\underline{D}_*(\alpha, F) = \inf\{D_*^f : f\colon F \to \mathbb{R} \text{ is locally non-constant and 1-Hölder-}\alpha\},$$

where the locally non-constant property is understood as f is non-constant on $U \cap F$ where U is any neighborhood of any accumulation point of F. As we are only concerned with nonnegative numbers, by convention the infimum of the empty set is 0. The value $\underline{D}_*(\alpha, F)$ concerns those functions for which "most" level sets are smallest possible.

As mentioned earlier we are also interested in level sets of generic 1-Hölder-α functions.

We denote by $\mathfrak{G}_{1,\alpha}(F)$, or by simply $\mathfrak{G}_{1,\alpha}$ the system of dense G_δ sets in $C_1^\alpha(F)$.

We put

$$D_*(\alpha, F) = \sup_{\mathcal{G} \in \mathfrak{G}_{1,\alpha}} \inf\{D_*^f : f \in \mathcal{G}\}. \tag{2.1.4}$$

In Theorem 2.2.2 we will show that there is a G_δ subset $\mathcal{G}$ of $C_1^\alpha(F)$ such that for every $f \in \mathcal{G}$ we have $D_*^f(F) = D_*(\alpha, F)$.

As we remarked in the introduction the existence of the above $\mathcal{G}$ shows that in the above definition the supremum is maximum, taken at this $\mathcal{G} \in \mathfrak{G}_{1,\alpha}$, and for this special $\mathcal{G}$ there is no need to take the infimum, since D_*^f takes this minimum for any $f \in \mathcal{G}$, which at the same time equals the maximum value. Combined with Theorem 2.2.4 for $0 < \alpha < 1$ in case of connected self-similar fractals one can think of $D_*(\alpha, F)$ as the Hausdorff dimension of almost every level set in the range of the generic $C_1^\alpha(F)$ function.

So far we have considered $0 < \alpha \leq 1$. To include generic continuous functions in our notation we set $D_*(0, F) = \max\{0, \dim_{tH} F - 1\}$. By Theorem 2.1.3, if f is the generic continuous function on F, then

$$D_*(0, F) = D_*^f(F).$$

For brevity, often we will omit F from our notation.

We will use the Mass Distribution Principle, see for example [18], Chapter 4.

Theorem 2.1.6. *Let μ be a mass distribution (a finite, non-zero Borel measure) on $F \subset \mathbb{R}^p$. Suppose that for some $s \geq 0$ there are numbers $c > 0$ and $\delta > 0$ such that $\mu(U) \leq c|U|^s$ for all sets U with $|U| \leq \delta$. Then $\mathcal{H}^s(F) \geq \mu(F)/c$ and $s \leq \dim F$.*

We will use the the following notion of separatedness:

Definition 2.1.7. For some $0 < \nu, \rho < 1$, a nonempty set $F \subseteq \mathbb{R}^p$ admits a (ν, ρ) separated structure, if there exists $K > 0$, and a sequence of finite families $\mathcal{S}_k$ such that

- $F \subset \bigcup \mathcal{S}_k$ for each k,
- for any k and $F' \in \mathcal{S}_k$ we have $|F'| < K\nu^k$,
- for any k and distinct $F_i, F_j \in \mathcal{S}_k$ we have $\frac{1}{K}\rho^k < d(F_i, F_j) = \inf\{|x - y| : x \in F_i,\ y \in F_j\}$.

This notion will be natural in Subsection 2.4.2. Such sets are fairly common in fractal geometry, for instance self-similar sets satisfying the strong separation condition admit such a structure, as we will see in Lemma 2.4.6.

2.2 Main Results

First we give a trivial upper bound for $D_*(\alpha, F)$. Observe that this upper bound does not depend on α.

Theorem 2.2.1. *For any bounded measurable set $F \subseteq \mathbb{R}^p$, we have*

$$D_*(\alpha, F) \leq \max\{0, \overline{\dim}_B(F) - 1\}.$$

The next theorem shows that in the complicated looking definition (2.1.4) for a suitable G_δ set one can skip taking inf and sup.

Theorem 2.2.2. *If $0 < \alpha \leq 1$ and $F \subset \mathbb{R}^p$ is compact, then there is a dense G_δ subset $\mathcal{G}$ of $C_1^\alpha(F)$ such that for every $f \in \mathcal{G}$ we have $D_*^f(F) = D_*(\alpha, F)$.*

From (2.1.1) it follows that $\underline{D}_*(\alpha, F)$ is monotone increasing in α, that is $\underline{D}_*(\alpha, F) \leq \underline{D}_*(\alpha', F)$ if $\alpha \leq \alpha'$. Next we state the same property for $D_*(\alpha, F)$.

Theorem 2.2.3. *Suppose that $F \subset \mathbb{R}^p$ is compact. Then the function $D_*(\alpha, F)$ is monotone increasing in α on $(0, 1]$.*

Our next theorem concerns with self-similar sets. Note that we do not assume the Open Set Condition.

Theorem 2.2.4. *Suppose that F is a connected self-similar set and $0 < \alpha < 1$. Then there exists a dense G_δ set $\mathcal{G}$ in $C_1^\alpha(F)$ such that for any $f \in \mathcal{G}$*

$$D_*(\alpha, F) = D_*^f(F) = D^f(r, F) \text{ for a.e. } r \in f(F).$$

This shows that in case of connected self-similar sets, like the Sierpiński triangle or the Sierpiński carpet one can think of $D_*(\alpha, F)$ as the Hausdorff dimension of almost every level set in the range of a generic 1-Hölder-α function.

The last main result in Section 2.4 is the following:

Theorem 2.2.5. *If F is the attractor of a bi-Lipschitz iterated function system satisfying the strong separation condition, then for small enough $\alpha > 0$ we have $\underline{D}_*(\alpha, F) = D_*(\alpha, F) = 0$.*

More specifically, if F is a self-similar set satisfying the strong separation condition, then for $0 < \alpha < 1$ we have $\underline{D}_*(\alpha, F) = D_*(\alpha, F) = 0$.

We start Section 2.5 with the following exact calculation:

Theorem 2.2.6. *Set $G_k := \bigcup_{j \in \mathbb{Z}} \left(j \cdot 2^{-k^2}, j \cdot 2^{-k^2} + 2^{-k^3}\right)$ for every $k \in \mathbb{N}$,*

$$F_0 := [0, 1/2] \setminus \bigcup_{k=2}^{\infty} G_k$$

and $F := F_0 \times F_0$. For every $\alpha \in (0, 1]$ we have $D_*(\alpha, F) = 1$, and $D_*(0, F) = 0$.

We also investigate the phenomenon of phase transition, i.e. when $D_*(\alpha, F)$ equals $D_*(0, F)$ for small αs, but exceeds it for larger αs.

Theorem 2.2.7. *There exists a compact subset of* $\mathbb{R}^2$ *which admits phase transition.*

We prove the following lower bound concerning the level sets of *any* 1-Hölder-α function defined on the Sierpiński triangle Δ.

Theorem 2.2.8. *Assume that* $f : \Delta \to \mathbb{R}$ *is a 1-Hölder-*α *function for some* $0 < \alpha \le 1$. *Then for Lebesgue almost every* $r \in f(\Delta)$ *we have*

$$\dim_H(f^{-1}(r)) \ge \frac{\frac{\alpha}{2}}{1 + \frac{1+\log\frac{3}{\alpha}}{\log 2} + \frac{2}{\alpha}} > 0. \tag{2.2.1}$$

Finally, an upper bound is verified only generically:

Theorem 2.2.9. *For any* $0 < \alpha < 1$, *we have* $D_*(\alpha, \Delta) \le 1 - 2^{-\alpha}$.

For $\alpha < 1$ from Theorem 2.2.1 one can obtain that $D_*(\alpha, \Delta) \le \frac{\log 3}{\log 2} - 1 \approx 0.584962500721$. Since $\lim_{\alpha \to 1-0} 1 - 2^{\alpha} = 1/2$ this upper estimate is better for any α.

Of course, it would be interesting to exactly determine $D_*(\alpha, \Delta)$, but this seems to be quite difficult.

Before proving these theorems, we would like to provide some intuition concerning these fractals. The closed set F defined in Theorem 2.2.6 almost "fills out" $[0, 1/2]^2$. We have selected $[0, 1/2]^2$, since we wanted to have a set of diameter not exceeding 1. It is looking like a "sponge" there is a dense system of narrow tubes in it and it is of zero topological dimension. If one considers the function $f_0(x, y) = y$ then its level sets are horizontal, running West-East. Taking a “generic continuous function" $f \in C^0(F)$ close to $f_0|_F$ almost all of its level sets are empty. We can also interpret it in the following way. Take a continuous extension of f onto $[0, 1/2]^2$, still denoted by f. Then its level sets are still "running almost West-East" but they are "flexible and compressible enough" to stay in the complement of F. This means that the topological Hausdorff dimension is not "sensing" the fact that F is a "large sponge". On the other hand, the theorem tells us that the level sets of generic Hölder-α functions cannot be squeezed into the thin tubes in the complement of F, this is reflected by the fact that $D_*(\alpha, F) = 1$ when $0 < \alpha \le 1$. For this fractal it is easy to carry out the calculations.

In case of connected fractals like the Sierpiński triangle or the Sierpiński carpet there are no tubes/holes in the complement in the fractal, but there are parts where it is thinner and there are parts where it is thicker. Level sets of generic 1-Hölder-α functions "try to run" at parts of the fractal where "it is thin". They give more precise information about these properties of the fractal than topological Hausdorff dimension.

2.3 Theoretical foundations

2.3.1 Some approximation and density results

We recall an extension theorem which is a consequence of Theorem 1 of [21].

Theorem 2.3.1. *Suppose that $F \subseteq \mathbb{R}^p$ and $f : F \to \mathbb{R}$ is a c-Hölder-α function. Then there exists a c-Hölder-α function $g : \mathbb{R}^p \to \mathbb{R}$ such that $g(x) = f(x)$ for $x \in F$.*

Next we prove the following general lemma, which will turn out to be rather useful in the study of generic properties of Hölder functions:

Lemma 2.3.2. *Assume that F is compact and $c > 0$ is fixed. Then the Lipschitz c-Hölder-α functions defined on F form a dense subset of the c-Hölder-α functions.*

Proof. Consider an arbitrary c-Hölder-α function $f : F \to \mathbb{R}$ and fix $\varepsilon > 0$. By using Theorem 2.3.1 we extend f to $\mathbb{R}^p$. The c-Hölder-α function obtained this way will be still denoted by f. It is known by the theory of mollifiers that if we consider the convolution $f_r = f * \eta_r$, it is a C^∞ function and $f_r \to f$ in the supremum norm on any compact subset of $\mathbb{R}^p$ as $r \to 0+$. Moreover, f_r restricted to F is c-Hölder-α as well. Indeed, for $x, y \in F$, due to the triangle inequality and the fact that the support of η_r

is $\{|z| < r\}$,

$$\begin{aligned} |f_r(x) - f_r(y)| &= \left| \int_{\mathbb{R}^p} \eta_r(z) f(x-z)dz - \int_{\mathbb{R}^p} \eta_r(z) f(y-z)dz \right| \\ &\leq \int_{\mathbb{R}^p} \eta_r(z) |f(x-z) - f(y-z)| dz \\ &= \int_{\{|z|<r\}} \eta_r(z) |f(x-z) - f(y-z)| dz \\ &\leq c|x-y|^\alpha \int_{\{|z|<r\}} \eta_r(z) dz \\ &= c|x-y|^\alpha. \end{aligned} \tag{2.3.1}$$

Consequently, we can fix r such that the restriction of f_r to F is a c-Hölder-α function in the ε-neighborhood of the restriction f to F in the supremum norm. Suppose that F' is a compact convex set containing F. As f_r is smooth, its derivative on F' is bounded. Consequently, f_r is K-Lipschitz on $F' \supset F$ for some $K > 0$. $\square$

Approximations by piecewise affine functions in the space $C_1^\alpha(F)$ are important as well. We will prove a lemma of this nature, but in order to avoid ambiguity, we first provide a precise definition:

Definition 2.3.3. A function $f : F \to \mathbb{R}$ is piecewise affine on $F \subseteq \mathbb{R}^p$, if we can find a system $\mathfrak{S}$ of non-overlapping (means disjoint interiors), non-degenerate closed p-simplices such that $F \subseteq \bigcup_{S \in \mathfrak{S}} S$, the set $\{S \in \mathfrak{S} : S \cap B \neq \emptyset\}$ is finite for every bounded $B \subset \mathbb{R}^p$, and for any $S \in \mathfrak{S}$ the restriction of f to any $S \cap F$ coincides with the restriction of an affine function to $S \cap F$.

Lemma 2.3.4. *Assume that F is compact, $0 < \alpha < 1$, and $0 < c$ are fixed. Then the locally non-constant piecewise affine c^--Hölder-α functions defined on F form a dense subset of the c-Hölder-α functions.*

Before proving this lemma, we state and prove an auxiliary proposition which is surely known in some form:

Proposition 2.3.5. *Assume that $S \subseteq \mathbb{R}^p$ is a non-degenerate p-simplex with vertices $x_0, ..., x_p$, and $\widetilde{f} : \{x_0, ..., x_p\} \to \mathbb{R}$ is K-Lipschitz for some $K > 0$. Let $a > 0$ be the* *length of the longest edge of S, and let $b = \min_{0 \le i \le p} b_i$, where $b_i > 0$ is the distance between x_i and the hyperplane determined by the remaining vertices. Then the function $\overline{f} : S \to \mathbb{R}$ defined by*

$$\overline{f}(x) = \sum_{i=0}^{p} \gamma_i \widetilde{f}(x_i)$$

for any convex combination $x = \sum_{i=0}^{p} \gamma_i x_i$ is M-Lipschitz, where

$$M = (p+1) \cdot K \cdot \frac{a}{b},$$

that is M depends on S only through $\frac{a}{b}$. In particular, it is invariant with respect to similarities.

Proof. As S is the convex hull of its vertices and any two vertices are connected by an edge, its diameter equals a. Moreover, adding a constant to $\widetilde{f}$ does not change the assumption, nor the implication of the proposition. Consequently, we can assume that $\min \widetilde{f} = 0$, and hence by the K-Lipschitz property we have $\max \widetilde{f} \le Ka$.

Consider now arbitrary points x, x' in S with

$$x = \sum_{i=0}^{p} \gamma_i x_i, \quad x' = \sum_{i=0}^{p} \gamma_i' x_i.$$

Without loss of generality, we can assume that $|\gamma_0 - \gamma_0'|$ is the maximal amongst the differences $|\gamma_i - \gamma_i'|$, as $i = 0, 1, ..., p$. Then

$$|\overline{f}(x) - \overline{f}(x')| \le \sum_{i=0}^{p} |\gamma_i - \gamma_i'| \widetilde{f}(x_i) \le (p+1) \cdot |\gamma_0 - \gamma_0'| \cdot Ka \tag{2.3.2}$$

where we use the bound on $\widetilde{f}$ in the last inequality. This quantity should be compared to the distance $|x - x'|$ to check the Lipschitz property of $\overline{f}$. However, one can easily see that the distance of x from the hyperplane determined by $x_1, ..., x_p$ is $\gamma_0 b_0$, while the distance of x' from the same hyperplane is $\gamma_0' b_0$. Consequently,

$$|x - x'| \ge |\gamma_0 - \gamma_0'| b_0 \ge |\gamma_0 - \gamma_0'| b. \tag{2.3.3}$$

Comparing estimates (2.3.2) and (2.3.3), we obtain

$$|\overline{f}(x) - \overline{f}(x')| \leq (p+1) \cdot K \cdot \frac{a}{b}|x - x'| = M|x - x'|.$$

$\square$

Proof of Lemma 2.3.4. Consider an arbitrary c-Hölder-α function $f : F \to \mathbb{R}$ and fix $\varepsilon > 0$. Since F is compact we can choose $0 < \gamma < 1$ such that $\|f - \gamma f\|_\infty < \varepsilon/4$. Then γf is c'-Hölder-α on F with $c' = c\gamma < c$. The proof starts similarly to the proof of Lemma 2.3.2: using Theorem 2.3.1 we extend γf to $\mathbb{R}^p$ such that it is still c'-Hölder-α. We select a closed hypercube F' containg F in its interior. By Lemma 2.3.2 we can find a K-Lipschitz, c'-Hölder-α function $\widetilde{f}$ for some $K > 0$ with domain F' such that on F' we have

$$\left\|\widetilde{f} - \gamma f\right\|_\infty < \frac{\varepsilon}{4}, \text{ which implies } \left\|\widetilde{f} - f\right\|_\infty < \frac{\varepsilon}{2}.$$

By introducing a further perturbation to $\widetilde{f}$ we will obtain a piecewise affine c-Hölder-α function $\overline{f}$ satisfying

$$\left\|\overline{f} - f\right\|_\infty < \varepsilon \text{ on } F. \tag{2.3.4}$$

To this end, fix any finite subdivision $\mathcal{U}$ of the unit hypercube into non-overlapping, non-degenerate p-simplices. (The existence of such a simplicial subdivision is simple to see.) Now divide F' into uniform, non-overlapping hypercubes such that their diameter is below some constant $\delta > 0$ to be fixed later. Let us divide these hypercubes further according to $\mathcal{U}$, that is denoting by Φ_Q a similarity from the unit hypercube onto a hypercube Q take the subdivision $\{\Phi_Q(S) : S \in \mathcal{U}\}$. Now if a simplex arising from this decomposition of F' has vertices $x_0, ..., x_p$, for any convex combination $x = \sum_{i=0}^{p} \gamma_i x_i$ let

$$\overline{f}(x) = \sum_{i=0}^{p} \gamma_i \widetilde{f}(x_i).$$

Observe that

$$|\overline{f}(x) - f(x)| \leq \sum_{i=0}^{p} \gamma_i |\widetilde{f}(x_i) - \widetilde{f}(x)| + |\widetilde{f}(x) - f(x)| \leq K\delta + \frac{\varepsilon}{2} < \frac{3\varepsilon}{4}, \tag{2.3.5}$$

if $\delta < \frac{\varepsilon}{4K}$. According to Proposition 2.3.5, the resulting function $\overline{f}$ is Lipschitz restricted to any of the small simplices, where the Lipschitz constant is invariant to similarities. However any of these small simplices is similar to a simplex $S \in \mathcal{U}$, and as $\mathcal{U}$ is finite, there are finitely many such Ss. Consequently, we can choose some M independently from δ, such that $\overline{f}$ is M-Lipschitz restricted to any of the small simplices. Hence $\overline{f}$ is clearly M-Lipschitz on F' as well, since any line segment in F' is the finite union of line segments contained by small simplices.

Choose and fix $c'' \in (c', c)$. Consider now arbitrary $x, y \in F'$. Due to the Lipschitz property of $\overline{f}$,

$$|\overline{f}(x) - \overline{f}(y)| \leq M|x-y| = M|x-y|^{1-\alpha}|x-y|^{\alpha} \leq c''|x-y|^{\alpha}$$

if $|x-y| \leq \left(\frac{c''}{M}\right)^{\frac{1}{1-\alpha}}$. That is, if x, y are close enough, the desired Hölder bound holds. Hence in what follows we can restrict our arguments to x, y with $|x-y| > \left(\frac{c''}{M}\right)^{\frac{1}{1-\alpha}}$, bounded away from 0.

We can find vertices x', y' of the small simplices which are at most δ apart from x, y, respectively. We have that

$$|\overline{f}(x) - \overline{f}(y)| \leq |\overline{f}(x) - \overline{f}(x')| + |\overline{f}(x') - \overline{f}(y')| + |\overline{f}(y') - \overline{f}(y)|.$$

By estimating the first and the third term using the Lipschitz bound, and the second term using the Hölder bound (as $\overline{f}(x') = \widetilde{f}(x')$ and $\overline{f}(y') = \widetilde{f}(y')$), we obtain

$$|\overline{f}(x) - \overline{f}(y)| \leq 2M\delta + c'|x'-y'|^{\alpha} \leq 2M\delta + c'(|x-y| + 2\delta)^{\alpha}.$$

As $\delta \to 0+$, the expression on the right hand side tends to $c'|x-y|^{\alpha}$. Consequently, as $|x-y|$ is bounded away from 0, for small enough δ it is always smaller than $c''|x-y|^{\alpha}$.

By using (2.3.5), the piecewise affine function $\overline{f}$ can be perturbed a bit to obtain a locally non-constant, piecewise affine c^{-}-Hölder-α function, still denoted by $\overline{f}$, for which (2.3.4) holds. □

2.3.2 Upper bound for $D_*(\alpha, F)$

Our goal now is to prove Theorem 2.2.1 which gives an upper bound for $D_*(\alpha, F)$ for an arbitrary $F \subseteq \mathbb{R}^p$. The next simple lemma is probably known. Since we were unable to find a reference to it we provide its short and simple proof.

Lemma 2.3.6. *For any bounded measurable set $F \subseteq \mathbb{R}^p$ and $(p-1)$-dimensional hyperplane L with unit normal vector v, we have that*

$$\overline{\dim}_B\left((L+tv)\cap F\right) \leq \max\{0, \overline{\dim}_B(F) - 1\}$$

for Lebesgue almost every $t \in \mathbb{R}$.

Proof. As non-degenerate affine transformations do not change the dimension of sets we can assume that L equals the hyperplane spanned by the first $p-1$ basis vectors of the standard basis $(e_i)_{i=1}^p$, and $v = e_p$.

Recall Definition 2.1.5 and let $a_N(F)$ denote the number of 2^{-N} grid hypercubes intersected by F, and set $s = \max\{1, \overline{\dim}_B(F)\}$. Due to the definition of the upper box dimension, for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for $N > N_0$ we have

$$a_N(F) \leq 2^{(s+\varepsilon)N}. \tag{2.3.6}$$

For $N > N_0$, define $E_N \subseteq \mathbb{R}$ such that $t \in E_N$ if

$$a_N((L+tv)\cap F) > 2^{(s-1+2\varepsilon)N}. \tag{2.3.7}$$

We claim that

$$\lambda(E_N) \leq 2^{-\varepsilon N}. \tag{2.3.8}$$

Indeed, if the reversed inequality holds, then E_N intersects the interior of at least $2^{(1-\varepsilon)N}$ grid intervals of length 2^{-N}, and then by (2.3.7), we can deduce

$$a_N(F) > 2^{(s+\varepsilon)N},$$

contradicting (2.3.6). Hence (2.3.8) is justified, which enables us to apply the Borel–Cantelli lemma to the sequence $(E_n)_{n=N_0+1}^{\infty}$. It yields that apart from a set of zero measure, for any $t \in \mathbb{R}$ we have

$$a_N((L+tv)\cap F) \leq 2^{(s-1+2\varepsilon)N}$$

for large enough N, yielding

$$\overline{\dim}_B\left((L+tv)\cap F\right) \leq \max\{0, \overline{\dim}_B(F)-1\}+2\varepsilon$$

for almost every t. It clearly gives the statement of the lemma. □

Proof of Theorem 2.2.1. Every $f \in C^{\alpha}(F)$ is uniformly continuous on F, hence it has a unique continuous extension f^* to $\overline{F}$ (where $\overline{F}$ is the closure of F). The function f^* is in $C^{\alpha}(\overline{F})$. Moreover, it is easy to see that $\phi\colon f \mapsto f^*$ is an isomorphism between $C^{\alpha}(F)$ and $C^{\alpha}(\overline{F})$. As $\overline{\dim}_B(F) = \overline{\dim}_B(\overline{F})$ and $f^{-1}(r) \subset (f^*)^{-1}(r)$, we can assume that F is closed.

We will prove a stronger statement, notably that for the generic 1-Hölder-α function $f : F \to \mathbb{R}$ and for almost every $r \in \mathbb{R}$ we have

$$\dim_H f^{-1}(r) \leq \underline{\dim}_B f^{-1}(r) \leq \max\{0, \overline{\dim}_B(F)-1\}.$$

Since the first inequality above is always true we need to verify the second one. We will calculate these box dimensions by estimating the number of 2^{-N} grid cubes intersected by $f^{-1}(r)$, which we denote by $a_N(f,r)$. Following this notation, we have

$$\underline{\dim}_B f^{-1}(r) = \underline{\lim} \frac{\log a_N(f,r)}{N\log 2}.$$

(Unless $a_N(f,r)$ is identically zero: in that case, this dimension is simply 0.)

Now for arbitrary $N \in \mathbb{N}$, $\varepsilon > 0$, $\delta > 0$ denote by $H_N(\varepsilon,\delta)$ the set of 1-Hölder-α functions, f for which there exists $E \subseteq \mathbb{R}$ with measure δ, such that for any $r \in E$ and for any $m \geq N$ we have

$$a_m(f, r) > (s + \varepsilon)^m\,,$$

where

$$s = \max\{1, \exp(\log 2 \cdot (\overline{\dim}_B(F) - 1))\}.$$

For the time being, assume that $H_N(\varepsilon, \delta)$ is nowhere dense for any N, ε, δ. Taking countable union for $\delta = \frac{1}{k}$ shows that for f not belonging to a meager set of 1-Hölder-α functions

$$a_m(f, r) > (s + \varepsilon)^m$$

holds for any $m \geq N$ only in a Lebesgue null-set of rs. Similarly, taking a countable union for $N \in \mathbb{N}$ shows that for f not belonging to a meager set of 1-Hölder-α functions we have that

$$a_m(f, r) \leq (s + \varepsilon)^m$$

for infinitely many m, except for a null-set of rs, and hence

$$\frac{\log a_m(f, r)}{m \log 2} \leq \frac{\log(s + \varepsilon)}{\log 2}.$$

However, it immediately yields that for any $\varepsilon > 0$, in a residual set of functions, f

$$\underline{\dim}_B f^{-1}(r) \leq \frac{\log(s + \varepsilon)}{\log 2},$$

except for a null-set of rs. Taking intersection for $\varepsilon = \frac{1}{l}$, $l \in \mathbb{N}$ then yields

$$\underline{\dim}_B f^{-1}(r) \leq \frac{\log s}{\log 2} = \max\{0, \overline{\dim}_B(F) - 1\}$$

in a residual set of 1-Hölder-α functions f for almost every r, which is the desired conclusion.

Consequently, to complete the proof of this theorem we need to show that $H = H_N(\varepsilon, \delta)$ is nowhere dense for any N, ε, δ.

To this end, using Lemma 2.3.4 fix a family $\mathcal{F}$ of locally non-constant piecewise affine 1-Hölder-α functions such that they form a dense subset of 1-Hölder-α functions, and fix N, ε, δ. Now it suffices to prove that any $f_0 \in \mathcal{F}$ has a neighborhood $B(f_0, R_0)$ such that for any $f \in B(f_0, R_0)$, we have $f \notin H_N(\varepsilon, \delta)$.

Assume that f_0 has k affine pieces. It yields that any level set $f_0^{-1}(r)$ consists of the intersection of F with pieces of at most k hyperplanes. These hyperplanes admit only a finite number of different directions, that is they arise as the translation of finitely many fixed $(p-1)$-dimensional hyperplanes. Consequently, according to Lemma 2.3.6, the upper box dimension of $f_0^{-1}(r)$ is at most $\max\{0, \overline{\dim}_B(F) - 1\}$ for almost every r. It yields that there exists a set $E \subseteq \mathbb{R}$ with

$$\lambda(E) < \frac{\delta}{2} \tag{2.3.9}$$

and $n_0 \in \mathbb{N}$ such that for any $r \notin E$ and $m > n_0$ we have

$$a_m(f_0, r) \leq (s + \varepsilon)^m .$$

Fix such an $m > N$.

Now let $\mathcal{H}$ be the family of 2^{-N} grid cubes intersected by F. For any $R > 0$, we can define

$$E_1(R) = \bigcup_{T \in \mathcal{H}} U_R(f_0(T \cap F)) \setminus f_0(T \cap F).$$

Since F is compact $f_0(T \cap F)$ is also compact. We can fix a sufficiently small $R > 0$ such that for $E_1 = E_1(R)$ we have

$$\lambda(E_1) < \frac{\delta}{2}. \tag{2.3.10}$$

However, if $r \notin E_1$, for any $f \in B(f_0, R)$ we have that $a_m(f_0, r) \geq a_m(f, r)$, as $f^{-1}(r) \cap T \neq \emptyset$ implies $f_0^{-1}(r) \cap T \neq \emptyset$. Putting together (2.3.9) and (2.3.10) we obtain that for any $f \in B(f_0, R)$, apart from the set

$$E' = E \cup E_1,$$

for any r and $f \in B(f_0, R)$ we have

$$a_m(f, r) \le (s+\varepsilon)^m .$$

Since $\lambda(E') < \delta$

it verifies that $H_N(\varepsilon, \delta)$ is nowhere dense. It concludes the proof. □

2.3.3 Dense G_δ sets in which $D_*^f(F) = D_*(\alpha, F)$ for any f

Lemma 2.3.7. *Suppose that $0 < \alpha \le 1$, $F \subset \mathbb{R}^p$ is compact, $E \subset \mathbb{R}^p$ is open or closed, and $\mathcal{U} \subset C_1^\alpha(F)$ is open. If $\{f_1, f_2, \ldots\}$ is a countable dense subset of $\mathcal{U}$, then there is a dense G_δ subset $\mathcal{G}$ of $\mathcal{U}$ such that*

$$\sup_{f\in\mathcal{G}} D_*^f(F\cap E) \le \sup_{k\in\mathbb{N}} D_*^{f_k}(F\cap E). \tag{2.3.11}$$

Proof. First we assume that E is closed. We can suppose that $E \subset F$.

Since countable union of sets of measure zero is still of measure zero we can choose a set $R_0 \subseteq \mathbb{R}$ such that $\lambda(\mathbb{R}\setminus R_0) = 0$ and for any k

$$D^{f_k}(r, E) \le \sup_{k'\in\mathbb{N}} D_*^{f_{k'}}(E) \text{ for any } r \in R_0. \tag{2.3.12}$$

Suppose that $D_1 > \sup_{k\in\mathbb{N}} D_*^{f_k}(E)$, and fix $k \in \mathbb{N}$ and $r \in R_0$. Recall (2.1.2). For every $\delta > 0$ there exists $\{U_{j,k,r}\}_{j=1}^\infty$, a δ-cover of $f_k^{-1}(r) \cap E$ such that $\sum_j |U_{j,k,r}|^{D_1} < 1$. As we remarked after (2.1.2) we can assume that the sets $U_{j,k,r}$ are open.

Next we suppose that $k, n \in \mathbb{N}$ are fixed and for $r \in R_0$ we consider $\delta = \frac{1}{n}$-covers, $\{U_{j,k,r,n}\}$ of $f_k^{-1}(r)\cap E$. Of course, if $f_k^{-1}(r)\cap E$ is empty then it may happen that these covers are also empty. As $E \setminus \bigcup_j U_{j,k,r,n}$ is compact, f_k is continuous and $f_k(x) \ne r$ for every $x \in E \setminus \bigcup_j U_{j,k,r,n}$, we have

$$0 < \rho_{k,n,r} := \min\left\{1, \inf\{|f_k(x) - r| : x \in E \setminus \bigcup_j U_{j,k,r,n}\}\right\} \text{ for any } r \in R_0 \tag{2.3.13}$$

(where the infimum of the empty set is $+\infty$ by convention). Since f_k is continuous,

$f_k(F)$ is bounded. Hence we can choose $\mathbf{M}_k$ such that

$$f_k^{-1}(r) \cap E = \emptyset, \text{ if } r \notin (-\mathbf{M}_k + 1, \mathbf{M}_k - 1). \tag{2.3.14}$$

Choose a compact subset

$$\mathbf{R}_{k,n} \subseteq R_0 \cap (-\mathbf{M}_k, \mathbf{M}_k) \text{ such that } \lambda(\mathbf{R}_{k,n}) > 2\mathbf{M}_k - 2^{-n}. \tag{2.3.15}$$

Then we can choose a finite subset $(R)_{k,n} \subseteq \mathbf{R}_{k,n}$, such that $\mathbf{R}_{k,n} \subseteq \bigcup_{r \in (R)_{k,n}} (r - \rho_{k,n,r}, r + \rho_{k,n,r})$. Moreover, the compactness of $\mathbf{R}_{k,n}$ also yields that we can choose $\rho_{k,n} \in (0,1)$ such that for any $r \in \mathbf{R}_{k,n}$ we can find $\mathbf{r}_{k,n}(r) \in (R)_{k,n}$ such that

$$(r - \rho_{k,n}, r + \rho_{k,n}) \subseteq (\mathbf{r}_{k,n}(r) - \rho_{k,n,\mathbf{r}_{k,n}(r)}, \mathbf{r}_{k,n}(r) + \rho_{k,n,\mathbf{r}_{k,n}(r)}). \tag{2.3.16}$$

Let $\mathcal{G}_n = \bigcup_k B(f_k, \rho_{k,n}) \cap \mathcal{U}$ and $\mathcal{G} = \bigcap_n \mathcal{G}_n$.

Suppose $f \in \mathcal{G}$. Then there exists a sequence k_n such that $f \in B(f_{k_n}, \rho_{k_n,n})$ for every n.

Set $R_\infty := \bigcap_m \bigcup_{n \geq m} (\mathbf{R}_{k_n,n} \cup (\mathbb{R} \setminus (-\mathbf{M}_{k_n}, \mathbf{M}_{k_n})))$. By (2.3.14), (2.3.15) and the Borel–Cantelli lemma, $\lambda(\mathbb{R} \setminus R_\infty) = 0$, and for every $r \in R_\infty$ either $f^{-1}(r) \cap E = \emptyset$ or for infinitely many n

$$\begin{aligned} f^{-1}(r) \quad &\subset \quad f_{k_n}^{-1}\big((r - \rho_{k_n,n}, r + \rho_{k_n,n})\big) \cap E \\ &\underset{(2.3.16)}{\subset} f_{k_n}^{-1}\big((\mathbf{r}_{k_n,n}(r) - \rho_{k_n,n,\mathbf{r}_{k_n,n}(r)}, \mathbf{r}_{k_n,n}(r) + \rho_{k_n,n,\mathbf{r}_{k_n,n}(r)})\big) \cap E \underset{(2.3.13)}{\subset} \bigcup_j U_{j,k_n,\mathbf{r}_{k_n,n}(r),n}, \end{aligned}$$

that is, the system $\{U_{j,k_n,\mathbf{r}_{k_n,n}(r),n}\}$ is a $\frac{1}{n}$-cover of $f^{-1}(r) \cap E$. Thus, using the inequality $\sum_j |U_{j,k_n,\mathbf{r}_{k_n,n}(r),n}|^{D_1} < 1$, we obtain $\dim_H(f^{-1}(r) \cap E) \leq D_1$ for a.e. $r \in \mathbb{R}$, and hence $D^f(r, F) \leq D_1$. As $D_1 > \sup_{k \in \mathbb{N}} D_*^{f_k}(E \cap F)$ was chosen arbitrarily (2.3.11) is satisfied.

Now suppose that E is open. For every $n \in \mathbb{N}$ set

$$E_n := \left\{ x \in E \cap F : \inf\{|x - y| : y \in F \setminus E\} \geq \frac{1}{n} \right\}.$$

Observe that $E_n \subseteq F$ is closed. We can apply the previously proved case to E_n. We obtain a dense G_δ subset $\mathcal{G}'_n$ of $\mathcal{U}$ such that $\sup_{f\in\mathcal{G}'_n} D_*^f(E_n) \le \sup_{k\in\mathbb{N}} D_*^{f_k}(E_n)$. Let $\mathcal{G} := \bigcap_{n=1}^\infty \mathcal{G}'_n$. If $f \in \mathcal{G}$ then

$$D_*^f(F\cap E) = \sup_{n\in\mathbb{N}} D_*^f(E_n) \le \sup_{n\in\mathbb{N}}\sup_{k\in\mathbb{N}} D_*^{f_k}(E_n) \le \sup_{k\in\mathbb{N}} D_*^{f_k}(F\cap E).$$

□

Proof of Theorem 2.2.2. Let $D_0 := D_*(\alpha, F)$.

For every $k \in \mathbb{N}$ choose a $\mathcal{G}^k \in \mathfrak{G}_{1,\alpha}(F)$ for which $D_0 - \frac{1}{k} \le \inf_{f\in\mathcal{G}^k} D_*^f(F)$. Set $\mathcal{G}_0 = \bigcap_{k=1}^\infty \mathcal{G}^k$. We have that $\mathcal{G}_0 \in \mathfrak{G}_{1,\alpha}(F)$ and $D_0 \le \inf_{f\in\mathcal{G}} D_*^f(F)$. It is enough to prove that for every $k \in \mathbb{N}$ there is a $\mathcal{G}_k \in \mathfrak{G}_{1,\alpha}$ such that

$$\sup_{f\in\mathcal{G}_k} D_*^f(F) \le D_k := D_0 + \frac{1}{k}, \tag{2.3.17}$$

since then $\mathcal{G} := \bigcap_{k=0}^\infty \mathcal{G}_k$ is a proper choice.

Fix $k \in \mathbb{N}$.

The set $H_k := \{f \in \mathcal{G}_0 : D_*^f(F) \le D_k\}$ cannot be nowhere dense in $C_1^\alpha(F)$, since otherwise $\mathcal{G}' := \mathcal{G}_0 \setminus cl(H_k)$ would be in $\mathfrak{G}_{1,\alpha}(F)$ and it would hold that

$$\inf_{f\in\mathcal{G}'} D_*^f(F) \ge D_k > D_0 = D_*(\alpha, F),$$

which contradicts the definition of $D_*(\alpha, F)$. Hence we can take $f_1 \in C_1^\alpha(F)$ and $\delta_1 > 0$ such that H_k is dense in $B(f_1, \delta_1) \cap C_1^\alpha(F)$. Choose a $\delta_2 > 0$ to satisfy $\delta_2^\alpha \le \delta_1/64$. As F is compact, we can take a finite set $A \subset F$ such that $\bigcup_{a\in A} B(a, \delta_2)$ covers F.

Suppose that a is fixed, $\varepsilon > 0$ and $g_0 \in C_1^\alpha(F)$ is an arbitrary function. Let $E := B(a, \delta_2) \cap F$. By the Hölder property, for every $f \in C_1^\alpha(F)$

$$\operatorname{diam}\big(f(E)\big) \le (2\delta_2)^\alpha \le \frac{\delta_1}{32}.$$

Thus setting

$$g_1(x) := \min\big\{ \max\{g_0(x) - g_0(a) + f_1(a), f_1(x) - \delta_1/2\}, f_1(x) + \delta_1/2\big\}.$$

we obtain

$$g_1|_E = g_0|_E - g_0(a) + f_1(a) \tag{2.3.18}$$

(since $g_1(a) = f_1(a)$ and $\operatorname{diam}\big(g_0(E)\big) + \operatorname{diam}\big(f_1(E)\big) \leq \delta_1/16$). As $g_1 \in B(f_1, \delta_1)$, we can take $g_2 \in H_k$ such that $\big|\big|g_1|_E - g_2|_E\big|\big| < \varepsilon/100$. Set

$$g_3(x) := \min\big\{ \max\{g_2(x) - g_2(a) + g_0(a), g_0(x) - \varepsilon\}, g_0(x) + \varepsilon\big\}.$$

Obviously $\big|\big|g_3 - g_0\big|\big| \leq \varepsilon$. By (2.3.18) and by the definition of g_2, for every $x \in E$

$$\begin{aligned} &\big|g_2(x) - g_2(a) + g_0(a) - g_0(x)\big| \\ \leq &\big|g_2(x) - g_1(x))\big| + \big|g_1(a) - g_2(a))\big| + \big|g_0(a) - g_1(a) + g_1(x) - g_0(x)\big| \\ &\leq \varepsilon/100 + \varepsilon/100 + 0 < \varepsilon, \end{aligned} \tag{2.3.19}$$

hence $g_3(x) = g_2(x) - g_2(a) + g_0(a)$ for every $x \in E$. Thus $D_*^{g_3}(E) = D_*^{g_2}(E) \leq D_k$ since $g_2 \in H_k$.

To sum up, for every $g_0 \in C_1^\alpha(F)$, $a \in A$ and $\varepsilon > 0$ we can find a $g_3 \in C_1^\alpha(F)$ such that $\|g_0 - g_3\| \leq \varepsilon$ and $D_*^{g_3}\big(B(a, \delta_2) \cap F\big) \leq D_k$. Consequently, by Lemma 2.3.7 for every $a \in A$ there is a $\mathcal{G}_a^k \in \mathfrak{G}_{1,\alpha}$ satisfying

$$\sup_{f \in \mathcal{G}_a^k} D_*^f(B(a, \delta_2) \cap F) \leq D_k.$$

Then (2.3.17) is true for $\mathcal{G}^k := \bigcap_{a \in A} \mathcal{G}_a^k$, which completes the proof. □

2.3.4 Monotonicity of $D_*(\alpha, F)$ in α

Proof of Theorem 2.2.3. Suppose that $\alpha' > \alpha > 0$. If $C_1^{\alpha'}(F)$ was dense in $C_1^\alpha(F)$, we could rely on the generic function in $C_1^{\alpha'}(F)$ determining $D_*(\alpha', F)$ to obtain conclusions about $D_*(\alpha, F)$ in a rather standard way. However, it is not the case, which raises certain technical difficulties in connecting these function spaces. We handle it as follows.

The set $C^{\alpha'}(F) \cap C^{\alpha}_{1-}(F)$ is dense in the separable space $C^{\alpha}_1(F)$. Hence we can select a sequence

$$(f_{k,1})_{k=1}^{\infty} \subseteq C^{\alpha'}(F) \cap C^{\alpha}_{1-}(F)$$

dense in $C^{\alpha}_1(F)$. Due to the two parts of this containment, we can find some $M_{k,1} > 0$ and $0 < c_{k,1} < 1$, $k = 1, 2, ...$ such that

$$f_{k,1} \in C^{\alpha'}_{M_{k,1}}(F) \cap C^{\alpha}_{c_{k,1}}(F) \text{ holds for } k = 1, 2, ...$$

Consequently, $\frac{1}{M_{k,1}} f_{k,1} \in C^{\alpha'}_1(F)$. Now due to Theorem 2.2.2, there exists a dense G_δ set $\mathcal{G}_0 \subseteq C^{\alpha'}_1(F)$ such that for any $f \in \mathcal{G}_0$ we have $D^f_*(F) = D_*(\alpha', F)$. This observation immediately yields the existence of a sequence $(f_{k,2})_{k=1}^{\infty} \in \mathcal{G}_0$ such that

$$\left\| \frac{1}{M_{k,1}} f_{k,1} - f_{k,2} \right\| < \delta_k \tag{2.3.20}$$

for some δ_k to be fixed later. By applying a simple rescaling, let

$$f_{k,3} := M_{k,1} f_{k,2} \in C^{\alpha'}_{M_{k,1}}(F).$$

For any k from $f_{k,2} \in \mathcal{G}_0$ it follows that $D^{f_{k,3}}_*(F) = D_*(\alpha', F)$.

Now let us set $c_{k,2} = \frac{1+c_{k,1}}{2} \in (c_{k,1}, 1)$. Our claim is that for some well-chosen δ_k, we have $f_{k,3} \in C^{\alpha}_{c_{k,2}}(F)$ as well. Momentarily assume that this claim holds. Then the proof can be concluded swiftly: by (2.3.20), we have

$$\|f_{k,1} - f_{k,3}\| < \delta_k M_{k,1} < \frac{1}{k}, \tag{2.3.21}$$

where the second inequality can be guaranteed by the choice of δ_k. This implies that the sequence $(f_{k,3})_{k=1}^{\infty}$ is dense in $C^{\alpha}_1(F)$ as well. Now to this sequence we can apply Lemma 2.3.7 with the roles $E = \mathbb{R}^p$ and $G = C^{\alpha}_1(F)$ to obtain a dense G_δ set $\mathcal{G} \subseteq C^{\alpha}_1(F)$ such that for any $f \in \mathcal{G}$ we have

$$D_*(\alpha, F) \leq \sup_{f \in \mathcal{G}} D_*^f(F) \leq \sup_{k \in \mathbb{N}} D_*^{f_{k,3}}(F) = D_*(\alpha', F),$$

where the second inequality follows from the lemma, while the equality follows from the construction of the sequence $(f_{k,3})_{k=1}^{\infty}$. Altogether we obtain the statement of the theorem indeed.

It only remains to prove the above claim, that is for any $x, y \in F$ and $f = f_{k,3}$ we have

$$|f(x) - f(y)| \leq c_{k,2}|x - y|^{\alpha}.$$

We use the standard technique of separating two cases based on the distance $|x - y|$. Notably, assume first that

$$|x - y| \leq \left(\frac{c_{k,2}}{M_{k,1}}\right)^{\frac{1}{\alpha' - \alpha}}. \tag{2.3.22}$$

Then due to $f = f_{k,3} \in C^{\alpha'}_{M_{k,1}}(F)$ we have

$$|f(x) - f(y)| \leq M_{k,1}|x - y|^{\alpha'} = M_{k,1}|x - y|^{\alpha' - \alpha}|x - y|^{\alpha} \leq c_{k,2}|x - y|^{\alpha},$$

where the last inequality directly follows from (2.3.22).

Now assume the opposite inequality concerning the distance $|x - y|$, that is

$$|x - y| > \left(\frac{c_{k,2}}{M_{k,1}}\right)^{\frac{1}{\alpha' - \alpha}}. \tag{2.3.23}$$

In this case, we appropriately substitute $f = f_{k,3}$ by $g = f_{k,1}$ and rely on $g \in C^{\alpha}_{c_{k,1}}(F)$. Notably,

$$|f(x) - f(y)| \leq |f(x) - g(x)| + |g(x) - g(y)| + |f(y) - g(y)| < c_{k,1}|x - y|^{\alpha} + 2\delta_k M_{k,1}$$

$$\leq c_{k,2}|x - y|^{\alpha},$$

where due to (2.3.23), the last inequality follows from

$$2\delta_k M_{k,1} \leq (c_{k,2} - c_{k,1}) \left(\frac{c_{k,2}}{M_{k,1}} \right)^{\frac{\alpha}{\alpha' - \alpha}},$$

which simply poses another restriction on the choice of δ_k. Consequently, if δ_k satisfies this, then $f \in C^{\alpha}_{c_{k,2}}(F)$ indeed, and if the assumption (2.3.20) holds as well, then the concluding step of the proof is also valid. $\square$

2.4 $D_*(\alpha, F)$ for various set families

2.4.1 Self-similar sets and $D_*(\alpha, F)$

In this subsection we prove Theorem 2.2.4.

Since generic continuous functions are non-constant on sets consisting of more than two points, for connected Fs containing at least two points the range of the generic continuous function contains an interval and hence is of positive Lebesgue measure.

As we mentioned in the introduction if F is the disjoint union of two fractals F_1 and F_2, with $D_*(\alpha, F_1) < D_*(\alpha, F_2)$ then it is easy to see that it is not necessarily true that for the generic 1-Hölder-α function $D_*(\alpha, F)$ equals the Hausdorff dimension of almost every level set in the range of the function.

Indeed, suppose that we put a scaled copy S of the Sierpiński triangle into $[0, 1/4] \times [0, 1/4]$, and T denotes $[1/2, 3/4] \times \{0\}$. Put $F = S \cup T$. Suppose that $f(x, y)$ is a function which is constant 0 on S and equals $1/8 + x$ on T. Results of [11] imply that $D_*(1/2, S) > 0$ and $D_*(1/2, T) = 0$ by Theorem 2.2.1. Then for some generic 1-Hölder-1/2 function g in the ball $B(f, 1/16)$ for almost every $r < 1/16$ with $r \in g(S)$ we have $\dim_H g^{-1}(r) = D_*(1/2, S) > 0$ and for almost every $r > 1/16$ we have $\dim_H g^{-1}(r) = 0$. As $g(S) \subset (-\infty, 1/16)$ and $g(T) \subset (1/16, \infty)$, and $\lambda(g(S)) > 0$ and $\lambda(g(T)) > 0$ for a generic g, this counterexample is valid.

We put $\Delta := D_*(\alpha, F)$ and

$$\kappa(f, \delta) := \frac{\lambda\{r \in f(F) : \dim_H f^{-1}(r) > \Delta - \delta\}}{\lambda(f(F))}. \tag{2.4.1}$$

The strategy of the proof of Theorem 2.2.4 is the following. First we reduce it to Proposition 2.4.1. In Lemma 2.4.2 we show that if we have a dense set of functions with relatively small portion of level sets with Hausdorff dimension close to Δ then there is a dense G_δ set of functions with the same property. Based on this lemma in Proposition 2.4.3 we show that for any $\delta_0 > 0$ we can find a $\kappa_0 > 0$ and an open ball in $C_1^\alpha(F)$ such that for any function f from this ball at least κ_0 portion of the range corresponds to level sets with Hausdorff dimension larger than $\Delta - \delta_0$. In the proof of Proposition 2.4.1 we use rescaled (both in range and domain) affine versions of the functions from the ball in Proposition 2.4.3. This way we obtain functions for which uniformly in any sufficiently large interval in the range of the function a portion of the range corresponds to level sets with Hausdorff dimension larger than $\Delta - \delta_0$. Finally, Lebesgue's density theorem will yield that almost every level set is of Hausdorff dimension larger than $\Delta - \delta_0$ for functions in a dense G_δ set. This will complete the proof of Proposition 2.4.1.

In the remainder of this subsection, we will assume that the self-similar set F is determined by the contractive similarities $\varphi_1, ..., \varphi_m$, $m \geq 2$ with ratios $0 < \mathbf{q}_1, ..., \mathbf{q}_m < 1$, that is, $F = \bigcup_i \varphi_i(F)$. We put $\mathbf{q}_{\min} = \min\{\mathbf{q}_1, ..., \mathbf{q}_m\}$.

Proposition 2.4.1. *Suppose that F is a connected self-similar set and $0 < \alpha < 1$. Then for every $\delta_0 > 0$ there exists a dense G_δ set $\mathcal{G}$ in $C_1^\alpha(F)$ such that for every $f \in \mathcal{G}$,*

$$\dim_H f^{-1}(r) \geq \Delta - \delta_0 \quad \textit{for a.e. } r \in f(F). \tag{2.4.2}$$

We prove this later. Using this proposition it is very easy to prove Theorem 2.2.4.

Proof of Theorem 2.2.4 based on Proposition 2.4.1. Using Theorem 2.2.2 choose a dense G_δ set $\mathcal{G}_0$ such that $D_*^f(F) = D_*(\alpha, F) = \Delta$ for any $f \in \mathcal{G}_0$. This implies that if $f \in \mathcal{G}_0$ then $\dim_H f^{-1}(r) \leq \Delta$ for a.e. $r \in f(F)$.

For $\delta_0 = 1/n$, $n \in \mathbb{N}$ select $\mathcal{G}_n$ by using Proposition 2.4.1 and set $\mathcal{G} = \bigcap_{n=0}^\infty \mathcal{G}_n$. Then for every $f \in \mathcal{G}$ we have $\dim_H f^{-1}(r) = \Delta$ for a.e. $r \in f(F)$. □

Before proving Proposition 2.4.1 we need the next lemma which is followed by Proposition 2.4.3.

Lemma 2.4.2. *Suppose that $0 < \kappa_0 < 1$ and there exists $\delta_0 > 0$ such that one can select a dense set $f_n \in C_1^\alpha(F)$ for which $\kappa(f_n, \delta_0) < \kappa_0$. Then there exists a dense G_δ set $\mathcal{G}^{\kappa_0}$ such that $\kappa(f, \delta_0) \le \kappa_0$ for every $f \in \mathcal{G}^{\kappa_0}$.*

Proof. Given $k \in \mathbb{N}$ using our dense set we will select radii $\delta_{n,k}$. We will define $\mathcal{G}_k = \bigcup_n B(f_n, \delta_{n,k})$ and $\mathcal{G}^{\kappa_0} = \bigcap_k \mathcal{G}_k$.

Suppose that n and k are given. Set

$$\mathbf{H}_n = \{r \in f_n(F) : \dim_H f_n^{-1}(r) \le \Delta - \delta_0\}.$$

By assumption $\kappa(f_n, \delta_0) < \kappa_0$ and hence

$$\lambda(\mathbf{H}_n) > (1 - \kappa_0)\lambda(f_n(F)).$$

Select a compact set $\Gamma_n \subseteq \mathbf{H}_n$ such that

$$\lambda(\Gamma_n) > (1 - \kappa_0)\lambda(f_n(F)). \tag{2.4.3}$$

Using the definition of the Hausdorff dimension for every $r \in \Gamma_n$ we select open sets $U_{n,k,r,j}$ such that $f_n^{-1}(r) \subseteq \bigcup_j U_{n,k,r,j}$, $|U_{n,k,r,j}| < 1/k$ and

$$\sum_j |U_{n,k,r,j}|^{\Delta - \delta_0 + \frac{1}{k}} < 1. \tag{2.4.4}$$

Put

$$\rho(n,k,r) := \min\{|f_n(x) - r| : x \in F \setminus \bigcup_j U_{n,k,r,j}\} > 0,$$

where the last inequality holds due to the compactness of F. Since Γ_n is also compact we can select finitely many $r_l \in \Gamma_n$ such that

$$\Gamma_n \subseteq \bigcup_l \Big(r_l - \frac{\rho(n,k,r_l)}{2}, r_l + \frac{\rho(n,k,r_l)}{2}\Big).$$

Let

$$\delta_{n,k} = \min\left\{\frac{1}{n+k}, \min_l\left\{\frac{\rho(n,k,r_l)}{2}\right\}\right\} > 0.$$

Suppose that $f \in B(f_n, \delta_{n,k})$ and $r \in \Gamma_n$. Then there exists an l such that $r \in (r_l - \frac{\rho(n,k,r_l)}{2}, r_l + \frac{\rho(n,k,r_l)}{2})$. Suppose that $x \in f^{-1}(r)$. Then

$$f_n(x) \in (r - \delta_{n,k}, r + \delta_{n,k}) \subseteq (r_l - \rho(n,k,r_l), r_l + \rho(n,k,r_l)).$$

Therefore $x \in \bigcup_j U_{n,k,r_l,j}$ and

$$f^{-1}(r) \subseteq \bigcup_j U_{n,k,r_l,j}. \tag{2.4.5}$$

Suppose that $f \in \mathcal{G}^{\kappa_0}$. Then there exists a sequence $n(k)$ such that $f \in \bigcap_k B(f_{n(k)}, \delta_{n(k),k})$. It is also clear that $\lim_{k\to\infty} \lambda\left(f_{n(k)}(F) \triangle f(F)\right) = 0$. We have

$$\lambda(\Gamma_{n(k)}) > (1-\kappa_0)\lambda(f_{n(k)}(F)).$$

Let $\Gamma_f := \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \Gamma_{n(k)}$. Then $\lambda(\Gamma_f) \geq (1-\kappa_0)\lambda(f(F))$. Suppose that $r \in \Gamma_f$.

From (2.4.4) and (2.4.5) we infer that

$$\dim_H f^{-1}(r) \leq \Delta - \delta_0.$$

This implies that

$$\lambda\{r \in f(F) : \dim_H f^{-1}(r) > \Delta - \delta_0\} \leq \kappa_0 \lambda(f(F)),$$

that is $\kappa(f, \delta_0) \leq \kappa_0$. □

In the sequel we will take balls in $C_1^\alpha(F)$ and hence, for ease of notation we will consider balls in this space, that is for example we will write $B(f_0, \rho_0)$ instead of $B(f_0, \rho_0) \cap C_1^\alpha(F)$.

Proposition 2.4.3. *For every $\delta_0 > 0$ there exist $0 < \kappa_0 \leq 1$, $f_0 \in C_1^\alpha(F)$, and $\rho_0 > 0$*

such that

$$\kappa(f,\delta_0) \geq \kappa_0 \quad \textit{for every } f \in B(f_0,\rho_0). \tag{2.4.6}$$

Proof. Proceeding towards a contradiction suppose that the statement of the proposition is not true. Then there exists δ_0 such that for every $0 < \kappa_0 \leq 1$ one can select a dense set $f_n \in C_1^\alpha(F)$ such that $\kappa(f_n,\delta_0) < \kappa_0$. For $\kappa_{0,n} = 1/n$ use Lemma 2.4.2 and take the dense G_δ sets, $\mathcal{G}^{\kappa_{0,n}}$ such that $\kappa(f,\delta_0) < \kappa_{0,n}$ for every $f \in \mathcal{G}^{\kappa_{0,n}}$.

Let $\mathcal{G}^0 = \bigcap_{n=1}^\infty \mathcal{G}^{\kappa_{0,n}}$. It is also dense G_δ. Suppose that $f \in \mathcal{G}^0$. Then

$$\lambda\{r \in f(F) : \dim_H f^{-1}(r) > \Delta - \delta_0\} \leq \kappa_{0,n}\lambda(f(F)) \text{ for all } n.$$

This implies $\lambda\{r \in f(F) : \dim_H f^{-1}(r) \geq \Delta - \delta_0\} = 0$, but $\Delta - \delta_0 < \Delta = D_*(\alpha, F)$ and this contradicts the definition of $D_*(\alpha, F)$. □

Now we are ready to prove Proposition 2.4.1.

Proof of Proposition 2.4.1. Without limiting generality we can suppose that $|F| = 1$. By using Lemma 2.3.4 select a dense set $\{f_n\}$ in $C_1^\alpha(F)$ consisting of locally non-constant piecewise affine 1^--Hölder-α functions. Since F is connected $f_n(F) = [m_n, M_n]$ with $m_n < M_n$. Since f_n is piecewise affine, it is Lipschitz-K_n. Without limiting generality we assume that $K_n \geq 1$. Since it is 1^--Hölder-α it is c_n-Hölder-α with a $c_n < 1$. We will select a sufficiently large $L_{n,k} > (n+k)(M_n - m_n + 1)$. Set

$$\mathbf{p}_{n,k}(t) = m_n + t \cdot \frac{M_n - m_n}{L_{n,k}}, \quad t = 0, ..., L_{n,k} - 1.$$

For each t choose $\mathbf{x}(t) \in F$ such that $f_n(\mathbf{x}(t)) = \mathbf{p}_{n,k}(t)$. If $t \neq t'$ then

$$K_n \geq K_n|\mathbf{x}(t) - \mathbf{x}(t')| \geq |f_n(\mathbf{x}(t)) - f_n(\mathbf{x}(t'))| \geq \frac{M_n - m_n}{L_{n,k}}$$

implies

$$|\mathbf{x}(t) - \mathbf{x}(t')| \geq \frac{M_n - m_n}{L_{n,k}K_n}. \tag{2.4.7}$$

By using self-similarity of F and $1/\alpha > 1$ select a sufficiently large $L_{n,k}$ and a similarity Φ_t such that $\mathbf{x}(t) \in \Phi_t(F)$ and

$$\frac{M_n - m_n}{3L_{n,k}K_n} > \Big(\frac{M_n - m_n}{3L_{n,k}}\Big)^{1/\alpha} \geq |\Phi_t(F)| > \mathbf{q}_{\min}\Big(\frac{M_n - m_n}{3L_{n,k}}\Big)^{1/\alpha}. \tag{2.4.8}$$

Observe that the sets $\Phi_t(F)$, $t = 1, ..., L_{n,k} - 1$ are pairwise disjoint due to (2.4.7) and (2.4.8). We denote by $\mathbf{q}(t)$ the similarity ratio of Φ_t. Since we supposed that $|F| = 1$, we also have

$$\Big(\frac{M_n - m_n}{3L_{n,k}}\Big)^{1/\alpha} \geq \mathbf{q}(t) > \mathbf{q}_{\min}\Big(\frac{M_n - m_n}{3L_{n,k}}\Big)^{1/\alpha}. \tag{2.4.9}$$

Given $\delta_0 > 0$, we select $\kappa_0 > 0$, $\rho_0 > 0$ and f_0^* according to Proposition 2.4.3 such that (2.4.6) holds for f_0^*. Without limiting generality we can suppose that $\rho_0 < 1$ and $\mathbf{0} \in F$ and $f_0^*(\mathbf{0}) = 0$, where $\mathbf{0}$ denotes the origin in $\mathbb{R}^p$.

Put $f_0 = (1/2)f_0^*$. From $|F| = 1$ and $f_0^* \in C_1^\alpha(F)$ it follows that $f_0 \in C_{1/2}^\alpha(F)$ and $|f_0(x)| \leq 1/2$ for all $x \in F$.

For $x \in \Phi_t(F)$ put

$$f_{n,k}(x) := f_n(\mathbf{x}(t)) + \mathbf{q}^\alpha(t)\big(f_0(\Phi_t^{-1}(x)) - f_0(\Phi_t^{-1}(\mathbf{x}(t)))\big), \quad t = 1, ..., L_{n,k} - 1. \tag{2.4.10}$$

This way $f_{n,k}$ is well-defined on $F_{n,k}^* = \bigcup_{t=1}^{L_{n,k}-1} \Phi_t(F)$, since as we noted, the sets $\Phi_t(F)$ are disjoint.

Claim 2.4.4. If $L_{n,k}$ is sufficiently large then

$$|f_{n,k}(x) - f_n(x)| < \frac{1}{n+k} \text{ for all } x \in F_{n,k}^*. \tag{2.4.11}$$

Proof of Claim 2.4.4. Take $x \in F_{n,k}^*$. Then there exists t such that $x \in \Phi_t(F)$. To obtain (2.4.11) we have the following chain of estimates

$$|f_{n,k}(x) - f_n(x)| \leq |f_{n,k}(x) - f_{n,k}(\mathbf{x}(t))| + |f_{n,k}(\mathbf{x}(t)) - f_n(\mathbf{x}(t))| + |f_n(\mathbf{x}(t)) - f_n(x)|$$

$$\leq \mathbf{q}^\alpha(t)|f_0(\Phi_t^{-1}(x)) - f_0(\Phi_t^{-1}(\mathbf{x}(t)))| + 0 + c_n|\mathbf{x}(t) - x|^\alpha$$

$$\leq \mathbf{q}^\alpha(t)\frac{1}{2}\Big(\frac{1}{\mathbf{q}(t)}|x - \mathbf{x}(t)|\Big)^\alpha + c_n|\mathbf{x}(t) - x|^\alpha \leq \Big(\frac{1}{2} + c_n\Big)|\mathbf{x}(t) - x|^\alpha$$

(using (2.4.8) and choosing a sufficiently large $L_{n,k}$)

$$\leq \frac{M_n - m_n}{3L_{n,k}}\Big(\frac{1}{2} + c_n\Big) < \frac{1}{n+k}. \tag{2.4.12}$$

This proves Claim 2.4.4. $\square$

Claim 2.4.5. If $L_{n,k}$ is sufficiently large then

$$|f_{n,k}(x) - f_{n,k}(y)| < \frac{1+c_n}{2}|x-y|^\alpha \text{ for all } x, y \in F^*_{n,k}. \tag{2.4.13}$$

Proof of Claim 2.4.5. Suppose that $x, y \in F^*_{n,k}$. If there exists t such that $x, y \in \Phi_t(F)$ then

$$\begin{aligned} |f_{n,k}(x) - f_{n,k}(y)| &= \mathbf{q}^\alpha(t)|f_0(\Phi_t^{-1}(x)) - f_0(\Phi_t^{-1}(y))| \\ &\leq \mathbf{q}^\alpha(t)\frac{1}{2}\frac{1}{\mathbf{q}^\alpha(t)}|x-y|^\alpha \leq \frac{1}{2}|x-y|^\alpha. \end{aligned} \tag{2.4.14}$$

Next suppose that $x \in \Phi_t(F)$ and $y \in \Phi_{t'}(F)$ with $t \neq t'$. We separate two subcases. First we suppose that x and y are not too far away. We mean by this that

$$|x-y|^{1-\alpha} \leq \frac{1-c_n}{2}. \tag{2.4.15}$$

We also need a lower estimate of the distance of x and y. We capitalize on (2.4.7) and (2.4.8)

$$|x-y| \geq |\mathbf{x}(t) - \mathbf{x}(t')| - |x - \mathbf{x}(t)| - |\mathbf{x}(t') - y| \tag{2.4.16}$$

$$\geq |\mathbf{x}(t) - \mathbf{x}(t')|\Big(1 - \Big(\frac{M_n - m_n}{L_{n,k}}\Big)^{\frac{1}{\alpha}}|\mathbf{x}(t) - \mathbf{x}(t')|^{-1}\Big)$$

$$\geq |\mathbf{x}(t) - \mathbf{x}(t')|\Big(1 - \Big(\frac{M_n - m_n}{L_{n,k}}\Big)^{\frac{1}{\alpha}-1}K_n\Big) \geq \frac{M_n - m_n}{L_{n,k}K_n}\Big(1 - \Big(\frac{M_n - m_n}{L_{n,k}}\Big)^{\frac{1}{\alpha}-1}K_n\Big)$$

(supposing that $L_{n,k}$ is sufficiently large)

$$\geq \frac{1}{2}\frac{M_n - m_n}{L_{n,k}K_n}.$$

For x and $\mathbf{x}(t)$, and for y and $\mathbf{x}(t')$ we use (2.4.14) to obtain

$$|f_{n,k}(x) - f_{n,k}(y)|$$

$$\begin{aligned}
&\le |f_{n,k}(x) - f_{n,k}(\mathbf{x}(t))| + |f_{n,k}(\mathbf{x}(t)) - f_{n,k}(\mathbf{x}(t'))| + |f_{n,k}(\mathbf{x}(t')) - f_{n,k}(y)| \\
&\le \frac{1}{2}|x - \mathbf{x}(t)|^\alpha + |f_n(\mathbf{x}(t)) - f_n(\mathbf{x}(t'))| + \frac{1}{2}|y - \mathbf{x}(t')|^\alpha \\
&\le \frac{M_n - m_n}{3L_{n,k}K_n} + c_n|\mathbf{x}(t) - \mathbf{x}(t')|^\alpha
\end{aligned} \tag{2.4.17}$$

(using (2.4.16))

$$\le \frac{2}{3}|x-y| + c_n|x-y|^\alpha\Big(1 - \Big(\frac{M_n - m_n}{L_{n,k}}\Big)^{\frac{1}{\alpha}-1}K_n\Big)^{-\alpha} \tag{2.4.18}$$

$$\le |x-y|^\alpha\Big(\frac{2}{3}|x-y|^{1-\alpha} + c_n\Big(1 - \Big(\frac{M_n - m_n}{L_{n,k}}\Big)^{\frac{1}{\alpha}-1}K_n\Big)^{-\alpha}\Big)$$

(using (2.4.15))

$$\le |x-y|^\alpha\Big(\frac{1-c_n}{3} + c_n\Big(1 - \Big(\frac{M_n - m_n}{L_{n,k}}\Big)^{\frac{1}{\alpha}-1}K_n\Big)^{-\alpha}\Big) \tag{2.4.19}$$

(if $L_{n,k}$ is sufficiently large)

$$< \Big(\frac{1+c_n}{2}\Big)|x-y|^\alpha.$$

This took care of the case when (2.4.15) holds.

Next we suppose that

$$|x-y| > \Big(\frac{1-c_n}{2}\Big)^{\frac{1}{1-\alpha}}. \tag{2.4.20}$$

We argue until (2.4.17) as before. At this point we can estimate the second term in (2.4.17) as we did it when we obtained (2.4.18). To estimate the first term using (2.4.20) we can choose $L_{n,k}$ sufficiently large such that

$$\frac{M_n - m_n}{3L_{n,k}K_n} < \frac{1-c_n}{3}\Big(\frac{1-c_n}{2}\Big)^{\frac{\alpha}{1-\alpha}} < \frac{1-c_n}{3}|x-y|^\alpha.$$

Then we can directly jump from (2.4.17) to (2.4.19) and then finish the estimate as before. This completes the proof of Claim 2.4.5. $\square$

In Claim 2.4.5 we have proved that $f_{n,k}$ is $\frac{1+c_n}{2}$-Hölder-α on $F^*_{n,k}$. Now, by using Theorem 2.3.1 we extend the definition of $f_{n,k}$ onto $\mathbb{R}^p$ such that its extension, still denoted by $f_{n,k}$ is still a $\frac{1+c_n}{2}$-Hölder-α function. Put

$$f^*_{n,k}(x) = \min\Big\{f_n(x) + \frac{1}{n+k}, \max\Big\{f_{n,k}(x), f_n(x) - \frac{1}{n+k}\Big\}\Big\}. \tag{2.4.21}$$

Since f_n is a c_n-Hölder-α function and $f_{n,k}$ is a $\frac{1+c_n}{2}$-Hölder-α function $f^*_{n,k}$ is also a $\frac{1+c_n}{2}$-Hölder-α function. Moreover by Claim 2.4.4 $f^*_{n,k} = f_{n,k}$ on $F^*_{n,k}$. Hence

$$f^*_{n,k}(\mathbf{x}(t)) = f_{n,k}(\mathbf{x}(t)) = f_n(\mathbf{x}(t)) = \mathbf{p}(t). \tag{2.4.22}$$

By Proposition 2.4.3 and by the choice of f^*_0 and ρ_0 for any $f \in B(f^*_0, \rho_0)$

$$\lambda\{r \in f(F) : \dim_H f^{-1}(r) > \Delta - \delta_0\} \geq \kappa_0 \lambda(f(F)). \tag{2.4.23}$$

By (2.4.10) the graph of $f^*_{n,k}|_{\Phi_t(F)}$ is an affine copy of the graph of f^*_0. In other words, $f^*_{n,k}|_{\Phi_t(F)}$ is a rescaled version of $f^*_0 = 2f_0$ with scaling ratio $\mathbf{q}(t)$ along domain directions and with scaling ratio $\frac{1}{2}\mathbf{q}^\alpha(t)$ along the range axis. This affine transformation also gives a correspondence between $B(f^*_0, \rho_0)$ and $\{f|_{\Phi_t(F)} : f \in B(f^*_{n,k}, \frac{1}{2}\mathbf{q}^\alpha(t)\rho_0)\}$. Consequently, by (2.4.23) for any $f \in B(f^*_{n,k}, \frac{1}{2}\mathbf{q}^\alpha(t)\rho_0)$

$$\lambda\{r \in f(\Phi_t(F)) : \dim_H f^{-1}(r) > \Delta - \delta_0\} \geq \kappa_0 \lambda(f(\Phi_t(F))). \tag{2.4.24}$$

Set $\ell_0 = \lambda(f^*_0(F)) > 0$. Then using (2.4.9)

$$\lambda(f^*_{n,k}(\Phi_t(F))) = \frac{1}{2}\mathbf{q}^\alpha(t) \cdot \ell_0 \geq \frac{1}{2} \cdot \frac{M_n - m_n}{3L_{n,k}} \mathbf{q}^\alpha_{\min} \ell_0. \tag{2.4.25}$$

Thus using (2.4.9) and $\ell_0 \leq 1 \cdot |F|^\alpha \leq 1$,

$$f^*_{n,k}(\Phi_t(F)) \subseteq \Big[\mathbf{p}_{n,k}(t) - \frac{1}{2} \cdot \frac{M_n - m_n}{3L_{n,k}}, \mathbf{p}_{n,k}(t) + \frac{1}{2} \cdot \frac{M_n - m_n}{3L_{n,k}}\Big].$$

We will select a sufficiently small

$$0 < \rho_{n,k} < \rho_0 \frac{1}{2} \cdot \frac{M_n - m_n}{3L_{n,k}} \mathbf{q}^{\alpha}_{\min} < \rho_0 \frac{1}{2} \cdot \mathbf{q}^{\alpha}_t \tag{2.4.26}$$

(the last inequality holds by (2.4.9)).

Suppose that $f \in B(f^*_{n,k}, \rho_{n,k}) \subseteq B(f^*_{n,k}, \rho_0 \mathbf{q}^{\alpha}(t)/2)$. Then

$$f(\Phi_t(F)) \subseteq \mathbf{I}_{n,k}(t) := \Big[\mathbf{p}_{n,k}(t) - \frac{M_n - m_n}{3L_{n,k}}, \mathbf{p}_{n,k}(t) + \frac{M_n - m_n}{3L_{n,k}}\Big].$$

By (2.4.24) we also obtain

$$\lambda\{r \in f(F) \cap \mathbf{I}_{n,k}(t) : \dim_H f^{-1}(r) > \Delta - \delta_0\} \geq \kappa_0 \lambda(f(\Phi_t(F))) \tag{2.4.27}$$

$$\geq \kappa_0(\lambda(f^*_{n,k}(\Phi_t(F))) - 2\rho_{n,k})$$

(by choosing a sufficiently small $\rho_{n,k}$ and using (2.4.25))

$$\geq \frac{\kappa_0}{2} \lambda(f^*_{n,k}(\Phi_t(F))) \geq \frac{\kappa_0}{2} \cdot \frac{1}{2} \cdot \frac{M_n - m_n}{3L_{n,k}} \mathbf{q}^{\alpha}_{\min} \ell_0.$$

For $t = 1, ..., L_{n,k} - 1$ the intervals $\mathbf{I}_{n,k}(t)$ are disjoint and equally spaced.

Set $\mathcal{G}_k = \bigcup_n B(f^*_{n,k}, \rho_{n,k})$. Since $\{f_n\}$ was dense in $C^{\alpha}_1(F)$ by (2.4.21) it is clear that the sets $\mathcal{G}_k$ are dense open in $C^{\alpha}_1(F)$ and hence $\mathcal{G} = \bigcap_k \mathcal{G}_k$ is dense G_δ.

Suppose that $f \in \mathcal{G}$. Then for any $k = 1, 2, ...$ there exists $n(k)$ such that $f \in B(f^*_{n(k),k}, \rho_{n(k),k})$.

Put

$$\mathbf{H} := \{r \in f(F) : \dim_H f^{-1}(r) < \Delta - \delta_0\}.$$

Proceeding towards a contradiction suppose that $\lambda(\mathbf{H}) > 0$.

By (2.4.22) we have

$$f(F) \supset \Big[m_{n(k)} + \rho_{n(k),k}, M_{n(k)} - \frac{M_{n(k)} - m_{n(k)}}{L_{n(k),k}} - \rho_{n(k),k}\Big],$$

and (2.4.26) implies

$$\rho_{n(k),k} < \frac{1}{2} \cdot \frac{M_{n(k)} - m_{n(k)}}{L_{n(k),k}}.$$

Then $\mathbf{p}_{n(k),k}(t) \in f(F)$ for $t = 1, ..., L_{n(k),k} - 2$.

By Lebesgue's Density Theorem for every $0 < \gamma < 1$ for large k there exists $t \in \{2, ..., L_{n(k),k} - 3\}$ such that letting $\mathbf{I}^* = [\mathbf{p}_{n(k),k}(t-1), \mathbf{p}_{n(k),k}(t+1)]$ we have

$$\lambda(\mathbf{I}^* \cap \mathbf{H}) \geq \gamma\lambda(\mathbf{I}^*). \tag{2.4.28}$$

On the other hand, $\mathbf{I}_{n(k),k}(t) \subseteq \mathbf{I}^*$. Set

$$\mathbf{H}^* := \{r \in f(F) \cap \mathbf{I}^* : \dim_H f^{-1}(r) \geq \Delta - \delta_0\}.$$

Using this notation from (2.4.27) it follows that

$$\lambda(\mathbf{H}^* \cap \mathbf{I}^*) = \lambda(\mathbf{H}^*) \geq \frac{\kappa_0}{12} \frac{M_{n(k)} - m_{n(k)}}{L_{n(k),k}} \mathbf{q}_{\min}^{\alpha} \ell_0 = \frac{\kappa_0}{12} \mathbf{q}_{\min}^{\alpha} \ell_0 \frac{\lambda(\mathbf{I}^*)}{2}. \tag{2.4.29}$$

Since $\mathbf{H} \cap \mathbf{H}^* = \emptyset$, adding (2.4.28) to (2.4.29) we obtain

$$\begin{aligned} \lambda(\mathbf{I}^*) &\geq \gamma\lambda(\mathbf{I}^*) + \frac{\kappa_0}{12} \mathbf{q}_{\min}^{\alpha} \ell_0 \frac{\lambda(\mathbf{I}^*)}{2} \\ 1 &\geq \gamma + \frac{\kappa_0}{12} \mathbf{q}_{\min}^{\alpha} \ell_0 \frac{1}{2} \\ 1 - \gamma &\geq \frac{\kappa_0}{12} \mathbf{q}_{\min}^{\alpha} \ell_0 \frac{1}{2}. \end{aligned}$$

This yields a contradiction, as γ can be chosen arbitrarily close to 1. □

2.4.2 Strongly separated fractals

In this subsection, our goal is to prove that $D_*(\alpha, F)$ vanishes for small α in the case when F admits a (ν, ρ) separated structure, to eventually yield Theorem 2.2.5.

Our first lemma shows how this separation condition is related to bi-Lipschitz iterated function systems.

Lemma 2.4.6. *Assume that $f_1, ..., f_m$ is an iterated function system satisfying the* *strong separation condition. Moreover, assume that each f_i is bi-Lipschitz, that is for $1 \leq i \leq m$ there exists $0 < \rho_i, \nu_i < 1$ with*

$$\nu_i|x-y| \leq |f_i(x) - f_i(y)| \leq \rho_i|x-y|.$$

Then the attractor F of the system admits a (ν, ρ) separated structure for some $\nu, \rho > 0$.

More specifically, if each f_i is a similarity, that is F is a self-similar set, then F admits a (ν, ν) separated structure for some $\nu > 0$.

Proof. For any $j_1, ..., j_k \in \{1, 2, ..., m\}$ we say that $f_{j_k}(...(f_{j_1}(F))...) = F_{j_1 j_2 ... j_k}$ is a kth level cylinder of F.

First we show that $\nu = \min_{1 \leq i \leq m} \nu_i$ is a valid choice. To establish this, we will define the required families for any k by considering smartly chosen cylinder sets. Notably, $\mathcal{S}_k$ will consist of cylinders $C_1, ..., C_l$ such that for the diameter $|C_j|$ of any of them,

$$\nu^{k+1}|F| \leq |C_j| \leq \nu^k|F|. \tag{2.4.30}$$

This condition is clearly satisfiable by iteratively splitting the cylinders we consider. In particular, start this procedure with the 0-level cylinder F, split it into m many first level cylinders. Later on, in each step split precisely those cylinders which have diameter larger than $\nu^k|F|$, and leave the others unchanged. Due to the bi-Lipschitz property of each f_i, this algorithm produces a finite system of cylinders in finitely many steps, such that each cylinder satisfies (2.4.30). It yields that the above choice of ν is valid indeed for large enough K.

Assume now that the minimal distance between any two of the sets $F_1, F_2, ..., F_m$ is r, and consider arbitrary cylinders $C_j, C_l \in \mathcal{S}_k$. Now let C be the smallest cylinder set containing both C_j and C_l. In this case, $C = g(F)$, where g is the composition of a finite sequence of functions $f_{i_1}, ..., f_{i_L}$ for some $1 \leq i_1, ..., i_L \leq m$. Consequently, $C_j \subseteq g(F_{j'})$ and $C_l \subseteq g(F_{l'})$ for some $1 \leq j', l' \leq m$. It yields that the distance between C_j and C_l is at least as large as the distance between $g(F_{j'})$ and $g(F_{l'})$. Moreover, C has diameter at least $\nu^k|F|$: otherwise it would not have been splitted during the procedure creating $\mathcal{S}_k$. Hence if $\rho_* = \max_{1\leq i\leq m} \rho_i$, we can deduce that for the number L of functions determining g we have

$$\begin{aligned}\rho_*^L|F| \geq |g(F)| &\geq \nu^k|F| \\ L\log\rho_* &\geq k\log\nu \\ L &\leq \frac{k\log\nu}{\log\rho_*}.\end{aligned}$$

That is $L \leq kL_*$ for

$$L_* = \frac{\log\nu}{\log\rho_*}.$$

Altogether it yields that as the distance between $F_{j'}$ and $F_{l'}$ is at least r, the distance between $g(F_{j'})$ and $g(F_{l'})$ is at least

$$r\nu^{kL_*} = r\left(\nu^{L_*}\right)^k,$$

which implies that ν^{L_*} is a valid choice for ρ with a large enough K, concluding the proof of the first part.

Concerning the statement for self-similar sets, capitalizing on the fact that g is a similarity, we are able to take a more comfortable route to conclude the proof from the observation that C has diameter at least $\nu^k|F|$. Notably, this implies that the similarity ratio of g is at least ν^k, and consequently, the distance between C_j and C_l cannot be smaller than $r\nu^k$. It verifies that in this case $\rho = \nu$ can be chosen. □

The essence of this subsection is the following lemma:

Lemma 2.4.7. *Assume that F admits a (ν, ρ) separated structure, and $0 < \alpha < \frac{\log\nu}{\log\rho}$. Then piecewise constant functions with finitely many pieces form a dense subset of the 1-Hölder-α functions.*

Proof. Taking union over $0 < c < 1$, c-Hölder-α functions clearly form a dense subset of 1-Hölder-α functions. Consequently, it is sufficient to prove that for any c-Hölder-α function f we can find a piecewise constant 1-Hölder-α function $\tilde{f}$ in the ε neighborhood of f in the supremum norm for fixed $\varepsilon > 0$. To this end, choose f_r according to Lemma

2.3.2 such that it is in the $\frac{\varepsilon}{2}$ neighborhood of f, M-Lipschitz and c-Hölder-α. Our aim is to introduce some further perturbation to obtain the 1-Hölder-α function $\tilde{f}$, which is piecewise constant on F. We will achieve this goal by using the covers granted by the separated structure of F. Notably, we will consider the covering $\mathcal{S}_k = \{F_1, ..., F_l\}$ guaranteed by Definition 2.1.7 for large enough k, and define $\tilde{f}$ separately on $F_1, ..., F_l$ by $\tilde{f}\Big|_{F_i} = f_r(x_i)$, using some reference points $x_i \in F_i$. Now we would like to prove that the function $\tilde{f}$ is 1-Hölder-α for large enough k. Choose points y, y' from distinct elements of the covering $\mathcal{S}_k$, where the reference points are x, x'. (If y, y' are in the same element of covering, we have nothing to prove.) We have

$$|\tilde{f}(y) - \tilde{f}(y')| = |f_r(x) - f_r(x')|.$$

Then by the triangle inequality, and the Hölder and Lipschitz properties of f

$$|\tilde{f}(y) - \tilde{f}(y')| \leq |f_r(y) - f_r(y')| + |f_r(y) - f_r(x)| + |f_r(y') - f_r(x')|$$

$$\leq c|y - y'|^\alpha + M|y - x| + M|y' - x'|.$$

Hence it is sufficient to prove

$$c|y - y'|^\alpha + M|y - x| + M|y' - x'| \leq |y - y'|^\alpha,$$

that is

$$M|y - x| + M|y' - x'| \leq (1 - c)|y - y'|^\alpha.$$

Now on the right hand side $|y - y'| \geq \frac{1}{K}\rho^k$, while on the left hand side both distances are at most $K\nu^k$, where K comes from Definition 2.1.7. Thus it suffices to prove that for large enough k we have

$$2K \cdot M\nu^k \leq \frac{1 - c}{K^\alpha}(\rho^k)^\alpha.$$

However, it immediately follows from the choice of α, as that guarantees $\rho^\alpha > \nu$. That is, $\tilde{f}$ is 1-Hölder-α if k is chosen sufficiently large. Moreover, by increasing k, f_r and $\tilde{f}$ can be arbitrarily close to each other. Consequently, $\tilde{f}$ can be in the ε neighborhood of f, which yields the statement of the lemma. $\square$

The following theorem is a straightforward consequence of Lemma 2.4.7:

Theorem 2.4.8. *Assume that F admits a (ν, ρ) separated structure, and $0 < \alpha < \frac{\log \nu}{\log \rho}$. Then for the generic 1-Hölder-α function we have that $\lambda(f(F)) = 0$, and consequently, $\underline{D}_*(\alpha, F) = D_*(\alpha, F) = 0$.*

Proof. Due to Lemma 2.4.7, the piecewise constant 1-Hölder-α functions form a dense subset of the 1-Hölder-α functions. Such a function f_0 has a finite range, hence for every $l \in \mathbb{N}$, in a small enough neighborhood of it, for any function f we have $\lambda(f(F)) < \frac{1}{l}$. By taking the union of all such neighborhoods we find an open, dense subset of the 1-Hölder-α functions, in which $\lambda(f(F)) < \frac{1}{l}$. Taking intersection of these open sets for $l = 1, 2, ...$ we obtain that generically, $\lambda(f(F)) = 0$. $\square$

Proof of Theorem 2.2.5. The statement directly follows from coupling Lemma 2.4.6 and Theorem 2.4.8. $\square$

Taking into consideration Theorem 2.2.8, we can see that in contrast with certain results of fractal geometry, this corollary does not extend to self-similar sets satisfying the open set condition instead of the strong separation condition.

2.5 Constructions and exact calculations

2.5.1 Computation of $D_*(\alpha, F)$ for an example

In this subsection we prove Theorem 2.2.6.

Lemma 2.5.1. *If $F \subset \mathbb{R}^2$ is closed, $f \colon \mathbb{R}^2 \to \mathbb{R}$ and $\lambda \times \lambda(\{(x, f(x,y)) : (x,y) \in F\}) > 0$, then $D_*^f(F) \geq 1$.*

Proof. By Fubini's theorem, there exists a set $H \subset \mathbb{R}$ of positive measure such that for every $r \in H$ we have

$$\lambda\{x \in \mathbb{R} : \text{there exists a } y \in \mathbb{R} \text{ such that } (x,y) \in f^{-1}(r)\} > 0.$$

That is the projection of $f^{-1}(r)$ onto the x axis is of positive measure, and hence $\dim_H(f^{-1}(r)) \geq 1$. This implies $D_*^f(F) \geq 1$. □

Proof of Theorem 2.2.6. As F is totally disconnected, its topological dimension is 0, hence every level set of the typical continuous function defined on it is a singleton, hence $D_*(0,F) = 0$ indeed.

Now fix an $\alpha \in (0,1]$. The upper estimate $D_*(\alpha, F) \leq 1$ is obvious by Theorem 2.2.1.

Using Lemma 2.3.4 we can select a countable dense subset $\{f_m : m \in \mathbb{N}\}$ of $C_{1-}^{\alpha}([0,1/2]^2)$ which consists of locally piecewise affine functions. As every $f \in C_1^{\alpha}(F)$ have an extension in $C_1^{\alpha}([0,1/2]^2)$, $\{f_m|_F : m \in \mathbb{N}\}$ is dense in $C_1^{\alpha}(F)$.

Next we suppose that $m \in \mathbb{N}$ is fixed. Since $\partial_y f_m(x,y)$ takes finitely many different values, we can perturb f_m by adding a function $\tau \cdot y$ with a suitably small τ to it such that $0 < |\partial_y f_m(x,y)|$ wherever $\partial_y f_m(x,y)$ exists. Thus we can assume that there is a $p_m > 0$ such that $p_m < |\partial_y f_m(x,y)|$ (wherever $\partial_y f_m(x,y)$ exists).

Fix $k \geq 2$ such that

$$\sum_{l \geq k} 2^{l^2} \cdot \left(2^{-l^3}\right)^{\alpha} \leq \frac{p_m \cdot 2^{-k^2}}{1000} = \frac{p_{m,k}}{1000}, \tag{2.5.1}$$

where $p_{m,k} := p_m \cdot 2^{-k^2}$.

Since f_m is piecewise affine on $[0,1/2]^2$, we can suppose that k is so large that we can take $j, j' \in \mathbb{Z}$ such that letting $I_i := \left((i-1) \cdot 2^{-k^2}, (i+1) \cdot 2^{-k^2}\right)$ for $i \in \mathbb{Z}$ the function f_m is affine on $Q_{j,j'} := I_j \times I_{j'}$ and $\lambda(F \cap Q_{j,j'}) > 0$.

Select a density point x_0 of $I_j \cap F_0$. By our assumptions, $\partial_y f_m(x_0, y)$ takes the same non-zero value for every $y \in I_{j'}$, and without limiting generality we can assume that it is positive. Set $y_0 := (j'-1)2^{-k^2}$ and $y_1 := (j'+1)2^{-k^2}$, that is $I_{j'} = [y_0, y_1]$. Then

$$f_m(x_0, y_1) - f_m(x_0, y_0) > 2p_{m,k}. \tag{2.5.2}$$

Let $\delta_m := p_{m,k}/10$. Suppose that $f \in B(f_m|_F, \delta_m) \cap C_1^\alpha(F)$. Denote still by f its 1-Hölder-α extension to $[0, 1/2]^2$. By (2.5.2),

$$f(x_0, y_1) - f(x_0, y_0) > p_{m,k}.$$

Since x_0 is a density point of F_0 and $f \in C_1^\alpha([0, 1/2]^2)$ we can choose $\delta_0 > 0$ such that $\lambda(F_0 \cap [x_0, x_0 + \delta_0]) > 0.99\delta_0$ and $|f(x_0, y_i) - f(x, y_i)| \le 0.01p_{m,k}$ for $x \in [x_0, x_0 + \delta]$ and $i = 0, 1$. This implies

$$[f(x_0, y_0) + 0.01p_{m,k}, f(x_0, y_1) - 0.01p_{m,k}] \subset \{f(x, t) : t \in [y_0, y_1]\} \text{ for } x \in [x_0, x_0 + \delta],$$

and by (2.5.2) we have $f(x, y_1) - f(x, y_0) > 0.98p_{m,k}$. Thus,

$$\begin{aligned} &\lambda\big(\{f(x, y) : y \in [y_0, y_1] \cap F_0\}\big) \\ &\ge \lambda\big(\{f(x, y) : y \in [y_0, y_1]\}\big) - \lambda\big(\{f(x, y) : y \in [y_0, y_1] \setminus F_0\}\big) \end{aligned}$$

(using the definition of F_0 and $f \in C_1^\alpha([0, 1/2]^2)$ we can estimate the jumps on the intervals contiguous to F_0)

$$\ge 0.98p_{m,k} - 2\sum_{l=k}^{\infty} 2^{l^2}\left(2^{-l^3}\right)^\alpha \underset{\text{by (2.5.1)}}{\ge} 0.98p_{m,k} - \frac{p_{m,k}}{500} > 0.9p_{m,k}.$$

By Fubini's theorem

$$\begin{aligned} 0 &< \lambda \times \lambda\Big(\big\{(x, f(x, y)) : (x, y) \in \big([x_0, x_0 + \delta_0] \cap F_0\big) \times \big([y_0, y_1] \cap F_0\big)\big\}\Big) \\ &\le \lambda \times \lambda\Big(\big\{(x, f(x, y)) : (x, y) \in F\big\}\Big). \end{aligned}$$

According to Lemma 2.5.1, this implies $D_*^f(F) \ge 1$. Put $\mathcal{G} = \bigcup_{m=1}^\infty B(f_m, \delta_m) \cap C_1^\alpha(F)$. Then $\mathcal{G}$ is an dense open subset of $C_1^\alpha(F)$ and for every $f \in \mathcal{G}$ we have $D_*^f(F) \ge 1$.

Therefore $D_*(\alpha, F) \geq 1$. Since we also know that $D_*(\alpha, F) \leq 1$, this completes the proof. □

2.5.2 Phase transition

In this subsection, our goal is to provide an example which verifies Theorem 2.2.7.

Looking at the example with the Sierpiński triangle our lower estimate for $D_*(\alpha, \Delta)$ guaranteed by Theorem 2.2.8 is positive for positive αs, and hence $D_*(\alpha, \Delta) > D_*(0, \Delta) = 0$. On the other hand, according to Theorem 2.2.5, if F is a self-similar set satisfying the strong separation condition, then $D_*(\alpha, F) = 0$ for $0 < \alpha < 1$. This phenomenon reflects the intuitive difference between these cases: informally speaking, while the Sierpiński triangle is a fairly "thick" fractal, self-similar sets satisfying the strong separation condition are quite loose. It raises the natural question whether there are fractals adhering to an intermediate behaviour in the following sense: for small values of α even the level sets of Hölder-α functions are sufficiently flexible and "compressible" and there exists $1 > \alpha_\phi > 0$ such that $D_*(\alpha, F) = D_*(0, F)$, holds for all $\alpha \in [0, \alpha_\phi)$ while $D_*(\alpha, F) > D_*(0, \Delta)$ holds for $\alpha > \alpha_\phi$. If this happens we say that there is a *phase transition* for $D_*(\alpha, F)$. In a very rough heuristic way we could say that if there is a phase transition then for small values of α the "traffic" corresponding to the level sets is not heavy enough to generate "traffic jams" and can go through the "narrowest" places, while for larger αs "traffic jams" show up and "thicker" parts of the fractal should be used to "accommodate" the level sets.

Next we construct a fractal F for which $D_*(\alpha, F) = D_*(0, F) = 0$ for some small values of α, while $D_*(\alpha, F) > 0$ for large values of α. Theorem 2.2.5 hints us that we can hope for simple examples displaying this phenomenon, however, probably not self-similar ones.

To this end, we construct a fat Cantor set $C = \bigcap_{n=0}^{\infty} C_n$, where (C_n) is a decreasing sequence of sets, such that C_n is the union of 2^n disjoint, closed intervals of the same length. Let $C_0 = [0, 1]$, and for $n > 0$ we obtain C_n by removing an open interval from the middle of each maximal subinterval of C_{n-1}. We make the construction explicit by specifying the length of the maximal subintervals at each level: let it be $l_n = \frac{1}{2^{n+1}-1}$.

Then $2l_n < l_{n-1}$, hence such a system can be constructed indeed by successive interval removals. Moreover, the Cantor set C in the limit is indeed a fat Cantor set in terms of Lebesgue measure, as $\lambda(C_n) = \frac{2^n}{2^{n+1}-1}$, thus

$$\lambda(C) = \lim_{n\to\infty} \lambda(C_n) = \frac{1}{2}.$$

Theorem 2.5.2. *$F = C \times C \subseteq \mathbb{R}^2$ admits phase transition. Notably, for $0 < \alpha < \frac{1}{2}$ we have $D_*(\alpha, F) = 0$, while for $\frac{1}{2} < \alpha \leq 1$ we have $D_*(\alpha, F) = 1$. In particular, Theorem 2.2.7 holds.*

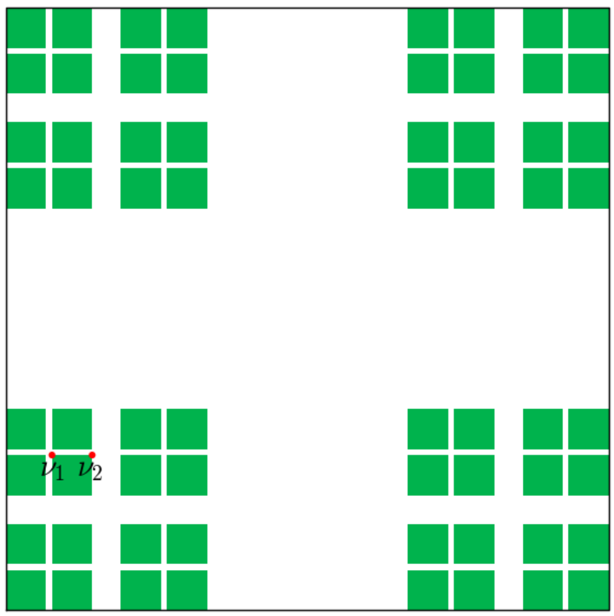

Figure 2.1: Step 3 of construction of F

While the statement concerning small exponents will easily follow from Theorem 2.4.8, the other part is more technical. It requires a lemma, for which we will need the notion of Hausdorff capacity:

Definition 2.5.3. The α dimensional Hausdorff capacity of a set $E \subseteq \mathbb{R}^p$ is

$$\Lambda^\alpha(E) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^\alpha : \ E \subseteq \bigcup_{i=1}^{\infty} U_i \quad \text{for some } (U_i)_{i=1}^{\infty} \right\}.$$

The Hausdorff capacity is closely related to the problem we consider: it gives an upper estimate for the measure $\lambda(f(E))$ if f is a 1-Hölder-α function.

Lemma 2.5.4. *Let I_k be a maximal subinterval of C_k and $\frac{1}{2} < \alpha \leq 1$. Then*

$$\frac{\Lambda^\alpha(I_k \setminus C)}{|I_k|} \to 0,$$

as $k \to \infty$.

Proof. Let $k \geq 1$. By construction, $|I_k| = \frac{1}{2^{k+1}-1} > \frac{1}{2^{k+1}}$. We also know that the length r_m of an interval removed from C_{m-1} to obtain C_m can be estimated from above by

$$r_m = l_{m-1} - 2l_m = \frac{1}{2^m - 1} - \frac{2}{2^{m+1} - 1} = \frac{1}{(2^m - 1)(2^{m+1} - 1)} < \frac{1}{2^{2m}} \tag{2.5.3}$$

for $m > 2$. Now cover the set $I_k \setminus C$ by intervals contiguous to C in I_k. It is easy to see that this covering consists of intervals of length r_m for some $m > k$, and the number of intervals with length r_m is 2^{m-k-1}. Consequently,

$$\Lambda^\alpha(I_k \setminus C) \leq \sum_{m=k+1}^{\infty} 2^{m-k-1} r_m^\alpha \leq \sum_{m=k+1}^{\infty} 2^{m-k-1-2m\alpha}, \tag{2.5.4}$$

where we use (2.5.3) for the second estimate. The geometric series is summable for $\frac{1}{2} < \alpha \leq 1$, and it yields

$$\Lambda^\alpha(I_k \setminus C) \leq 2^{-k-1} \frac{2^{(k+1)(1-2\alpha)}}{1 - 2^{1-2\alpha}} = \frac{2^{-2\alpha(k+1)}}{1 - 2^{1-2\alpha}}. \tag{2.5.5}$$

Consequently, for $\frac{1}{2} < \alpha \leq 1$

$$\frac{\Lambda^\alpha(I_k \setminus C)}{|I_k|} \leq \frac{2^{(k+1)(1-2\alpha)}}{1 - 2^{1-2\alpha}} \to 0, \text{ as } k \to \infty, \tag{2.5.6}$$

which concludes the proof. $\square$

Proof of Theorem 2.5.2. The first statement about $0 < \alpha < \frac{1}{2}$ simply follows from Theorem 2.4.8, as F has a $\left(\frac{1}{2}, \frac{1}{4}\right)$ separated structure. This observation follows easily from calculations carried out in the proof of Lemma 2.5.4: notably, if $\mathcal{S}_k$ consists of the sets of the form $F \cap (I_j \times I_{j'})$, where I_j and $I_{j'}$ are (not necessarily different) maximal subintervals of C_k then each element of $\mathcal{S}_k$ has diameter

$$\sqrt{2} \cdot \frac{1}{2^{k+1} - 1} \leq \sqrt{2} \cdot 2^{-k}.$$

Moreover, for $k \geq 2$ one can easily deduce that the distance between different elements of $\mathcal{S}_k$ is at least

$$\frac{1}{(2^k - 1)(2^{k+1} - 1)} \geq \frac{1}{4} 4^{-k},$$

using the first part of (2.5.3) and the fact that as the elements of $\mathcal{S}_k$ are product sets, they differ in one of their factors. It verifies that F has a $\left(\frac{1}{2}, \frac{1}{4}\right)$ separated structure, and yields the first part of the theorem due to Theorem 2.4.8 and

$$\frac{\log \frac{1}{2}}{\log \frac{1}{4}} = \frac{1}{2}.$$

For the second statement, by Theorem 2.2.1 we have $D_*(\alpha, F) \leq 2 - 1 = 1$ and hence it is sufficient to show that $D_*(\alpha, F) \geq 1$ holds for $\frac{1}{2} < \alpha \leq 1$.

Recall that the union of all the c-Hölder-α functions for $0 < c < 1$ defined on F is a dense subset of 1-Hölder-α functions in the supremum norm. Consequently it would be sufficient to verify that for a fixed c-Hölder-α function f and $\varepsilon > 0$ we can find a 1-Hölder-α function $\tilde{f} \in B(f, \varepsilon)$ and $\varepsilon' > 0$ such that for any 1-Hölder-α function $g \in B(\tilde{f}, \varepsilon')$ we have $\dim_H(g^{-1}(r)) \geq 1$ in a set of positive measure of rs. In fact, it would verify that $D_*^g(F) = 1$ on a dense open set, which clearly yields that it is the generic behaviour.

As f is c-Hölder-α, $f + h$ is a 1-Hölder-α function if h is $(1-c)$-Hölder-α. We will use this property to introduce the perturbed function $\tilde{f}$, for which some kth level cylinder of $C \times C$ (which is a square) has adjacent vertices $v_1 = (x_1, y_1), v_2 = (x_2, y_1)$ such that

$$|\tilde{f}(v_1) - \tilde{f}(v_2)| \geq (1-c)|v_1 - v_2| = (1-c)|x_1 - x_2|. \tag{2.5.7}$$

More explicitly, choose k large enough such that for a maximal subinterval $I = [x_1, x_2] \subseteq$

C_k we have

$$\frac{\Lambda^\alpha(I \setminus C)}{|x_1 - x_2|} < \delta, \tag{2.5.8}$$

where δ is to be fixed later. By Lemma 2.5.4, this estimate holds for large enough k. We can assume without loss of generality that $f(v_1) \le f(v_2)$ for the vertices $v_1 = (x_1, y_1), v_2 = (x_2, y_1)$ of some kth level cylinder of $C \times C$, as the other case is similar. We can also assume that these vertices are top vertices of that kth level cylinder see Figure 2.1. Hence if we define

$$h(x,y) = \begin{cases} 0, & \text{if } x < x_1, \\ (1-c)(x - x_1), & \text{if } x_1 \le x \le x_2, \\ (1-c)(x_2 - x_1) & \textit{otherwise}, \end{cases} \tag{2.5.9}$$

then $\tilde{f} = f + h$ satisfies (2.5.7).

We take a $\delta' > 0$ which will be specified later. By continuity, we can choose r such that for any $y \in [y_1 - r, y_1] \cap C$ we have

$$|\tilde{f}(x_1, y_1) - \tilde{f}(x_1, y)| < \delta' \text{ and } |\tilde{f}(x_2, y_1) - \tilde{f}(x_2, y)| < \delta'.$$

Consequently, if $g \in B(\tilde{f}, \delta')$, then

$$|g(x_1, y_1) - g(x_1, y)| < 3\delta' \text{ and } |g(x_2, y_1) - g(x_2, y)| < 3\delta'.$$

Besides that, as $(x_1, y_1), (x_2, y_1)$ were chosen as top vertices of cylinders of $C \times C$,

$$\lambda(C \cap [y_1 - r, y_1]) =: \eta > 0.$$

Now by Theorem 1 of [21] we can extend g to a 1-Hölder-α function defined on $[0,1]^2$. Denote the extended function by g as well. Due to the choice of r, the continuity of the extended function, and the intermediate value theorem, we have that the g-image of the planar line segment $[(x_1, y), (x_2, y)]$ for any $y \in [y_1 - r, y_1] \cap C$ contains the interval $[\tilde{f}(v_1) + 3\delta', \tilde{f}(v_2) - 3\delta']$. This interval has measure at least $(1-c)(x_2 - x_1) - 6\delta'$.

Moreover, as $[(x_1, y), (x_2, y)] \setminus F \subseteq [(x_1, y), (x_2, y)]$ is congruent to $I \setminus C \subseteq I$, due to (2.5.8) and the fact that g is 1-Hölder-α, we have

$$\lambda(g([(x_1, y), (x_2, y)] \setminus F)) \leq \delta(x_2 - x_1).$$

Consequently, the remainder measure of values is taken on $[(x_1, y), (x_2, y)] \cap F$, yielding that $g([(x_1, y), (x_2, y)] \cap F) \cap [\tilde{f}(v_1) + 3\delta', \tilde{f}(v_2) - 3\delta']$ has measure at least $(1 - c - \delta)(x_2 - x_1) - 6\delta'$. Fix now the values of δ and δ' such that this quantity is positive.

By the above calculations, we can conclude that we have

$$\begin{aligned}\lambda_2 \Big\{(g(x, y), y) : g(x, y) \in [\tilde{f}(v_1) + 3\delta', \tilde{f}(v_2) - 3\delta'],\ y \in [y_1 - r, y_1] \cap C,\ x \in [x_1, x_2] \cap C\Big\} \\ \geq \eta \cdot ((1 - c - \delta)(x_2 - x_1) - 6\delta') > 0,\end{aligned} \tag{2.5.10}$$

where λ_2 denotes the two-dimensional Lebesgue measure. Note that this set is measurable indeed as it is the image of the compact set

$$([x_1, x_2] \cap C) \times ([y_1 - r, y_1] \cap C),$$

under the continuous mapping $(x, y) \to (g(x, y), y)$. However, by Fubini's theorem, we can rewrite the measure in (2.5.10) as

$$\int_{t=\tilde{f}(v_1)+4\delta'}^{\tilde{f}(v_2)-4\delta'} \lambda \{y : y \in [y_1 - r, y_1] \cap C \text{ and } g(x, y) = t \text{ for an } x \in [x_1, x_2] \cap C\}\, dt. \tag{2.5.11}$$

As this integral is positive, the integrand is positive on a set A of positive measure. That is, for any $t \in A$ we have that

$$\lambda \{y : y \in [y_1 - r, y_1] \cap C \text{ and } g(x, y) = t \text{ for an } x \in [x_1, x_2] \cap C\} > 0.$$

which is equivalent to that the projection of

$$g^{-1}(t) \cap ([x_1, x_2] \times [y_1 - r, y_1]) \cap F$$

to the second coordinate has positive measure. That is, the projection has Hausdorff dimension 1, which obviously yields that $g^{-1}(t) \cap ([x_1, x_2] \times [y_1 - r, y_1]) \cap F$ has Hausdorff dimension at least 1 as well for a set of ts with positive measure. It concludes the proof. $\square$

2.6 Estimates for the Sierpiński triangle

2.6.1 Lower estimate for arbitrary functions

In this section, our aim is to prove Theorem 2.2.8. As Δ is a connected self-similar set, hence by Theorem 2.2.4 $D_*(\alpha, \Delta)$ equals the Hausdorff dimension of almost every level set of a generic 1-Hölder-α function.

Some people prefer to work with different versions of the Sierpiński triangle. We work with the one which is obtained by starting with an equilateral triangle of side length one. Hence it satisfies our earlier assumptions about the fractals considered since its diameter equals one. Its topological Hausdorff dimension equals one and this implies that for a generic continuous function every level set is zero-dimensional, see [7]. The level sets of continuous functions are very flexible, and very "compressible", hence during the proof of this theorem one can capitalize on the fact that the Sierpiński triangle is very "thin" near the vertices of the small triangles appearing during its construction. As Hölder-α functions do not have this flexibility, one can expect that their level sets generically exhibit a different behaviour.

By its definition the Sierpiński triangle is expressible as $\Delta = \bigcap_{n=0}^{\infty} \Delta_n$ where Δ_n is the union of the triangles appearing at the nth step of the construction. The set of triangles on the nth level is τ_n. For $T \in \tau_n$ we denote by $V(T)$ the set of its vertices. Moreover, let $\mathbf{V}_n$ be the set of the points which are vertices of some $T \in \tau_n$, and their union is $\mathbf{V} = \bigcup_{n=0}^{\infty} \mathbf{V}_n$. We are interested in the Hausdorff dimension of the level sets of a 1-Hölder-α function $f : \Delta \to \mathbb{R}$ for $0 < \alpha \leq 1$.

Suppose $l \in \mathbb{N}$. It will be useful for us to define the self-similar set $\Delta^l \subseteq \Delta$ as well. It is induced by the similarities which map Δ_0 to any triangle $T \in \tau_l$ on the boundary of Δ_0. For example, $\Delta^1 = \Delta$, while the case $l = 3$ is shown by Figure 2.2 where the

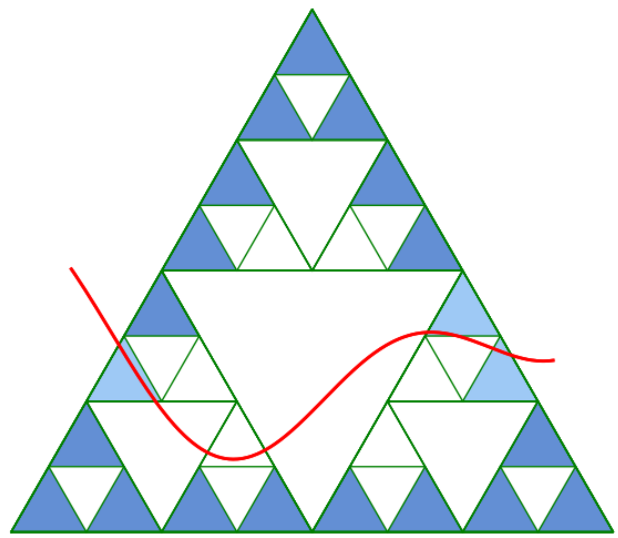

Figure 2.2: Sierpiński triangle and a crossing level set, shaded blue triangles are used during the definition of Δ^3, the lighter shaded triangles correspond to the first level approximation $G_1^3(r)$ of the red level set at level r

shaded triangles on the sides of Δ_0 are used in the definition of Δ^3. (The lighter shaded triangles will have importance later.)

One can easily check that the number of triangles used in the construction of Δ^l is $3(2^l - 1)$, and the nth level of Δ^l consists of certain triangles of τ_{nl}. Let us denote the family of these triangles by τ_n^l, the union of their vertices for fixed n by $\mathbf{V}_n^l$, and the union of vertices for all the triangles in some τ_n^l by $\mathbf{V}^l = \bigcup_{n=1}^{\infty} \mathbf{V}_n^l$. It is clear that a 1-Hölder-α function $f : \Delta \to \mathbb{R}$ restricted to Δ^l is still a 1-Hölder-α function.

Suppose that $f : \Delta^l \to \mathbb{R}$ is a 1-Hölder-α function and $r \notin f(\mathbf{V})$. We can define the nth approximation of $f^{-1}(r)$ denoted by $G_n^l(r)$ for any n and $r \in f(\Delta^l)$ as the union of some triangles in τ_n^l. More explicitly, $T \in \tau_n^l$ is taken into $G_n^l(r)$ if and only if T has vertices v and v' such that $f(v) < r < f(v')$, that is, r is in the interior of the convex hull $\mathrm{conv}(f(V(T)))$. The idea is that in this case $f^{-1}(r)$ necessarily intersects T. On Figure 2.2 the level set corresponding to $f^{-1}(r)$ is the intersection of the red curve with Δ. The set $G_1^3(r)$ consists of the light shaded triangles. Now it is easy to check that, using the notation conv for the convex hull,

$$\operatorname{conv}(f(V(T))) \subseteq \bigcup_{T' \in \tau_{n+1}^l,\ T' \subseteq T} \operatorname{conv}(f(V(T'))), \tag{2.6.1}$$

hence if $G_n^l(r)$ contains a triangle $T \in \tau_n^l$ then $G_{n+1}^l(r)$ contains a triangle $T' \in \tau_{n+1}^l$ such that $T' \subseteq T$. We introduce the following terminology: we say that $T' \in \tau_{n+k}^l$ is the l, r-descendant of $T \subseteq G_n^l(r)$ if there exists a sequence $T_0 = T \supseteq T_1 \supseteq \ldots \supseteq T_k = T'$ of triangles such that $T_i \in \tau_{n+i}^l$ and $T_i \subseteq G_{n+i}^l(r)$ for $i = 0, 1, \ldots, k$. We denote the set of l, r-descendants of T by $\mathcal{D}_r^l(T)$.

Observe the obvious property that for any T we can label the vertices in $V(T)$ such that $f(v_0) \le f(v_2) \le f(v_1)$. We refer to v_0, v_1 as the extreme vertices of T. Since we supposed that $r \in \operatorname{intconv}(f(V(T)))$ we have $f(v_1) > f(v_0)$. If $f(v_0) = f(v_2)$ then we call only one vertex v_0 as an extreme vertex, the other vertex denoted by v_2 will not regarded to be an extreme vertex. We proceed analogously if $f(v_1) = f(v_2)$.

We define the conductivity $\kappa_n^l(T) = \kappa_n^l(T, f)$ of any triangle $T \in \tau_n^l$ inductively (as f is fixed during most of our arguments, it will be omitted from the notation unless it might cause ambiguity). If $n = 0$, we define $\kappa_0^l(T) = 1$. On the other hand, if $n \ge 1$, there is a unique triangle $T' \in \tau_{n-1}^l$ such that $T \subseteq T'$. Now if T is one of the two triangles at an extreme vertex of T', then let $\kappa_n^l(T) = \kappa_{n-1}^l(T')$ (in this case we say that T is an extreme triangle of T'), while in any other case we let $\kappa_n^l(T) = \frac{1}{2}\kappa_{n-1}^l(T')$. The following lemma can be thought of as the weak conservation of conductivity:

Lemma 2.6.1. *Assume that $T \in G_n^l(r)$ and $k \ge 1$. Then we have*

$$\sum_{T' \in \tau_{n+k}^l \cap \mathcal{D}_r^l(T)} \kappa_{n+k}^l(T') \ge \kappa_n^l(T).$$

Proof. By induction, it suffices to work with $k = 1$. Consider the vertices ν_0 and ν_1 on which f is minimal and maximal respectively in $V(T)$. Since $r \notin f(\mathbf{V})$ and $f(\nu_0) < r < f(\nu_1)$ there are at least two edges of T containing points of $f^{-1}(r)$. One of them is the one connecting ν_0 and ν_1.

If there is a $T' \in \tau_{n+1}^l$ which contains all the intersection points of the edges of T

and $f^{-1}(r)$ then it should contain ν_0, or ν_1. Hence, it is an extreme triangle of T and the conductivity of T' equals that of T.

Otherwise we have at least two triangles of $G^l_{n+1}(r)$ which are in T and the sum of their conductivity is at least the conductivity of T. $\square$

Now we have enough tools to prove Theorem 2.2.8.

Proof of Theorem 2.2.8. Since Δ is compact and connected $f(\Delta)$ is a closed interval. Moreover as $\mathbf{V}$ is a countable and dense subset of Δ the set $f(\mathbf{V})$ is countable and dense in $f(\Delta)$. Suppose that $r \in \operatorname{int}(f(\Delta)) \setminus f(\mathbf{V})$. Then we can find $T \in \bigcup_n \tau_n$ such that $r \in \operatorname{intconv}(f(V(T)))$. Due to self-similarity properties, we can assume $T = \Delta_0$.

Restrict f to some Δ^l. The number l will be fixed later, it is useful to think of it as something large. Roughly speaking, in order to bound the dimension, we would like to obtain that for Lebesgue almost every $r \in f(\Delta)$ we have that $f^{-1}(r)$ does not intersect triangles with high conductivity on the nth level for large n. Consequently, by Lemma 2.6.1 we could deduce that $f^{-1}(r)$ intersects "many" triangles, which yields "high" Hausdorff dimension due to the Mass Distribution Principle (Theorem 2.1.6). In order to formalize this idea, we would like to estimate the number of triangles with high conductivity.

For any $T \in \tau^l_n$ we can consider the chain of triangles $T_1, T_2, \ldots, T_n$ such that $T_i \supseteq T$ and $T_i \in \tau^l_i$. We bound from above the number of triangles in τ^l_n, whose conductivity is at least 2^{-nd_1}, where $0 < d_1 \le \frac{1}{2}$ is chosen to be a small rational number, hence nd_1 is an integer for infinitely many n. From this point on we restrict our arguments to such ns, that is we suppose that $n = n'\mathbf{q}$ for some $n' = 1, 2, ...$ where $\mathbf{q} = \min\{m \in \mathbb{N} : md_1 \in \mathbb{N}\}$. The conductivity is at least 2^{-nd_1} if T_i is an extreme triangle for at least $n - nd_1$ of the indices $i = 1, 2, ..., n$.

The number of such triangles is estimated from above by

$$\binom{n}{n - nd_1} \left(3\left(2^l - 1\right)\right)^{nd_1} 2^{n-d_1 n} = \binom{n}{nd_1} \left(3\left(2^l - 1\right)\right)^{nd_1} 2^{n-d_1 n},$$

as we can choose the $n - nd_1$ places where we use one of the two extreme triangles, and in the remaining places we allow the usage of any of the $3\left(2^l - 1\right)$ triangles, hence giving an upper bound. By standard bounds on binomial coefficients, this can be estimated from above by

$$\left(\frac{en}{nd_1}\right)^{nd_1}\left(3\left(2^l-1\right)\right)^{nd_1}2^{n-d_1n}. \tag{2.6.2}$$

The diameter of the triangles in τ_n^l is 2^{-ln}. Consequently, due to f being 1-Hölder-α and by (2.6.2), we know that the f-image of the union of the well conducting triangles has Lebesgue measure at most

$$\left(\frac{e}{d_1}\right)^{nd_1}\left(3\left(2^l-1\right)\right)^{nd_1}2^{n-d_1n}2^{-ln\alpha}=\left(\left(\frac{e}{d_1}\right)^{d_1}\left(3\left(2^l-1\right)\right)^{d_1}2^{1-d_1-l\alpha}\right)^n=:c^n. \tag{2.6.3}$$

Assume that $c<1$. Then the corresponding series is convergent, hence we can apply the Borel–Cantelli lemma to deduce that almost every $r\in\operatorname{conv}(f(V(T))$ appears in the image of well conducting triangles only on finitely many levels. Consequently, for almost every r, if n is large enough, $f^{-1}(r)$ must intersect at least 2^{nd_1} triangles of τ_n^l, as the sum of the conductivities of triangles in $T\in\tau_n^l$ for which $r\in\operatorname{conv}(f(V(T)))$, is at least 1. We will use this observation paired with the Mass Distribution Principle, Theorem 2.1.6 to give a lower bound on the dimension of almost every level set, but first, let us consider the question how to choose l,d_1 in order to guarantee that $c<1$. Elaborating (2.6.3), we would like to assure

$$\left(\frac{e}{d_1}\right)^{d_1}\left(3\left(2^l-1\right)\right)^{d_1}2^{1-d_1}2^{-l\alpha}<1. \tag{2.6.4}$$

If this inequality holds for 2^l instead of 2^l-1, that is still fine for our purposes. Rewriting our powers in base e, it leads to

$$\exp\left(d_1-d_1\log d_1+d_1\log 3+d_1l\log 2+\log 2-d_1\log 2-\alpha l\log 2\right)<1,$$

that is after taking logarithm

$$d_1(1-\log d_1+\log 3-\log 2)+\log 2+l(d_1-\alpha)\log 2<0.$$

We clearly need $d_1 < \alpha$ to satisfy this inequality, as only the third term can be negative. Fixing this assumption, after rearrangement we obtain that it holds if and only if

$$\frac{d_1(1 - \log d_1 + \log 3 - \log 2) + \log 2}{(\alpha - d_1)\log 2} = \frac{d_1(1 + \log \frac{3}{2d_1}) + \log 2}{(\alpha - d_1)\log 2} < l. \tag{2.6.5}$$

No matter how we fix the rational number $0 < d_1 < \alpha$, such an l implies $c < 1$. We notice that d_1 can be chosen arbitrarily close to $\frac{\alpha}{2}$, and due to the continuity of the left hand side of (2.6.5), if they are sufficiently close to each other, then we can choose l so that

$$\frac{\frac{\alpha}{2}(1 + \log \frac{3}{\alpha}) + \log 2}{\frac{\alpha}{2}\log 2} < l \leq 1 + \frac{\frac{\alpha}{2}(1 + \log \frac{3}{\alpha}) + \log 2}{\frac{\alpha}{2}\log 2}. \tag{2.6.6}$$

We recall that for such l, d_1 we have that for almost every r, if n is large enough, $f^{-1}(r)$ can only intersect triangles of τ_n^l with conductivity smaller than 2^{-nd_1}. Fix such an r and consider only such large enough ns. We define a probability measure μ on Δ^l.

Due to Kolmogorov's extension theorem (see for example [33], [42] or [28]) it suffices to define consistently $\mu(T \cap \Delta^l)$ for any triangle T in τ_n^l. First, if T is not an l, r-descendant of Δ_0, let $\mu(T \cap \Delta^l) = 0$. For descendants, we proceed by recursion. Notably, if T is an l, r-descendant in τ_n^l, and $\mu(T \cap \Delta)$ is already defined, then we divide its measure among its l, r-descendants in τ_{n+1}^l proportionally to their conductivity. More explicitly, for an l, r-descendant $T^* \in \tau_{n+1}^l$ of T we define

$$\mu(T^* \cap \Delta^l) = \mu(T \cap \Delta^l) \frac{\kappa_{n+1}^l(T^*)}{\sum_{T' \in \tau_{n+1}^l,\ T' \text{ is an } l, r\text{-descendant of } T} \kappa_{n+1}^l(T')}.$$

Then using Lemma 2.6.1 by induction it is clear that

$$\mu(T \cap \Delta^l) \leq \kappa_n^l(T) \text{ for any } l, r\text{-descendant } \Delta_0.$$

Hence,

$$\mu(T \cap \Delta^l) \leq 2^{-nd_1}. \tag{2.6.7}$$

Next we want to use the Mass Distribution Principle. Recall that we assumed that we work with ns of the form $n'\mathbf{q}$. Now assume that we have a Borel set $U \in \Delta^l$ such that for its diameter we have $2^{-n'\mathbf{q}l} \le |U| \le 2^{-(n'-1)\mathbf{q}l}$. By a simple geometric argument one can show that U might intersect at most C triangles in $\tau^l_{n'\mathbf{q}}$ for some constant C not depending on n'. (One can consider the triangular lattice formed by triangles with side length $2^{-ln'\mathbf{q}}$ and it is easy to see that a Borel set with diameter $2^{-(n'-1)\mathbf{q}l}$ can intersect only a limited number of the triangles.) Consequently, the number of l, r-descendants of Δ_0 in $\tau^l_{n'\mathbf{q}}$ intersected by U is also bounded by C. For such an l, r-descendant T we can apply (2.6.7), hence

$$\mu(U) \le 2^{-n'\mathbf{q}d_1}C.$$

As $|U| \ge 2^{-n'\mathbf{q}l}$, the mass distribution principle tells us that if there exists C', $s > 0$ independent of n' with

$$2^{-n'\mathbf{q}d_1}C \le \left(2^{-n'\mathbf{q}l}\right)^s C',$$

then $s \le \dim_H(f^{-1}(r))$. Such a C' exists if and only if

$$s \le \frac{d_1}{l}.$$

Hence the expression on the right hand side of this inequality is a good choice for s in the mass distribution principle, thus it is a lower estimate for $\dim_H(f^{-1}(r))$ for any valid pair l, d_1. Using (2.6.6) and the argument leading to it, we can approximate $\frac{\alpha}{2}$ by possible d_1s and for sufficiently good approximations we can use

$$l \le 1 + \frac{\frac{\alpha}{2}(1 + \log\frac{3}{\alpha}) + \log 2}{\frac{\alpha}{2}\log 2} = 1 + \frac{1 + \log\frac{3}{\alpha}}{\log 2} + \frac{2}{\alpha}.$$

Consequently,

$$\dim_H(f^{-1}(r)) \ge \frac{\frac{\alpha}{2}}{1 + \frac{1+\log\frac{3}{\alpha}}{\log 2} + \frac{2}{\alpha}} > 0.$$

□

2.6.2 Upper estimate for the generic function

This subsection is dedicated to the proof of Theorem 2.2.9. We will need the following definition, specific to the Sierpiński triangle.

Definition 2.6.2. We say that $f : \Delta \to \mathbb{R}$ is a *piecewise affine function at level* $n \in \mathbb{N}$ *on the Sierpiński triangle* if it is affine on any $T \in \tau_n$.

If a piecewise affine function at level $n \in \mathbb{N}$ on the Sierpiński triangle satisfies the property that for any $T \in \tau_n$ one can always find two vertices of T where f takes the same value, then we say that f is a *standard piecewise affine function at level* $n \in \mathbb{N}$ *on the Sierpiński triangle.*

A function $f : \Delta \to \mathbb{R}$ is a *strongly piecewise affine function on the Sierpiński triangle* if there is an $n \in \mathbb{N}$ such that it is a piecewise affine function at level n.

Here we state a specialized version of Lemma 2.3.4 valid for the Sierpiński triangle.

Lemma 2.6.3. *Assume that* $0 < \alpha < 1$*, and* $0 < c$ *are fixed. Then the locally non-constant standard strongly piecewise affine* c^-*-Hölder-*α *functions defined on* Δ *form a dense subset of the* c*-Hölder-*α *functions.*

Before proving this lemma we need to state and prove another one.

Recall that $\mathbf{V}_n = \bigcup_{T \in \tau_n} V(T)$.

Lemma 2.6.4. *Suppose,* $0 < \varepsilon$*,* $0 < \alpha < 1$*,* $0 < c$*,* $f : \Delta \to \mathbb{R}$ *is Lipschitz-*M *and* c^-*-Hölder-*α *on* Δ*. Then there exists* $N \in \mathbb{N}$ *such that for any fixed* $n \geq N$ *if for any* $T \in \tau_n$ g *is* $c/8$*-Hölder-*α *on* $T \cap \Delta$ *and* $g(x) = f(x)$ *for all* $x \in \mathbf{V}_n$ *then*

$$\|g - f\|_\infty < \varepsilon \text{ and } g \text{ is } c^-\text{-Hölder-}\alpha \text{ in } \Delta. \tag{2.6.8}$$

Proof. Since f is c^--Hölder-α on Δ we can choose $0 < c' < c$ such that f is c'-Hölder-α on Δ.

If

$$|x - y| < \left(\frac{c}{16M}\right)^{\frac{1}{1-\alpha}} =: \overline{M}$$

then

$$|f(x) - f(y)| \leq M|x - y|^{1-\alpha}|x - y|^\alpha < \frac{c}{16}|x - y|^\alpha.$$

Choose $c'' \in (c', c)$.

First we prove that g is c^--Hölder-α.

Suppose that $|x-y| \geq \overline{M}/4$, $x \in T_x \in \tau_n$, and $y \in T_y \in \tau_n$ and select vertices

$$\nu_x \in V(T_x) \text{ and } \nu_y \in V(T_y). \tag{2.6.9}$$

Then by our assumption $f(\nu_x) = g(\nu_x)$ and $f(\nu_y) = g(\nu_y)$. Since the diameter of T_x and T_y equals 2^{-n} we obtain

$$\begin{aligned}|g(x)-g(y)| &\leq |g(x)-g(\nu_x)| + |g(\nu_x)-g(\nu_y)| + |g(\nu_y)-g(y)| \\ &\leq 2c2^{-n\alpha} + |f(\nu_x)-f(\nu_y)| \leq 2c2^{-n\alpha} + c'|\nu_x-\nu_y|^\alpha \\ &\leq 2c2^{-n\alpha} + c'\big(|x-y| + 2\cdot 2^{-n}\big)^\alpha \xrightarrow[n\to\infty]{} c'|x-y|^\alpha,\end{aligned}$$

where the convergence is uniform due to $|x-y|$ being separated from zero. Thus, we can choose N large enough (independently of x and y) such that

$$|g(x)-g(y)| \leq c''|x-y|^\alpha. \tag{2.6.10}$$

Suppose $|x-y| < \overline{M}/4$. If $T_x = T_y$ then by our assumption

$$|g(x)-g(y)| < \frac{c}{8}|x-y|^\alpha. \tag{2.6.11}$$

If $T_x \neq T_y$, but T_x and T_y has a common vertex v then by geometric properties of the Sierpiński triangle $xvy\sphericalangle \geq \pi/6$, hence by the Law of sines

$$|x-v| = \frac{|x-y|\sin(vyx\sphericalangle)}{\sin(xvy\sphericalangle)} \leq |x-y|\frac{2}{\sqrt{3}}$$

and similarly $|y-v| \leq |x-y|\frac{2}{\sqrt{3}}$. Hence,

$$\begin{aligned}|g(x)-g(y)| \leq |g(x)-g(v)| + |g(v)-g(y)| &\leq \frac{c}{8}|x-v|^\alpha + \frac{c}{8}|v-y|^\alpha \\ &\leq 2\frac{c}{8}\Big(\frac{2}{\sqrt{3}}\Big)^\alpha |x-y|^\alpha < \frac{c}{2}|x-y|^\alpha.\end{aligned} \tag{2.6.12}$$

If T_x and T_y does not have a common vertex then $|x-y| \geq 2^{-n}\frac{\sqrt{3}}{2}$. Choose $v_x \in$

$V(T_x)$ and $v_y \in V(T_y)$. Then

$$|v_x - v_y| \leq |x-y| + 2 \cdot 2^{-n} \leq |x-y|\Big(1 + 2 \cdot \frac{2}{\sqrt{3}}\Big) < 4|x-y|.$$

Thus

$$\begin{aligned} |g(x)-g(y)| &\leq |g(x)-g(v_x)| + |g(v_x)-g(v_y)| + |g(v_y)-g(y)| \\ &\leq 2\frac{c}{8}2^{-n\alpha} + |f(v_x)-f(v_y)| < 2\frac{c}{8}\Big(\frac{2}{\sqrt{3}}\Big)^{\alpha}|x-y|^{\alpha} + \frac{c}{16}|v_x-v_y|^{\alpha} \qquad (2.6.13) \\ < 2\frac{c}{8}\Big(\frac{2}{\sqrt{3}}\Big)^{\alpha}|x-y|^{\alpha} &+ \frac{c}{16}4^{\alpha}|x-y|^{\alpha} \leq c\Big(\frac{1}{2\sqrt{3}} + \frac{1}{4}\Big)|x-y|^{\alpha} < \frac{c}{\sqrt{3}}|x-y|^{\alpha}. \end{aligned}$$

From (2.6.10), (2.6.11), (2.6.12) and (2.6.13) it follows that g is c^{-}-Hölder-α on Δ.

To see the inequality in (2.6.8) select $v_x \in V(T_x)$. Then

$$|f(x)-g(x)| \leq |f(x)-f(v_x)| + |f(v_x)-g(v_x)| + |g(x)-g(v_x)| < M|x-v_x| + 0$$

$$+\frac{c}{8}|x-v_x|^{\alpha} \leq M \cdot 2^{-n} + \frac{c}{8}2^{-n\alpha} \leq M \cdot 2^{-N} + \frac{c}{8}2^{-N\alpha} < \varepsilon,$$

if N is chosen sufficiently large. □

Proof of Lemma 2.6.3 . With a rather straightforward modification of the proof of Lemma 2.3.4 one can verify the following weaker form of Lemma 2.6.3: locally non-constant strongly piecewise affine c^{-}-Hölder-α functions defined on Δ form a dense subset of the c-Hölder-α functions.

Hence suppose that $f \in C_c^{\alpha}(\Delta)$ and $\varepsilon > 0$ are given. By the previous remark choose a locally non-constant $f_1 \in C_{c^-}^{\alpha}(\Delta)$ and $n \in \mathbb{N}$ such that

$$\|f - f_1\|_{\infty} < \varepsilon/2 \text{ and } f_1 \text{ is piecewise affine on any } T \in \tau_n. \qquad (2.6.14)$$

By our assumption about the diameter of Δ, the triangles in τ_n are of side length 2^{-n}.

Since f_1 is piecewise affine on any $T \in \tau_n$ it is Lipschitz-M for a suitable M and c^{-}-Hölder-α on Δ. By Lemma 2.6.4 used with f_1 and $\varepsilon/2$ instead of f and ε choose N.

We want to obtain a locally non-constant *standard* strongly piecewise affine c^{-}-

Hölder-α function f_2 which is ε-close to f. Since f_1 is a locally non-constant strongly piecewise affine c^--Hölder-α function which is $\varepsilon/2$-close to f we will modify f_1 to obtain f_2, ε-close to f.

Select a sufficiently large $n' \geq \max\{N, n\}$ which satisfies

$$\frac{4}{\sqrt{3}} M (2^{-n'})^{1-\alpha} < \frac{c}{8}. \tag{2.6.15}$$

To obtain f_2 we will modify f_1 on the triangles $T \in \tau_{n'}$. On $\mathbf{V}_{n'}$ the functions f_2 and f_1 will coincide.

Suppose that $T \in \tau_{n'}$ is arbitrary. Denote its vertices by v_1, v_2 and v_3. Suppose that v_4, v_5 and v_6 are the midpoints of the segments v_1v_2, v_2v_3 and v_1v_3, respectively. We denote by T_1, T_2 and T_3 the triangles $v_1v_4v_6$, $v_4v_2v_5$ and $v_5v_3v_6$, respectively. The triangles T_j, $j = 1, 2, 3$ belong to $\tau_{n'+1}$.

We define $f_2(v_1) = f_2(v_4) = f_1(v_1)$, $f_2(v_2) = f_2(v_5) = f_1(v_2)$ and $f_2(v_3) = f_2(v_6) = f_1(v_3)$. We also assume that f_2 is affine on any triangle $T' \in \tau_{n'+1}$.

By our choice of M we have

$$|f_1(v_i) - f_1(v_j)| \leq M \cdot 2^{-n'} \text{ for any } i, j \in \{1, 2, 3\}.$$

Suppose that $x, y \in T_1 \cap \Delta$ (a similar argument works for the triangles T_2 and T_3). Denote by π the orthogonal projection onto the second coordinate "$\mathbf{y}$"-axis then

$$|f_2(x) - f_2(y)| \leq \frac{|f_1(v_3) - f_1(v_1)|}{\frac{\sqrt{3}}{2} \cdot 2^{-n'-1}} |\pi(x) - \pi(y)| \leq \frac{4}{\sqrt{3}} M |x - y|$$

$$= \Big(\frac{4}{\sqrt{3}} M |x - y|^{1-\alpha}\Big) |x - y|^{\alpha} \leq \frac{4}{\sqrt{3}} M (2^{-n'})^{1-\alpha} |x - y|^{\alpha} < \frac{c}{8} |x - y|^{\alpha},$$

where at the last step we used (2.6.15). Hence if Lemma 2.6.4 is applied with the constants fixed above to the function f_2 as g, we obtain a standard strongly piecewise affine c^--Hölder-α function which is ε-close to f. $\square$

We denote by Δ^* the rescaled and translated copy of Δ in a way that the vertices of Δ^* are $v_1^* = (0, 0)$, $v_2^* = (2/\sqrt{3}, 0)$ and $v_3^* = (1/\sqrt{3}, 1)$.

It is clear that $\mathbb{R}^2$ can be tiled by translated copies of the triangle $v_1^* v_2^* v_3^*$ and its mirror image about the x-axis. We denote the system of these triangles by $\mathbf{T}_0^*$. For $n \in \mathbb{N}$ we also use the scaled copies of this triangular tiling consisting of triangles of the form $2^{-n}T$, $T \in \mathbf{T}_0^*$. The system of triangles belonging to this tiling is denoted by $\mathbf{T}_n^*$. During the definition of box dimension many different concepts can be used, see for example [18]. Given a set $F \subseteq \mathbb{R}^2$ we denote by $\mathbf{N}_n^*(F)$ the number of those triangles $T \in \mathbf{T}_n^*$ which intersect F. It is an easy exercise to see that

$$\overline{\dim}_B F = \limsup_{n\to\infty} \frac{\log \mathbf{N}_n^*(F)}{n \log 2} \text{ and } \underline{\dim}_B F = \liminf_{n\to\infty} \frac{\log \mathbf{N}_n^*(F)}{n \log 2}. \tag{2.6.16}$$

Lemma 2.6.5. *Suppose* $0 < \alpha < 1$. *There exists* $\phi : \Delta^* \to [0,1]$, $\phi \in C_3^\alpha(\Delta^*)$ *such that* $\phi(v_1^*) = \phi(v_2^*) = 0$, $\phi(v_3^*) = 1$ *and there exists an exceptional set* $\mathbf{E}^*$ *such that* $\lambda(\mathbf{E}^*) = 0$ *and for any* $y \in \mathbb{R} \setminus \mathbf{E}^*$

$$\overline{\dim}_B \phi^{-1}(y) \le 1 - 2^{-\alpha}, \text{ that is } \limsup_{n\to\infty} \frac{\log \mathbf{N}_n^*(\phi^{-1}(y))}{n \log 2} \le 1 - 2^{-\alpha}. \tag{2.6.17}$$

Since the lower, and hence the upper box dimension is never less than the Hausdorff dimension we also have

$$\dim_H \phi^{-1}(y) \le 1 - 2^{-\alpha}. \tag{2.6.18}$$

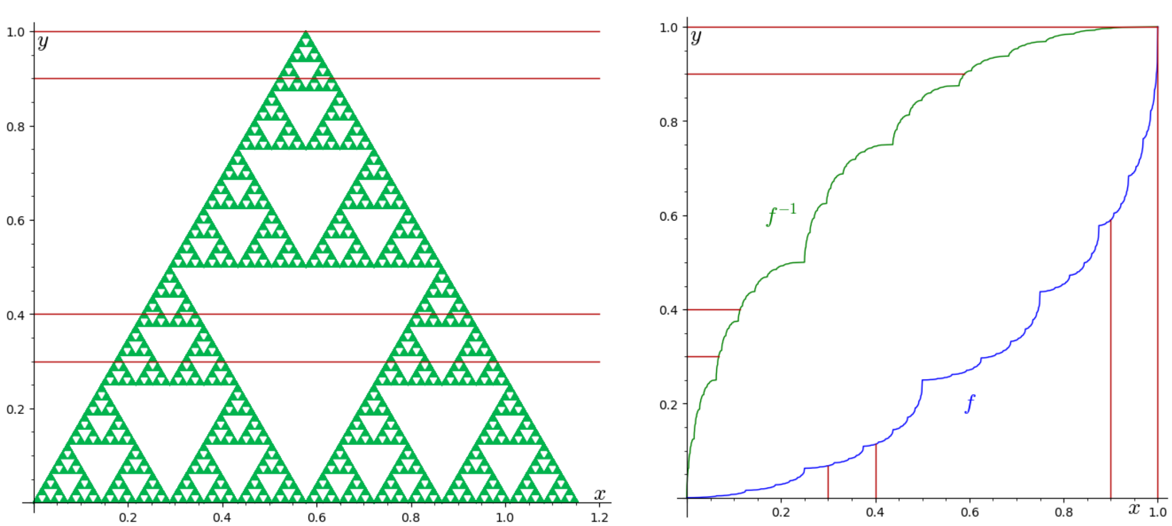

Figure 2.3: the rescaled and translated Sierpiński triangle Δ^*, the function $f(x) = \lambda_p([0,x))$ and its inverse f^{-1}

Proof. The basic concepts and results of ergodic theory we use in the sequel can be found for example in [26]. Suppose that $1/2 < p < 1$. Denote by σ the doubling map on $[0,1)$, that is $\sigma(x) = \{2x\}$, where $\{.\}$, denotes the fractional part. Denote by λ_p, the σ invariant ergodic measure for which

$$\lambda_p\Big(\sum_{k=1}^{n} \mathbf{e}_k 2^{-k} + 2^{-n}[0,1)\Big) = p^{\sum_{k=1}^{n}\mathbf{e}_k}(1-p)^{\sum_{k=1}^{n}(1-\mathbf{e}_k)}, \text{ where } \mathbf{e}_k \in \{0,1\},\ k = 1,2,.... \tag{2.6.19}$$

Set $f(x) = \lambda_p([0,x))$. Suppose that $x, y \in [0,1]$, $x < y$ and $2^{-n} \le y - x < 2^{-n+1}$, $k \in \mathbb{N}$. By (2.6.19) any interval of the form $\sum_{k=1}^{n} \mathbf{e}_k 2^{-k} + 2^{-n}[0,1)$ is of λ_p measure at most p^n. Since $[x,y]$ can be covered by no more than three such intervals we have

$$|f(x) - f(y)| \le 3 \cdot p^n = 3 \cdot 2^{n \log_2 p} \le 3|x-y|^{-\log_2 p}. \tag{2.6.20}$$

Since σ is ergodic with respect to λ_p by the Birkhoff Ergodic Theorem (see for example Chapter 4 of [26]) we have for λ_p almost every x

$$\frac{\sum_{k=1}^{n} \mathbf{e}_k(x)}{n} = \frac{\sum_{k=1}^{n} \chi_{[1/2,1)}(\sigma^k x)}{n} \to \lambda_p([1/2,1)) = p \tag{2.6.21}$$

(where $\mathbf{e}_k(x)$ denotes the kth digit after the binary point in the binary representation of x). We denote by X_p the set of xs satisfying (2.6.21). Since for any $x, y \in [0,1]$, $x < y$ we have $\lambda(f([x,y))) = \lambda_p([0,y)) - \lambda_p([0,x)) = \lambda_p([x,y))$ and the intervals $[x,y)$ generate the Borel sigma algebra we have $\lambda_p(A) = \lambda(f(A))$ for any Borel set $A \subseteq [0,1)$. Hence $\lambda(f(X_p)) = 1$ and for λ almost every $y \in [0,1]$ we have $f^{-1}(y) \in X_p$.

For $(x,y) \in \Delta^*$ set $\phi(x,y) = f(y)$ and select p such that $-\log_2 p = \alpha$. From (2.6.20) it follows that ϕ is a 3-Hölder-α function.

The definition of the Sierpiński triangle implies that if $y = \sum_{k=1}^{\infty} \mathbf{e}_k 2^{-k}$ is not a dyadic rational then the horizontal line

$$\{(x,y) : x \in \mathbb{R}\} \text{ intersects } 2^{\sum_{k=1}^{n}(1-\mathbf{e}_k)} \text{ many triangles of } \mathbf{T}_n^*. \tag{2.6.22}$$

If needed, by removing a countable set we can assume that $f(X_p)$ does not contain dyadic rational numbers. Set $\mathbf{E}^* = [0,1] \setminus f(X_p)$. If $y \notin [0,1]$ then $\phi^{-1}(y) = \emptyset$ and (2.6.17) is obvious. If $y \in [0,1] \setminus \mathbf{E}^*$ then by (2.6.21) and (2.6.22) we have

$$\limsup_{n\to\infty} \frac{\log \mathbf{N}_n^*(\phi^{-1}(y))}{n \log 2} = \limsup_{n\to\infty} \frac{\log 2^{\sum_{k=1}^{n}(1-\mathbf{e}_k)}}{-\log 2^{-n}} \tag{2.6.23}$$

$$= \lim_{n\to\infty} 1 - \frac{\sum_{k=1}^{n} \mathbf{e}_k(x)}{n} = 1 - p = 1 - 2^{-\alpha}.$$

□

Now we can prove Theorem 2.2.9

Proof of Theorem 2.2.9. Suppose that $\{g_k : k \in \mathbb{N}\}$ consists of locally non-constant standard strongly piecewise affine 1^--Hölder-α functions defined on Δ, and this set is dense in the space of 1-Hölder-α functions defined on Δ. It is also clear that each g_k is Lipschitz with a constant which we denote by M_k.

We suppose that n_k is selected in a way that g_k is piecewise affine on each $T \in \tau_{n_k}$ and there exist $v_1(T),\ v_2(T) \in V(T)$ such that

$$g_k(v_1(T)) = g_k(v_2(T)) \text{ and if } v_3(T) \text{ denotes the third vertex of } T \tag{2.6.24}$$

$$\text{then } g_k(v_3(T)) \neq g_k(v_1(T)).$$

Observe that if we take subdivisions, that is we take an $n_k' \geq n_k$ then (2.6.24) holds for suitably chosen vertices of triangles $T \in \tau_{n_k'}$. Later in the proof we will select a sufficiently large n_k'.

Next we define a function f_k satisfying

$$\|f_k - g_k\|_\infty < 2^{-k}. \tag{2.6.25}$$

First using Lemma 2.6.4 with $\alpha = \alpha$, $c = 1$, $f = g_k$, $M = M_k$ and $\varepsilon = 2^{-k}$, select $N_k \geq n_k$.

We define f_k such that with $n'_k > N_k$

$$f_k(x) = g_k(x) \text{ for any } x \in \bigcup_{T \in \tau_{n'_k}} V(T). \tag{2.6.26}$$

Since g_k is M_k-Lipschitz if $T \in \tau_{n'_k}$, $x, y \in V(T)$ are different then $|x - y| = 2^{-n'_k}$ and

$$|g_k(x) - g_k(y)| \leq M_k|x - y| = M_k \cdot 2^{-n'_k(1-\alpha)}|x - y|^\alpha < \frac{1}{100}|x - y|^\alpha, \tag{2.6.27}$$

if we suppose that n'_k is chosen large enough to satisfy

$$M_k \cdot 2^{-n'_k(1-\alpha)} < \frac{1}{100}. \tag{2.6.28}$$

Suppose that $T \in \tau_{n'_k}$. For ease of notation we will write v_i instead of $v_i(T)$ for $i = 1, 2, 3$. Using notation from the second paragraph before Lemma 2.6.5, denote by Ψ_T the similarity for which $\Psi_T(T \cap \Delta) = \Delta^*$ and the vertices of T for which we have (2.6.24) satisfied are mapped in a way that

$$\Psi_T(v_i) = v_i^* \text{ for } i = 1, 2, 3. \tag{2.6.29}$$

Then for every $x, y \in \Delta^*$

$$|\Psi_T(x) - \Psi_T(y)| = 2^{n'_k}\frac{2}{\sqrt{3}}|x - y|. \tag{2.6.30}$$

Let $\phi \in C_3^\alpha(\Delta^*)$ be given by Lemma 2.6.5. For $x \in T \cap \Delta$ we put

$$f_k(x) = \phi(\Psi_T(x))(g_k(v_3) - g_k(v_1)) + g_k(v_1). \tag{2.6.31}$$

Suppose $x, y \in T \cap \Delta$ then

$$|f_k(x) - f_k(y)| \leq |g_k(v_3) - g_k(v_1)| \cdot 3|\Psi_T(x) - \Psi_T(y)|^\alpha \tag{2.6.32}$$

$$\leq M_k 2^{-n'_k} \cdot 3 \cdot 2^{n'_k \alpha} \cdot \left(\frac{2}{\sqrt{3}}\right)^{\alpha} |x-y|^{\alpha} = M_k 2^{-n'_k(1-\alpha)} \cdot 3 \cdot \left(\frac{2}{\sqrt{3}}\right)^{\alpha} |x-y|^{\alpha} < \frac{1}{8}|x-y|^{\alpha},$$

where at the last step we used (2.6.28).

Now by (2.6.26), (2.6.32) and $n'_k \geq N_k$ we can apply Lemma 2.6.4. Thus f_k is a 1^--Hölder-α function satisfying (2.6.25).

Since $\tau_{n'_k}$ consists of finitely many triangles T, finite union of exceptional sets of measure zero is still of measure zero, and affine transformations are not changing the Hausdorff dimension we obtain from (2.6.18) and (2.6.31) that $\dim_H f_k^{-1}(y) \leq 1 - 2^{-\alpha}$ for almost every $y \in \mathbb{R}$ and for every k. Therefore $D_*^{f_k}(\Delta) \leq 1 - 2^{-\alpha}$ for every k and by the density of the functions g_k and (2.6.25) the functions f_k are also dense in $C_1^{\alpha}(\Delta)$. Hence we can apply Lemma 2.3.7 with the compact set Δ and the dense set of functions f_k to obtain a dense G_δ set $\mathcal{G}_1$ such that $D_*^f(\Delta) \leq 1 - 2^{-\alpha}$ for any $f \in \mathcal{G}_1$. Since in (2.1.4) there is also a supremum a little extra care is needed. Using Theorem 2.2.2 select and denote by $\mathcal{G}_2$ a dense G_δ subset of $C_1^{\alpha}(\Delta)$ such that for every $f \in \mathcal{G}_2$ we have $D_*^f(\Delta) = D_*(\alpha, \Delta)$. Since $\mathcal{G}_1 \cap \mathcal{G}_2$ is non-empty we can select a function f from it. For this function $D_*^f(\Delta) = D_*(\alpha, \Delta) \leq 1 - 2^{-\alpha}$. This completes the proof of the theorem. $\square$

Chapter 3

Generic Birkhoff spectra

3.1 Preliminaries

3.1.1 Notation and terminology

Let $\Omega = \{0,1\}^{\mathbb{N}}$, and σ be the shift map.

We introduce the usual metric d on Ω defined by

$$d(\omega, \omega') = \sum_{k=1}^{\infty} \frac{|\omega_k - \omega'_k|}{2^k},$$

where ω_k (resp. ω'_k) denotes the coordinates/entries of ω (resp. ω'). If $k \in \mathbb{N} \cup \{\infty\}$ and A is a finite string of 0s and 1s then A^k denotes the k-fold concatenation of A and $[A]$ denotes the cylinder set $\{\omega : A\omega',\ \omega' \in \Omega\}$. Given $k, l \in \mathbb{N}$ and $\omega = (\omega_1\omega_2...) \in \Omega$ we put $\omega|k = \omega_1...\omega_k$ and $(\omega)_k^l := \omega_k\omega_{k+1}\dots\omega_{l-1}\omega_l$, if $k \leq 0$ then $\omega|k$ is the empty string and analogously if $k > l$ then $(\omega)_k^l$ is also the empty string. The "conjugate" $\overline{\omega}$ is the string which we obtain from ω by swapping 0s and 1s, that is $\overline{\omega}_k = 1 - \omega_k$ for all k.

We recall the definition of the s-dimensional Hausdorff measure $\mathcal{H}^s$ and the Hausdorff dimension $\dim_H$. Notably, for $A \subseteq \Omega$, $\mathcal{H}^s(A) = \lim_{\delta\to 0+} \mathcal{H}^s_\delta(A)$ where $\mathcal{H}^s_\delta(A) = \inf\{\sum_i (\text{diam } U_i)^s :$ where $A \subseteq \bigcup_i U_i$ and diam $U_i < \delta\}$. The Hausdorff dimension of $A \subseteq \Omega$ is $\dim_H A = \inf\{s : \mathcal{H}^s(A) = 0\}$. From this definition, it is a standard exercise to show that $\dim_H \Omega = 1$.

The complement of a set A is denoted by A^c.

Let $\mathrm{PCC}^k(\Omega)$ be the set of those piecewise constant continuous functions in $C(\Omega)$, which depend only on cylinders of length/depth k. While the set of piecewise constant continuous functions in $C(\Omega)$, is denoted by $\mathrm{PCC}(\Omega)$. Obviously $\mathrm{PCC}(\Omega) = \bigcup_k \mathrm{PCC}^k(\Omega)$.

The $(1/2, 1/2)$-Bernoulli measure, the "Lebesgue measure" on Ω is denoted by λ. In case we write $\int f$ for an $f : \Omega \to \mathbb{R}$ we always mean $\int_\Omega f d\lambda$.

We denote by $C_0(\Omega)$ the set of continuous functions for which $\int f = 0$, and $\mathrm{PCC}_0^k(\Omega) = \mathrm{PCC}^k(\Omega) \cap C_0(\Omega)$.

Given $f \in C(\Omega)$, we denote $\|f\| = \sup_{\omega\in\Omega} |f(\omega)|$, and for any $\delta > 0$, $B(f, \delta) = \{g \in C(\Omega) : \|f - g\| < \delta\}$.

Recall

$$E_f(\alpha) := \left\{\omega \in \Omega : \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\sigma^n\omega) = \alpha\right\}, \tag{3.1.1}$$

and $S_f(\alpha) := \dim_H E_f(\alpha)$ We remark that our definition of $S_f(\alpha)$ is a bit different from the usual notation in multifractal analysis, since quite often $S_f(\alpha)$ is defined to be $-\infty$ when $E_f(\alpha)$ is empty.

As previously defined, we set $\alpha^*_{f,\max} = \sup\{\alpha \in \mathbb{R} : E_f(\alpha) \neq \emptyset\}$, where $\alpha^*_{f,\min} = \inf\{\alpha \in \mathbb{R} : E_f(\alpha) \neq \emptyset\}$. In general we have $\alpha_{f,\min} \leq \alpha^*_{f,\min} \leq \alpha^*_{f,\max} \leq \alpha_{f,\max}$, and it is possible for the strict inequalities to hold (including the first and the third inequality), as we will see in an example (cf. Example 3.1.4). In fact, as Theorem 3.2.7 shows this property is true for the generic continuous functions as well.

The σ-invariant Borel probability measures are denoted by $\mathcal{M}_\sigma$. By Birkhoff's Ergodic Theorem, we know that $\lambda(E_f(\int f)) = 1$. Furthermore, if $\{C_i\}_{i=1}^\infty$ are cylinders in Ω of length at least $k \in \mathbb{N}$ and $E_f(\int f) \subset \bigcup_{i=1}^\infty C_i$ then

$$1 = \lambda\left(E_f\left(\int f\right)\right) \leq \sum_{i=1}^{\infty} \lambda(C_i) = \sum_{i=1}^{\infty} \mathrm{diam}(C_i),$$

which implies that $1 \leq \mathcal{H}^1_{2^{-k}}(E_f(\int f)) \leq \mathcal{H}^1_{2^{-k}}(\Omega)$ for any $k \in \mathbb{N}$, and thus $S_f(\int f d\lambda) =$

1. Given $f \in C(\Omega)$ and $\alpha \in \mathbb{R}$ we will also use the following subsets of $\mathcal{M}_\sigma$

$$\mathcal{F}_f(\alpha) := \Big\{\mu \in \mathcal{M}_\sigma : \int f\, d\mu = \alpha\Big\}. \tag{3.1.2}$$

3.1.2 Examples

We present a few examples of Birkhoff spectra of certain $PCC(\Omega)$ functions. We will first provide an example for a function with continuous spectrum.

Example 3.1.1. *Let $f \in C(\Omega)$ be the function given by $f(\omega) = 1$ if $\omega_1 = 1$ and $f(\omega) = 0$ if $\omega_1 = 0$. Then for any $\alpha \in (0,1)$ we have*

$$S_f(\alpha) = -\frac{\alpha \log(\alpha) + (1-\alpha)\log(1-\alpha)}{\log 2},$$

if $\alpha \notin (0,1)$ then $S_f(\alpha) = 0$. In particular, f has continuous spectrum, as $\alpha^*_{f,\min} = 0$, $\alpha^*_{f,\max} = 1$, and furthermore, $\partial^+ S_f(\alpha^*_{f,\min}) = +\infty$ and $\partial^- S_f(\alpha^*_{f,\max}) = -\infty$.

Verification of the properties of Example 3.1.1. We will prove two inequalities using suitably defined Hölder functions and Eggleston's formula on dimension of real subsets determined by their digit density ([17]). First, let us consider the function $h_1 : \Omega \to [0,1]$ defined by

$$h_1(\omega) = \sum_{i=1}^{\infty} \frac{\omega_i}{2^i}.$$

That is, h_1 takes a 0-1 sequence to the number with the corresponding binary expansion. We claim that h_1 is a Lipschitz function in fact. Indeed, if ω' differs from ω in its nth coordinate, but not before that point, then $d(\omega, \omega') \geq 2^{-n}$, while $|h_1(\omega) - h_1(\omega')| \leq 2^{-n+1}$, hence h_1 has Lipschitz constant 2. Moreover, $h_1(E_f(\alpha))$ equals the set of numbers in $[0,1]$ having a binary expansion in which the density of 1s equals α. Thus due to [17], the dimension of $h_1(E_f(\alpha))$ is given by the formula in the statement of the lemma, yielding

$$S_f(\alpha) \geq -\frac{\alpha \log(\alpha) + (1-\alpha)\log(1-\alpha)}{\log 2},$$

as h_1 is Lipschitz.

Concerning the other inequality, define $h_2 : C \to \Omega$ for the triadic Cantor set $C \subset [0,1]$: if the triadic expansion of $x \in C$ is

$$x = \sum_{i=1}^{\infty} \frac{x_i}{3^i},$$

then let $\omega = h_2(x)$ have coordinates $\frac{x_1}{2}, \frac{x_2}{2}, \ldots$. That is, h_2 is a one-to-one mapping between Ω and C. Now if x differs from x' in its nth coordinate, but not before that point, then $|x - x'| \geq 3^{-n}$. On the other hand, $d(h_2(\omega), h_2(\omega')) \leq 2^{-n+1}$. It quickly yields that h_2 is a Hölder function with exponent $\frac{\log 2}{\log 3}$. Moreover, $h_2^{-1}(E_f(\alpha))$ is the set of numbers in $[0,1]$ having a ternary expansion with no 1s, in which the density of 2s is α and the density of 0s is $1-\alpha$. Hence $h_2^{-1}(E_f(\alpha))$ is contained by the set of numbers in $[0,1]$ having a ternary expansion in which the density of 2s is α and the density of 0s is $1-\alpha$. Thus due to [17], the dimension of $h_2^{-1}(E_f(\alpha))$ is at most

$$-\frac{\alpha \log(\alpha) + (1-\alpha)\log(1-\alpha)}{\log 3}.$$

Hence as h_2 is $\frac{\log 2}{\log 3}$-Hölder, we obtain an upper estimate for $S_f(\alpha)$, that is the dimension of $E_f(\alpha)$, notably

$$S_f(\alpha) \leq -\frac{\alpha \log(\alpha) + (1-\alpha)\log(1-\alpha)}{\log 2}.$$

This shows that the desired equality holds, and the remaining claims clearly follow. $\square$

Remark 3.1.2. One can use the fact that S_f is the Legendre transform of the topological pressure function $P(tf)$ to obtain a less direct argument that verifies the formula in Example 3.1.1.

Next, we will see examples of continuous functions with discontinuous spectra.

Example 3.1.3. *If f is a constant function, i.e. $f \equiv C \in \mathbb{R}$, then $S_f(C) = 1$ and $S_f(\alpha) = 0$ otherwise. The same is true if f is cohomologous to a constant, i.e. there exists $g \in C(\Omega)$ for which $f = C + g - g \circ \sigma$ (we recall that if C is zero, f is called a coboundary).*

Finally, we give an example where $\alpha_{f,\min} < \alpha^*_{f,\min} < \alpha^*_{f,\max} < \alpha_{f,\max}$ (that is, strict inequalities are satisfied), and the Birkhoff spectrum is discontinuous.

Example 3.1.4. *There exists* $f \in \mathrm{PCC}_0^3(\Omega)$ *satisfying* $\alpha_{f,\min} < \alpha^*_{f,\min} < \alpha^*_{f,\max} < \alpha_{f,\max}$ *and* $S_f(\alpha^*_{f,\min}), S_f(\alpha^*_{f,\max}) > 0$.

Proof. As $f \in \mathrm{PCC}_0^3(\Omega)$ we can define it by giving its values on 3-cylinders by abusing a bit the notation for f. We define f by $f([000]) = f([010]) = -2$, $f([001]) = -3$, $f([100]) = -1$, and $f(\overline{\omega}) = -f(\omega)$. Then we clearly have $\alpha_{f,\min} = -3$ while $\alpha_{f,\max} = 3$.

Now we claim $\alpha^*_{f,\min} = -2$, while $\alpha^*_{f,\max} = 2$, which would yield the inequalities $\alpha_{f,\min} < \alpha^*_{f,\min} < \alpha^*_{f,\max} < \alpha_{f,\max}$. Due to symmetry reasons, it suffices to verify $\alpha^*_{f,\min} = -2$. To this end, consider an arbitrary $\omega \in \Omega$. Now we are interested in the averages $\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega)$. In the sequence $f(\sigma^n\omega)$ each value is at least -2, except for the cases when the first three coordinates of $\sigma^n\omega$ are 001. However, in this case the first three coordinates of $\sigma^{n+2}\omega$ contain at least two 1s, or they are 100. In either case, $f(\sigma^{n+2}\omega) \geq -1$. This argument shows that in the sum $\sum_{n=1}^{N} f(\sigma^n\omega)$ the summands with value -3 can be paired with summands with value at least -1, except for possibly the last one, whose pair does not appear in the sum. Besides that, all the other summands have value at least -2. Consequently, the average $\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) \geq -2 - \frac{3}{N}$, hence the limit is at least -2, verifying $\alpha^*_{f,\min} \geq -2$. For the other inequality, we may simply consider the identically 0 sequence, hence $\alpha^*_{f,\min} = -2$. It proves the claim of this paragraph.

It remains to show that $S_f(\alpha^*_{f,\min}), S_f(\alpha^*_{f,\max}) > 0$. Due to symmetry reasons, these quantities are clearly equal, hence $S_f(\alpha^*_{f,\min}) > 0$ would be sufficient. Consider the following subset of Ω:

$$B = \{\omega \in \Omega : \omega_k = 0 \text{ for } k \equiv 1, 2 \mod 3\}.$$

Then for any $\omega \in B$ and n we have that at least two of the first three coordinates of $\sigma^n\omega$ equals 0. Consequently, $f(\sigma^n\omega) < 0$. Moreover, similarly to the previous argument we find that the in the sum $\sum_{n=1}^{N} f(\sigma^n\omega)$ the summands with value -3 can be paired with summands with value -1, except for possibly the last one. All the other summands have value -2. Hence we find

$$-2 - \frac{1}{N} \leq \frac{1}{N} \sum_{n=1}^{N} f(\sigma^n \omega) \leq -2.$$

It proves that $B \subset E_f(-2)$, hence $\dim_H B > 0$ would conclude the proof. However, this dimension can be calculated explicitly as B is a self-similar set, which equals the disjoint union of its 2 similar images, where the similarities have ratio $\frac{1}{8}$. Thus $\dim_H B = \frac{\log 2}{\log 8} = \frac{1}{3}$ by Hutchinson's Theorem [22]. $\square$

3.1.3 Variational formula

The following result was obtained by Fan, Feng, and Wu. We present this result in the context of the full-shift on an alphabet of two symbols (Ω, σ) (in [19], they proved the result for a topologically mixing subshift of finite type).

Theorem 3.1.5 ([19, Theorem A]). *Suppose that $f : \Omega \to \mathbb{R}^d$ is a continuous function. We denote $L_f := \{\alpha \in \mathbb{R}^d : \alpha = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\sigma^n \omega)$ for some $\omega \in \Omega\}$. There exists a concave and upper semi-continuous function Λ_f such that for any $\alpha \in L_f$*

$$S_f(\alpha) := \dim_H(E_f(\alpha)) = \Lambda_f(\alpha),$$

and

$$\Lambda_f(\alpha) = \max_{\mu \in \mathcal{F}_f(\alpha)} \frac{h_\mu}{\log 2}$$

where h_μ is the metric entropy of μ, and $\mathcal{F}_f(\alpha)$ can be defined analogously to (3.1.2).

The function $\Lambda_f(\alpha)$ is defined in the same paper [19, Proposition 5] using the cardinality of the cylinders of large length that contain at least one point ω for which the Birkhoff average of f of that length is close to α. It was later shown that the quantity $\Lambda_f(\alpha)$ indeed agrees with $S_f(\alpha)$ for all $\alpha \in L_f$ [19, Proposition 6].

3.2 Main results

We first prove that two Birkhoff spectra of two continuous functions are close (except near the endpoints) if those two functions are close in the supremum norm.

Theorem 3.2.1. *Let $f \in C(\Omega)$ for which $\alpha^*_{f,\min} < \alpha^*_{f,\max}$, and $\varepsilon \in \left(0, \frac{\alpha^*_{f,\max} - \alpha^*_{f,\min}}{2}\right)$ be given. Then there exists $\delta > 0$ such that for any $g \in B(f, \delta)$, we have $|S_f(\alpha) - S_g(\alpha)| < \varepsilon$ for all $\alpha \in (\alpha^*_{f,\min} + \varepsilon, \alpha^*_{f,\max} - \varepsilon)$.*

Remark 3.2.2. We will later learn that the generic continuous function satisfies the hypothesis of this theorem; see Theorem 3.2.7.

Recall an example of a $\mathrm{PCC}^3(\Omega)$ function with discontinuous spectrum from Example 3.1.4. Our next theorem tells that functions in $\mathrm{PCC}(\Omega)$ with discontinuous spectrum form a dense subset of $C(\Omega)$.

Theorem 3.2.3. *Functions $h \in \mathrm{PCC}(\Omega)$ with $S_h(\alpha^*_{h,\max}) > 0$ are dense in $C(\Omega)$.*

Remark 3.2.4. Of course, a similar theorem is valid with $S_h(\alpha^*_{h,\min}) > 0$ in the conclusion and with a little extra technical effort one can show density in $C(\Omega)$ of those $f \in \mathrm{PCC}(\Omega)$ for which $S_h(\alpha^*_{h,\max}) > 0$ and $S_h(\alpha^*_{h,\min}) > 0$ hold simultaneously. As Theorem 3.2.5 shows functions satisfying the conclusion of Theorem 3.2.3, or any of its above mentioned variants form a first category set in $C(\Omega)$.

Next we will show that the set of functions with discontinuous spectrum is of first category.

Theorem 3.2.5. *For the generic continuous function $f \in C(\Omega)$, we have that S_f is continuous on $\mathbb{R}$.*

Remark 3.2.6. This theorem implies that the set of continuous functions with discontinuous Birkhoff spectrum is a set of first category. This set includes functions which are cohomologous to a constant, as we observed in Example 3.1.3, hence this is a possible way to see that these functions form a set of first category.

In Example 3.1.1 we saw a very simple PCC function for which the range of the function $[\alpha_{f,\min}, \alpha_{f,\max}]$ coincides with the support of the spectrum $[\alpha^*_{f,\min}, \alpha^*_{f,\max}]$. Our next theorem states that generically, this coincidence does not hold. In fact, we prove a little more; we show that the set of functions for which $[\alpha^*_{f,\min}, \alpha^*_{f,\max}] \subset (\alpha_{f,\min}, \alpha_{f,\max})$ is open and dense.

Theorem 3.2.7. *For a dense open set $\mathcal{G} \subseteq C(\Omega)$ we have*

$$\alpha_{f,\min} < \alpha^*_{f,\min} < \alpha^*_{f,\max} < \alpha_{f,\max} \tag{3.2.1}$$

hence the generic $f \in C(\Omega)$ satisfies (3.2.1).

For the generic continuous function we have already seen in Theorem 3.2.5 that the spectrum is continuous at these endpoints, and as in the direction of the exterior of L_f the spectrum is constant zero, the one-sided derivative is also zero. On the other hand, towards the interior of the support it is of infinite absolute value as we see in the next theorem.

Theorem 3.2.8. *For the generic continuous function $f \in C(\Omega)$, we have $\partial^- S_f(\alpha^*_{f,\max}) = -\infty$, while $\partial^+ S_f(\alpha^*_{f,\min}) = \infty$.*

Our next theorem tells that there exist functions which are exceptional in Theorem 3.2.8.

Theorem 3.2.9. *There exists $f \in C_0(\Omega)$ such that S_f is continuous, $\alpha^*_{f,\min} = -1$ and $\alpha^*_{f,\max} = 1$, and $\partial^- S_f(\alpha^*_{f,\max}) > -\infty$, while $\partial^+ S_f(\alpha^*_{f,\min}) < \infty$. Moreover, these derivatives can be arbitrarily close to -1 and 1, respectively.*

It is natural to ask whether Theorem 3.2.9 holds if we restrict our attention to the class of Hölder functions, or PCC functions. While we do not know whether there is a PCC function with finite one-sided derivatives at the endpoints of the spectrum, our final theorem might make one believe that the answer to this question is negative:

Theorem 3.2.10. *If $f \in \mathrm{PCC}(\Omega)$ and S_f is continuous, then $\partial^- S_f(\alpha^*_{f,\max}) = -\infty$, while $\partial^+ S_f(\alpha^*_{f,\min}) = \infty$.*

3.3 Tools

3.3.1 Norm Continuity Theorem

The goal of this subsection is to prove Theorem 3.2.1.

If one considers $f, g \in C(\Omega)$ with continuous spectrum then the above theorem can be used to show that for given $\varepsilon > 0$ one can find $\delta > 0$ such that $\|f - g\| < \delta$ implies that $\|S_f - S_g\| < \varepsilon$. On the other hand, if f has discontinuous spectrum, say $S_f(\alpha^*_{f,\max}) > 0$ then the density of functions with continuous spectrum (Theorem 3.2.5) and Remark 3.3.2 imply that arbitrary close to f one can find functions g such that $\|S_f - S_g\| > S_f(\alpha^*_{f,\max})/2$.

To proceed, we first prove the following lemma.

Lemma 3.3.1. *Let $\varepsilon > 0$ be given. Suppose that $f \in C(\Omega)$, and $\alpha \in [\alpha^*_{f,\min}, \alpha^*_{f,\max}]$. Then for any $g \in C(\Omega)$ such that $\|f - g\| < \varepsilon$, there exists $\alpha' \in (\alpha - \varepsilon, \alpha + \varepsilon)$ for which $S_g(\alpha') \geq S_f(\alpha)$. If $S_f(\alpha) = 0$, but $E_f(\alpha) \neq \emptyset$ then $E_g(\alpha') \neq \emptyset$.*

Remark 3.3.2. This implies that if $\|f - g\| < \varepsilon$ then $|\alpha^*_{f,\max} - \alpha^*_{g,\max}| < \varepsilon$ and $|\alpha^*_{f,\min} - \alpha^*_{g,\min}| < \varepsilon$.

Proof. Recall the definition of $\mathcal{F}_f(\alpha)$ from (3.1.2). By Theorem 3.1.5 there exists $\mu_0 \in \mathcal{F}_f(\alpha)$ for which

$$S_f(\alpha) = \frac{h_{\mu_0}}{\log 2} = \frac{\max_{\mu \in \mathcal{F}_f(\alpha)} h_\mu}{\log 2}.$$

Defining $\alpha' = \int g \, d\mu_0$, the bound $\|f - g\| < \varepsilon$ yields $\alpha' \in (\alpha - \varepsilon, \alpha + \varepsilon)$, and from Theorem 3.1.5 we can quickly conclude $S_g(\alpha') \geq S_f(\alpha)$, as

$$S_g(\alpha') = \frac{\max_{\mu \in \mathcal{F}_g(\alpha')} h_\mu}{\log 2} \geq \frac{h_{\mu_0}}{\log 2} = S_f(\alpha).$$

If $S_f(\alpha) = 0$, but $E_f(\alpha) \neq \emptyset$, we will obtain the desired conclusion by integrating g with respect to another measure to get α'. First, consider the map $f_* : \mathcal{M}_\sigma \to L_f$ for which $f_*(\mu) = \int f \, d\mu$. Since this map is affine and continuous, and $f_*(\mu_0) = \alpha$ is an extremal point of its range, we can conclude the existence of an extremal point

$\widetilde{\mu}_0$ of the convex set $\mathcal{M}_\sigma$, that is an ergodic measure, for which $f_*(\widetilde{\mu}_0) = \alpha$. Thus for $\alpha' = \int g\, d\widetilde{\mu}_0$, by Birkhoff's Ergodic Theorem we have $\widetilde{\mu}_0$-a.e.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(\sigma^n \omega) = \alpha',$$

and hence $E_g(\alpha') \neq \emptyset$. $\square$

Using this lemma, we will prove the theorem by using concavity of the spectrum.

Proof of Theorem 3.2.1. For some $L \in \mathbb{N}$, we consider a partition

$$\alpha^*_{f,\min} = \alpha_1 < \alpha_2 < \cdots < \alpha_L = \alpha^*_{f,\max}$$

for which for every $i = 1, 2, \ldots, L-1$, $|\alpha_{i+1} - \alpha_i| < \varepsilon/4$ is small enough such that for every $t \in [0,1]$, we have

$$(1-t)S(\alpha_i) + tS(\alpha_{i+1}) > S((1-t)\alpha_i + t\alpha_{i+1}) - \varepsilon/2.$$

For each α_i, we choose a positive number $\delta(\alpha_i) < \varepsilon/8$ as follows: For any $\alpha'_i \in (\alpha_i - \delta(\alpha_i), \alpha_i + \delta(\alpha_i))$, and $\beta'_i \geq S_f(\alpha_i)$, the line segments connecting the points (α'_i, β'_i) and $(\alpha'_{i+1}, \beta'_{i+1})$ are above the graph of $S_f(\alpha) - \varepsilon$ for $i = 2, ..., L-2$. We can also suppose that the intervals $(\alpha_i - \delta(\alpha_i), \alpha_i + \delta(\alpha_i))$ are disjoint. Then we set

$$\delta = \min\{\varepsilon/8, \delta(\alpha_1), \delta(\alpha_2), \ldots, \delta(\alpha_L)\}.$$

We apply Lemma 3.3.1 with $\varepsilon = \delta$ to show that there exists $\alpha'_i \in (\alpha_i - \delta, \alpha_i + \delta) \subseteq (\alpha_i - \delta(\alpha_i), \alpha_i + \delta(\alpha_i))$ such that $S_g(\alpha'_i) \geq S_f(\alpha_i)$ for $i = 1, ..., L-1$. Since $|\alpha'_1 - \alpha^*_{f,\min}| = |\alpha'_1 - \alpha_1| < \varepsilon/8$ and $|\alpha'_L - \alpha^*_{f,\max}| = |\alpha'_L - \alpha_L| < \varepsilon/8$ by using the concavity of S_g one can show that $S_g(\alpha) > S_f(\alpha) - \varepsilon$ for all $\alpha \in (\alpha^*_{f,\min} + \varepsilon/2, \alpha^*_{f,\max} - \varepsilon/2)$. By reversing the roles of f and g, by an analogous argument we can conclude that $S_f(\alpha) > S_g(\alpha) - \varepsilon$ for all $\alpha \in (\alpha^*_{g,\min} + \varepsilon/2, \alpha^*_{g,\max} - \varepsilon/2)$. Using Remark 3.3.2 we can conclude the proof. $\square$

3.3.2 Piecewise constant (PCC) functions

We start with a lemma in which we show that $\alpha^*_{f,\max}$ is a uniform upper bound of the limit of the Birkhoff averages of any $f \in PCC^k$.

Lemma 3.3.3. *Assume $f \in \mathrm{PCC}^k(\Omega)$ and $\varepsilon > 0$. Then there exists N_0 such that for any $N \geq N_0$, for any $\omega \in \Omega$, we have*

$$\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) \leq \alpha^*_{f,\max} + \varepsilon, \tag{3.3.1}$$

which implies that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) \leq \alpha^*_{f,\max} \text{ uniformly for any } \omega \in \Omega. \tag{3.3.2}$$

Proof. Choose N_0 such that for any $N > N_0$

$$\frac{-k\,\|f\| + N(\alpha^*_{f,\max} + \varepsilon)}{N+k} > \alpha^*_{f,\max} + \frac{\varepsilon}{2}. \tag{3.3.3}$$

We claim that this N_0 satisfies the statement of the lemma. Proceeding towards a contradiction, assume the existence of a configuration ω and $N > N_0$ which refutes this claim, that is

$$\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) > \alpha^*_{f,\max} + \varepsilon. \tag{3.3.4}$$

Our goal is to construct $\omega' \in \Omega$, periodic by $N + k$ which will satisfy

$$\sum_{n=1}^{N} f(\sigma^n\omega') = \sum_{n=1}^{N} f(\sigma^n\omega) > N(\alpha^*_{f,\max} + \varepsilon), \tag{3.3.5}$$

and this will contradict the definition of $\alpha^*_{f,\max}$ as we will see in (3.3.7). In the ergodic sums we consider, the first coordinate has no importance, thus it is sufficient to construct $\sigma\omega'$. Let it be periodic with period $N + k$ (that is $\sigma^{N+k+1}\omega' = \sigma\omega'$), and define its first $N + k$ coordinates to be $\omega_2, \omega_3, ..., \omega_{N+k+1}$. Now if N' is arbitrary, express it modulo $N + k$ as $N' = p(N + k) + q$, where p is a nonnegative integer, while $0 \leq q < N + k$.

Then the corresponding ergodic sum can be written as

$$\frac{1}{N'}\sum_{n=1}^{N'} f(\sigma^n\omega') = \frac{1}{N'}\sum_{n=1}^{p(N+k)} f(\sigma^n\omega') + \frac{1}{N'}\sum_{n=1}^{q} f(\sigma^{p(N+k)+n}\omega')$$
$$= \frac{p(N+k)}{N'}\left(\frac{1}{p(N+k)}\sum_{n=1}^{p(N+k)} f(\sigma^n\omega')\right) + \frac{1}{N'}\sum_{n=1}^{q} f(\sigma^{p(N+k)+n}\omega') = \circledast \tag{3.3.6}$$

Using the periodicity of $\sigma\omega'$ in the first sum, and the boundedness of f in the second one we infer

$$\circledast = \frac{p(N+k)}{N'}\left(\frac{1}{N+k}\sum_{n=1}^{N+k} f(\sigma^n\omega')\right) + o(N').$$

Hence if $N' \to \infty$, the ergodic sum $\frac{1}{N'}\sum_{n=1}^{N'} f(\sigma^n\omega')$ converges to $\frac{1}{N+k}\sum_{n=1}^{N+k} f(\sigma^n\omega')$. Now by (3.3.4) and $f \in \mathrm{PCC}^k(\Omega)$, we have (3.3.5). Thus by (3.3.3), we deduce

$$\frac{1}{N+k}\sum_{n=1}^{N+k} f(\sigma^n\omega') > \frac{-k\|f\| + N(\alpha^*_{f,\max}+\varepsilon)}{N+k} > \alpha^*_{f,\max} + \frac{\varepsilon}{2}, \tag{3.3.7}$$

Hence $E_f(\alpha) \neq \emptyset$ for some $\alpha > \alpha^*_{f,\max} + \frac{\varepsilon}{2}$, which is obviously a contradiction. It concludes the proof. $\square$

Remark 3.3.4. More general version of Lemma 3.3.3 can be found in [40, Theorem 1.9]. In particular, the result would hold for continuous functions, rather than PCC functions. We will not, however, require such general result in our subsequent argument.

Next, we will show that if $f \in PCC(\Omega)$, then there exists a periodic point in Ω for which the limit of the Birkhoff averages of f equals $\alpha^*_{f,\max}$.

Lemma 3.3.5. *Let $f \in \mathrm{PCC}^k(\Omega)$. Then there exists a periodic configuration ω such that $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) = \alpha^*_{f,\max}$.*

Proof. We define a directed graph $G = (V, E)$ as follows: $V = \{0,1\}^k$, and there is an edge from $u \in V$ to $v \in V$ if roughly speaking v is one of the possible shifted images of u, that is $v_i = u_{i+1}$ for $i = 1, ..., k-1$. Now we can think of the values of f as weights on the vertices of G, while an arbitrary $\omega \in \Omega$ corresponds to an infinite walk Γ_ω in G.

Moreover, the ergodic averages are simply the averages of weights along the vertices of finite subwalks of Γ_ω.

For technical reasons, it is advantageous to put the weights on the edges and work with those ones: one of the convenient ways to do so is putting weight $f(u)$ on all the edges *leaving* the vertex u. Denote the function $E \to \mathbb{R}$ obtained this way by f, too. Now the ergodic averages can be considered as the averages of weights along the edges of finite subwalks of ω.

Consider now $\omega \in \Omega$ such that $\frac{1}{N}\sum_{i=1}^{N} f(\sigma^i \omega) \to \alpha^*_{f,\max}$. Take the corresponding path Γ_ω. As V is finite, there exists a vertex which appears infinitely many times in Γ_ω. By erasing the first few entries of ω, or equivalently, erasing the first few edges of Γ_ω, we might assume by abuse of notation that the first vertex v of Γ_ω recurs infinitely many times. Now based on the recurrences of v, we can partition the infinite walk Γ_ω into closed, finite walks $\Gamma_\omega^{(1)}, \Gamma_\omega^{(2)}, ...$ such that each such walk starts and ends with v, and in the meantime it does not hit v. Now it is simple to verify that the edge set (counted with multiplicities from now on) of each $\Gamma_\omega^{(i)}$ is the union of graph cycles, or in other words, it is the union of closed walks containing each of their edges precisely once. (One cycle might also appear multiple times in this decomposition.) Indeed, we can find a subpath $e_1 e_2 ... e_r$ such that $e_1 = e_r$, and there is no other repetition of edges in this subpath. Then $e_1 e_2 ... e_{r-1}$ is a cycle, and its removal from $\Gamma_\omega^{(i)}$ results in a shorter closed walk starting and ending with v. Thus we can repeat the previous reasoning to find another cycle, if such exists and this procedure ends in finitely many steps.

Let us note now that there are only finitely many cycles in G as it is a finite graph. Denote their set by $\mathcal{C}$. By the previous paragraph, up to the last edge of any $\Gamma_\omega^{(i)}$, the edge set of Γ_ω can be written as the union of these cycles, such that $C \in \mathcal{C}$ is used $\rho_{C,i}$ times. Thus the ergodic average corresponding to the subpath of the Γ_ω up to the last edge of $\Gamma_\omega^{(i)}$ is the following:

$$\frac{\sum_{C\in\mathcal{C}} \rho_{C,i} \sum_{e\in C} f(e)}{\sum_{C\in\mathcal{C}} \rho_{C,i}|C|} = \frac{\sum_{C\in\mathcal{C}} \rho_{C,i}|C| \sum_{e\in C} \frac{f(e)}{|C|}}{\sum_{C\in\mathcal{C}} \rho_{C,i}|C|}. \tag{3.3.8}$$

Notice that it is simply a convex combination of the cycle averages $\sum_{e\in C} \frac{f(e)}{|C|}$. Hence the ergodic average in (3.3.8) can be bounded from above by $\max_{C\in\mathcal{C}}\sum_{e\in C}\frac{f(e)}{|C|}$. Now by the choice of ω we also know that this ergodic average tends to $\alpha^*_{f,\max}$ as $i\to\infty$, hence

$$\alpha^*_{f,\max} \le \max_{C\in\mathcal{C}}\sum_{e\in C}\frac{f(e)}{|C|} \tag{3.3.9}$$

also holds.

Now consider the infinite walk which goes along a cycle C_0 over and over again, where C_0 is chosen so that the above maximum is attained. Then C_0 together with a starting point uniquely determines a periodic configuration $\omega^*\in\Omega$ for which $\sigma^i\omega^*$ always equals the respective vertex of C_0. Moreover, it is simple to check that the ergodic averages tend to $\sum_{e\in C_0}\frac{f(e)}{|C_0|}$. Hence this limit must be $\alpha^*_{f,\max}$ by (3.3.9), as it is an upper estimate for all ergodic limits. $\square$

3.4 Continuity, discontinuity and support of the spectrum

By [19], we know that S_f is necessarily upper semi-continuous for any continuous function. Moreover, it is continuous on $[\alpha^*_{f,\min},\alpha^*_{f,\max}]$, while it vanishes outside of this interval. However it is not necessarily continuous at the endpoints of this interval.

3.4.1 Denseness of PCC functions with discontinuous spectra

The goal of this subsection is to prove Theorem 3.2.3. The main idea of the proof of Theorem 3.2.3 is to show that given any continuous function, we can approximate it by a PCC function, and we further "perturb" that PCC function in an appropriate way so that its spectrum will be discontinuous.

Proof of Theorem 3.2.3. Suppose $\varepsilon>0$ and $f_0\in C(\Omega)$ are arbitrary. We need to find an $h\in\mathrm{PCC}(\Omega)$ such that

$$\|f_0-h\|<\varepsilon \text{ and } S_h(\alpha^*_{h,\max})>0. \tag{3.4.1}$$

We will achieve this by the following. We first find $f \in \mathrm{PCC}^{\mathbf{k}}(\Omega)$ for suitably large $\mathbf{k}$ that approximates f_0. By Remark 3.3.2, $\alpha^*_{f_0,\max} \approx \alpha^*_{f,\max}$. Next we will "perturb" the function f by adding another PCC function g. This function g will be small in a sense that it does nothing to perturb f for many points, but it will perturb just slightly near the points where the limit of Birkhoff averages of f attains the maximum (i.e. $\alpha^*_{f,\max}$) to the point where S_{f+g} discontinuous at the boundary of L_{f+g}. The sum $f+g$ will be our candidate for h.

By using a suitably large $\mathbf{k}$ choose $f \in \mathrm{PCC}^{\mathbf{k}}(\Omega)$ such that $\|f - f_0\| < \varepsilon/2$. By Lemma 3.3.5 select a periodic ω' such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega') = \alpha^*_{f,\max}. \tag{3.4.2}$$

In this proof, as in (3.4.2) we prefer to take Birkhoff sums with indices between 0 and $N-1$, when taking limits it makes no difference. We can assume that there is a finite string of 0s and 1s, denoted by A such that $\omega' = A^\infty$, by not necessarily using the prime period we can also suppose that $k_A = |A|$, the length of A is a multiple of $\mathbf{k}$.

Now we select a string B of length k_A. If $A \neq 0^{k_A}$ then we let $B = 0^{k_A}$, if $A = 0^{k_A}$ then we let $B = 1^{k_A}$. Without limiting generality in the sequel we assume that $B = 0^{k_A}$.

By using a suitably large number ℓ, to be fixed later, we consider strings $X = (A^{2\ell})AABAA$ and $Y = (A^{2\ell})ABAAA$.

Set $\mathbf{H} = \{X, Y\}^\infty$. We will later show that this set will be contained in $E_h(\alpha^*_{h,\max})$ (where h will be defined by perturbing f slightly). We note that it is easy to see that $\dim_H \mathbf{H} > 0$, since by Hutchinson's theorem, $2 \cdot (2^{-(2\ell+5)k_A})^{\dim_H \mathbf{H}} = 1$, which gives $\dim_H \mathbf{H} = \frac{1}{(2\ell+5)k_A}$. This would imply $E_h(\alpha^*_{h,\max}) > 0$.

Claim 3.4.1. There exists $\alpha_\ell \in \mathbb{R}$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega) = \alpha_\ell \leq \alpha^*_{f,\max} \text{ for any } \omega \in \mathbf{H}. \tag{3.4.3}$$

Proof. First we consider the sum $\sum_{n=0}^{2(\ell+5)k_A-1} f(\sigma^n \omega)$ for any $\omega \in \mathbf{H}$. We select $\omega^A \in [AA]$, $\omega^{AB} \in [ABA]$. Since $f \in \mathrm{PCC}^{\mathbf{k}}$, and $\mathbf{k}$ divides k_A, the values of $f(\sigma^n \omega^A)$; $n =$

$0, ..., k_A - 1$ and $f(\sigma^n \omega^{AB})$; $n = 0, ..., 2k_A - 1$ are independent of our choice of $\omega^A \in [AA]$ and $\omega^{AB} \in [ABA]$. Hence, there exists a constant Σ_ℓ such that for any $\omega \in \mathbf{H}$, we have

$$\Sigma_\ell = \sum_{n=0}^{(2\ell+5)k_A-1} f(\sigma^n \omega) = (2\ell + 3) \sum_{n=0}^{k_A-1} f(\sigma^n \omega^A) + \sum_{n=0}^{2k_A-1} f(\sigma^n \omega^{AB}).$$

Define α_ℓ so that it satisfies $2(\ell + 5)k_A \alpha_\ell = \Sigma_\ell$. Let N be greater than $2(\ell + 5)k_A$, and write $N = 2(\ell + 5)k_A M_N + R_N$ for some positive integer M_N and $R_N \in \{0, 1, \ldots, 2(\ell + 5)k_A - 1\}$. Thus

$$\sum_{n=0}^{N-1} f(\sigma^n \omega) = 2(\ell + 5)k_A M_N \alpha_\ell + \sum_{n=N-R_N}^{N} f(\sigma^n \omega)$$

for any $\omega \in \mathbf{H}$. Thus, we obtain (3.4.3) by dividing both sides of the equation by N and letting $N \to \infty$. $\square$

Now we return to the proof of Theorem 3.2.3. Next we construct the perturbation function g. Put $m = \ell + 7$ and

$$C_m = \{U_1 U_2 ... U_m \omega_0 \omega_1 ... : U_i \in \{X, Y\},\ i = 1, ..., m,\ \omega_j \in \{0, 1\},\ j = 0, 1, ...\}. \quad (3.4.4)$$

We take the following finite union of cylinder sets in Ω

$$\mathbf{P} = \bigcup_{i=0}^{\ell-1} \sigma^{ik_A} C_m.$$

Next we define our perturbation function $g \in \mathrm{PCC}^{mk_A}(\Omega)$. If $\omega \in \mathbf{P}$ then we set $g(\omega) = \varepsilon/4$, otherwise put $g(\omega) = 0$.

Claim 3.4.2. If ℓ is sufficiently large then for $h = f + g$ and for any $\omega \in \mathbf{H}$ we have

$$b^* := \frac{1}{(2\ell + 5)k_A} \sum_{j=0}^{(2\ell+5)k_A-1} h(\sigma^{j+t(2\ell+5)k_A} \omega) > \alpha^*_{f,\max} + \frac{\varepsilon}{32k_A} \text{ for } t = 0, 1, \ldots . \quad (3.4.5)$$

Proof. Take and fix an arbitrary $\omega \in \mathbf{H}$. Recall that $|X| = |Y| = (2\ell + 5)k_A$. By our definition of X and Y we have

$$\frac{1}{(2\ell+5)k_A}\sum_{j=0}^{(2\ell+5)k_A-1} f(\sigma^{j+t(2\ell+5)k_A}\omega)=\alpha_\ell \text{ for any } t\in\{0,1,...\}. \tag{3.4.6}$$

From the choice of ω and A it is also clear that

$$\frac{1}{2\ell k_A}\sum_{j=0}^{2\ell k_A-1} f(\sigma^{j+t(2\ell+5)k_A}\omega)=\alpha^*_{f,\max} \text{ for any } t\in\{0,1,...\}. \tag{3.4.7}$$

Hence,

$$\alpha_\ell \geq \frac{2\ell k_A\cdot\alpha^*_{f,\max}+5k_A\alpha_{f,\min}}{(2\ell+5)k_A}\to\alpha^*_{f,\max} \text{ as } \ell\to\infty. \tag{3.4.8}$$

Next we look at the averages of g. Observe that if $U_i\in\{X,Y\}$ then there is a maximal substring of U_i which consists of consecutive zeros. This is the one which contains B, and of course might contain some zeros from the end/beginning of the As before/after B in U_i. This and the definition of $\mathbf{P}$ and g imply that for $\omega\in\mathbf{H}$

$$g(\sigma^j\omega)>0 \text{ holds iff } j=ik_A+t(2\ell+5)k_A,\ i=0,...,\ell-1,\ t=0,1,\dots. \tag{3.4.9}$$

Therefore,

$$\frac{1}{(2\ell+5)k_A}\sum_{j=0}^{(2\ell+5)k_A-1} g(\sigma^{j+t(2\ell+5)k_A}\omega)=\frac{\ell\varepsilon}{4(2\ell+5)k_A} \text{ for any } t\in\{0,1,...\}. \tag{3.4.10}$$

Now we determine how large ℓ should be. Indeed, we select an ℓ such that

$$\ell\cdot\frac{\varepsilon}{8}>5k_A(\alpha^*_{f,\max}-\alpha_{f,\min}) \text{ and } \frac{\ell}{8(2\ell+5)}>\frac{1}{32}. \tag{3.4.11}$$

These inequalities will grant us that

$$\begin{aligned}\frac{2\ell k_A\alpha^*_{f,\max}+5k_A\alpha_{f,\min}+\ell\frac{\varepsilon}{4}}{(2\ell+5)k_A} &> \frac{2\ell k_A\alpha^*_{f,\max}+5k_A\alpha^*_{f,\max}+\ell\frac{\varepsilon}{8}}{(2\ell+5)k_A} \\ &> \alpha^*_{f,\max}+\frac{\varepsilon}{32k_A}.\end{aligned} \tag{3.4.12}$$

From (3.4.7), (3.4.8), (3.4.10) and (3.4.12), it follows that if $h = f + g$ then for $\omega \in \mathbf{H}$ we have (3.4.5).

$\square$

Now we return again to the proof of Theorem 3.2.3. Claim 3.4.2 implies that $\mathbf{H} \subseteq E_h(b^*)$ and hence $S_h(b^*) = \dim_H E_h(b^*) > 0$.

If we can verify that $b^* = \alpha^*_{h,\max}$ then we are done. We need to show that if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} h(\sigma^n \omega) = \alpha \text{ then } \alpha \leq b^*. \tag{3.4.13}$$

Suppose that we have a fixed $\omega \in \Omega$ for which the limit in (3.4.13) exists and equals α.

Now we subdivide ω into finitely or infinitely many substrings in the following way

$$\omega = Z_0 W_1 Z_1 W_2 Z_2 ...$$

where Z_0 might be the empty string, the other strings are non-empty. For any j the strings $W_j \in \{X, Y\}^{d_j}$, where $1 \leq d_j \leq +\infty$. The strings Z_j do not contain any substring of the form X or Y and they can be finite, or infinite. In case one of the Z_js is infinite then there exists N_1 such that for all $n \geq N_1$, $g(\sigma^n \omega) = 0$ and hence $\alpha \leq \alpha^*_{f,\max} < b^*$.

Hence from now on we can suppose that the Z_js are finite.

If one of the W_js is infinite then one can find N_1 such that $\sigma^{N_1} \omega \in \mathbf{H}$ and hence $\alpha = b^*$ by (3.4.5).

Hence from now on we can suppose that all the W_js are finite.

Since for any $k \in \mathbf{N}$ we have $\omega \in E_h(\alpha)$ iff $\sigma^k \omega \in E_h(\alpha)$ we can suppose that $Z_0 = \emptyset$ and hence $\omega = W_1 Z_1 W_2 Z_2 ....$ Choose k_j, $j = 1, 2, ...$ such that the substring $W_j Z_j$ of ω starts at ω_{k_j}, that is $W_j Z_j = \omega_{k_j} \omega_{k_j+1} ... \omega_{k_{j+1}-1}$. We denote by k'_j the place where Z_j starts, that is, $W_j = \omega_{k_j} \omega_{k_j+1} ... \omega_{k'_j-1}$ and $Z_j = \omega_{k'_j} \omega_{k'_j+1} ... \omega_{k_{j+1}-1}$.

Suppose that we have a j for which

$$\text{there exists } n \in \{k_j, ..., k_{j+1} - 1\} \text{ such that } g(\sigma^n \omega) > 0. \tag{3.4.14}$$

We denote the set of such js by $\mathbf{J}$.

Then $g(\sigma^n\omega) = \varepsilon/4$. We define n_j to be the maximal n satisfying the inequality in (3.4.14). Since Z_j does not contain a substring of the form X or Y, $\sigma^n\omega \notin \mathbf{P}$ for any $k_j' < n < k_{j+1}$. Hence $n_j < k_j'$. Moreover, by the definition of g and $\mathbf{P}$ we have

$$n_j = k_j' - m(2\ell+5)k_A + (\ell-1)k_A.$$

Put

$$k_j'' = n_j - (\ell-1)k_A + (2\ell+5)k_A.$$

Then by the definition of g

$$\sigma^{k_j''}\omega|(2\ell+5)k_A \in \{X,Y\} \text{ and } \sigma^{k_j}\omega|(k_j''-k_j) \in \{X,Y\}^{\frac{k_j''-k_j}{(2\ell+5)k_A}}, \tag{3.4.15}$$

where $\frac{k_j''-k_j}{(2\ell+5)k_A}$ is an integer, that is $\sigma^{k_j}\omega|(k_j''-k_j)$ starts with a long string of Xs and Ys. Hence

$$\frac{1}{k_j''-k_j}\sum_{n=k_j}^{k_j''-1} g(\sigma^n\omega) = \frac{\ell\varepsilon}{4(2\ell+5)k_A}. \tag{3.4.16}$$

It is also clear that

$$\frac{1}{k_{j+1}-k_j''}\sum_{n=k_j''}^{k_{j+1}-1} g(\sigma^n\omega) = 0. \tag{3.4.17}$$

Suppose that $\delta > 0$ is given. We want to find N_δ such that for $N \geq N_\delta$ we have

$$\frac{1}{N}\sum_{n=0}^{N-1} h(\sigma^n\omega) < b^* + \delta. \tag{3.4.18}$$

We can suppose that $\mathbf{J}$ is infinite since otherwise there exists N_1 such that $h(\sigma^n\omega) = f(\sigma^n\omega)$ for $n \geq N_1$ and $\alpha \leq \alpha^*_{f,\max} < b^*$ holds.

We will obtain N_δ by splitting ω into two infinite substrings: The "good part" and the "bad part." The "good part" can be obtained as the concatenation of the substrings $\sigma^{k_j}\omega|(k_j''-k_j)$, $j \in \mathbf{J}$, while the "bad part" of ω is the "rest" of ω, that is what is left of ω if we delete from it the good part. We denote this bad part ω^b. To be more specific if

$j \notin \mathbf{J}$ then we take the string $\sigma^{k_j}\omega|(k_{j+1}-k_j)$, otherwise if $j \in \mathbf{J}$ then we take the string $\sigma^{k''_j}\omega|(k_{j+1}-k''_j)$ and concatenate these strings. To achieve our goals of obtaining N_δ, we will be observing Birkhoff averages along the "good" part $\mathbf{N}^g := \bigcup_{j\in\mathbf{J}}\{k_j, ..., k''_j - 1\}$ and the "bad" part $\mathbf{N}^b = \{0, 1, ...\} \setminus \mathbf{N}^g$. In particular, we will be at some point evaluating Birkhoff sum on the point ω^b rather than ω; we will explain why this works, particularly when we verify equation (3.4.23).

Using (3.4.5), (3.4.16), and the definition of the strings X and Y it is clear that if $j \in J$ then

$$\frac{1}{k''_j - k_j}\sum_{n=k_j}^{k''_j-1} h(\sigma^n\omega) = b^*. \tag{3.4.19}$$

We also know that if $n \in \mathbf{N}^b$ then $g(\sigma^n\omega) = 0$ and hence $h(\sigma^n\omega) = f(\sigma^n\omega)$.

Moreover, whenever $t \in \mathbb{N}$ satisfies the inequality $(t+1)(2\ell+5)k_A \leq k''_j - k_j$, for some $j \in \mathbf{J}$, then

$$\frac{1}{(2\ell+5)k_A}\sum_{n=k_j+t(2\ell+5)k_A}^{k_j+(t+1)(2\ell+5)k_A-1} h(\sigma^n\omega) = b^* \tag{3.4.20}$$

holds as well.

From (3.4.20) and the boundedness of h it follows that we can select N'_δ such that for $N > N'_\delta$

$$\frac{1}{\#\{n \in \mathbf{N}^g : n < N\}}\sum_{n\in\mathbf{N}^g,\ n<N} h(\sigma^n\omega) < b^* + \frac{\delta}{2}. \tag{3.4.21}$$

Denote $\#\{n \in \mathbf{N}^b : n < N\}$ by $\nu_b(N)$.

Next we need to estimate

$$\frac{1}{\nu_b(N)}\sum_{n\in\mathbf{N}^b,\ n<N} h(\sigma^n\omega) = \frac{1}{\nu_b(N)}\sum_{n\in\mathbf{N}^b,\ n<N} f(\sigma^n\omega). \tag{3.4.22}$$

A little later we will show that

$$\frac{1}{\nu_b(N)}\sum_{n\in\mathbf{N}^b,\ n<N} f(\sigma^n\omega) = \frac{1}{\nu_b(N)}\sum_{n=0}^{\nu_b(N)-1} f(\sigma^n\omega^b). \tag{3.4.23}$$

Next we show that if we verified this then we can complete our proof. Indeed by Lemma

3.3.3

$$\limsup_{N'\to\infty} \frac{1}{N'} \sum_{n=0}^{N'-1} f(\sigma^n \omega^b) \le \alpha^*_{f,\max}$$

and hence we can select $N_\delta \ge N'_\delta$ such that if $N \ge N_\delta$ then $\nu_b(N)$ is sufficiently large to have

$$\frac{1}{\nu_b(N)} \sum_{n=0}^{\nu_b(N)-1} f(\sigma^n \omega^b) \le \alpha^*_{f,\max} + \frac{\delta}{2}.$$

By (3.4.23) this yields that

$$\frac{1}{\nu_b(N)} \sum_{n\in\mathbf{N}^b,\ n<N} f(\sigma^n \omega) < \alpha^*_{f,\max} + \frac{\delta}{2} < b^* + \frac{\delta}{2}.$$

From this, (3.4.21), and (3.4.22), it follows that for $N > N_\delta$

$$\frac{1}{N} \sum_{n=0}^{N-1} h(\sigma^n \omega) < b^* + \frac{\delta}{2}.$$

Since a suitable N_δ can be chosen for any $\delta > 0$ we proved that $\alpha \le b^*$.

Hence, to complete the proof of the theorem we need to verify (3.4.23). But this is not difficult. Since $f \in \mathrm{PCC}^{\mathbf{k}}(\Omega)$ we know that $f(\sigma^n\omega)$ depends only on the string $\sigma^n\omega|\mathbf{k}$. Observe that during the definition of ω^b we concatenate strings which start with a string A and A is of length $k_A > \mathbf{k}$. Indeed, if $j \notin \mathbf{J}$ then during the definition we concatenate the string $\sigma^{k_j}\omega|(k_{j+1} - k_j) = W_j Z_j$, and W_j starts with X or Y and they both start with A. If $j \in \mathbf{J}$ then we take the string $\sigma^{k''_j}\omega|(k_{j+1} - k''_j)$ and by (3.4.15) this string starts with A.

We can define a function $\psi : \{0, 1, ...\} \to \mathbf{N}^b$ the following way. For $n \in \{0, 1, ...\}$ if we take ω^b_n then this entry corresponded to exactly one entry $\omega_{\psi(n)}$ of ω and belonged to a concatenated string making up ω^b. Suppose that $k_j \le \psi(n) < k_{j+1}$. If $\psi(n) \le k_{j+1} - \mathbf{k}$ then the strings $\sigma^n\omega^b|\mathbf{k}$ and $\sigma^{\psi(n)}\omega|\mathbf{k}$ are identical and hence $f(\sigma^n\omega^b) = f(\sigma^{\psi(n)}\omega)$. If $\psi(n) > k_{j+1} - \mathbf{k}$ then there is an $n' < n + \mathbf{k}$ such that $\psi(n') = k_{j+1}$. By our concatenation procedure it is clear that the strings $\sigma^n\omega^b|(n' - n)$ and $\sigma^{\psi(n)}\omega|(n' - n)$ are identical. It is also clear that $\psi(n') = k_{j+1}$ and $\sigma^{\psi(n')}\omega|k_A = A$, since we take the first k_A entries of a string which equals X or Y. Now recall our earlier observation that ω^b was obtained by the concatenation of strings which start with A. Hence $\sigma^{n'}\omega^b$ starts with the string A. This implies again that $f(\sigma^n\omega^b) = f(\sigma^{\psi(n)}\omega)$. $\square$

3.4.2 A generic continuous function has a continuous Birkhoff spectrum

To prove Theorem 3.2.5, we need the following lemma, which shows that one can "perturb" a PCC function so that the new function would have a continuous spectrum.

Lemma 3.4.3. *Let $f \in \mathrm{PCC}^k(\Omega)$ and let $\varepsilon > 0$. Then there exists $g \in C_0(\Omega)$ such that $\|g\| < \varepsilon$, S_{f+g} vanishes at $\alpha^*_{f+g,\max}$ and $\alpha^*_{f,\min} - \varepsilon \leq \alpha^*_{f+g,\min} \leq \alpha^*_{f+g,\max} \leq \alpha^*_{f,\max} + \varepsilon$.*

Proof. Let $f \in PCC^k(\Omega)$ and let $\varepsilon > 0$. Let ω^* be a periodic point with prime period p for which $\frac{1}{p}\sum_{n=1}^{p} f(\sigma^n\omega^*) = \alpha^*_{f,\max}$ (which exists by Lemma 3.3.5). Let $g_0(\omega) = \min_{i=1,\ldots,p}\{d(\omega, \sigma^i\omega^*)\}$, and let $g = -\varepsilon g_0 + c$, where $c = \varepsilon \int g_0\, d\lambda$, which implies $\int g\, d\lambda = 0$. Since $\lambda(\Omega) = \mathrm{diam}(\Omega) = 1$, it is clear that $\|g\| < \varepsilon$.

Given $E \subset \mathbb{N}$, we denote by $\mathbf{d}(E)$ the density of the set E, that is $\lim_{N\to\infty} \frac{\#(E\cap[1,N])}{N}$ (if it exists). We let

$$H_{\omega^*} := \{\omega \in \Omega : \omega|_E = \omega^* \text{ for some } E \subset \mathbb{N} \text{ for which } \mathbf{d}(E^c) = 0\},$$

where $\omega|_E$ denotes the concatenation of ω_j, $j \in E$. We will show that $E_{f+g}(\alpha^*_{f+g,\max}) \subset H_{\omega^*}$, and then we observe that $\dim_H H_{\omega^*} = 0$.

By using (3.3.2) from Lemma 3.3.3 one can see that $\alpha^*_{f+g,\max} \leq \alpha^*_{f,\max} + c$. Since $g_0(\sigma^n\omega^*) = 0$ for any n, we obtain $\alpha^*_{f+g,\max} \geq \alpha^*_{f,\max} + c$, and hence $\alpha^*_{f+g,\max} = \alpha^*_{f,\max} + c$. Let $\omega \in E_{f+g}(\alpha^*_{f+g,\max})$. Then we must have

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) - \frac{\varepsilon}{N}\sum_{n=1}^{N} g_0(\sigma^n\omega)\right) = \alpha^*_{f,\max}\,,$$

and this is only possible if $\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) \to \alpha^*_{f,\max}$, and, in particular,

$$\frac{1}{N}\sum_{n=1}^{N} g_0(\sigma^n\omega) \to 0 \text{ as } N \to \infty.$$

This implies that the set

$$J_\omega := \{n \in \mathbb{N} : g_0(\sigma^n \omega) \geq 2^{-p}\}$$

has zero density. Observe that if $g_0(\sigma^n\omega) < 2^{-p}$ for $n = j', ..., j' + l$ then there exists $i \in \{0, ..., p-1\}$ such that

$$(\sigma^n \omega)_{j'}^{j'+l+p} = \sigma^i \omega^* | l + p + 1. \tag{3.4.24}$$

The case when J_ω is finite is much easier and is left to the reader, we detail only the case when J_ω is infinite.

Suppose we enumerate $J_\omega = \{j_1, j_2, j_3, \ldots\}$ in the increasing order and we set $j_0 := 1$. Then for each $k \in \mathbb{N} \cup \{0\}$, there exists $i_k \in \{0, ..., p-1\}$ such that the (possibly empty) string $\gamma(j_k) := (\omega)_{j_k+1}^{j_{k+1}-1}$ equals $\sigma^{i_k}\omega^* | j_{k+1} - j_k - 1$. Hence, we have

$$\omega|_{J_\omega^c} = \gamma(j_0)\gamma(j_1)\gamma(j_2)\cdots .$$

Since ω^* is periodic we can choose $m_k \in \{0, ..., p-1\}$ such that if $\gamma^*(j_k) = \sigma^{m_k}\gamma(j_k)$, that is we throw away the first m_k entries of $\gamma(j_k)$, then

$$\gamma^*(j_0)\gamma^*(j_1)\gamma^*(j_2)\cdots = \omega^* .$$

Put $F = \bigcup_k \{j_k, ..., j_k + m_k\}$. Then $F \subset \bigcup_{i=0}^{p-1} J_\omega + i$ (where $A + b = \{a + b : a \in A\}$ for any $A \subset \mathbb{N}$ and $b \in \mathbb{N}$), which has a zero density. Setting $E = F^c$, we get $\omega|_E = \omega^*$. Hence, $\omega \in H_{\omega^*}$, which shows that $E_{f+g}(\alpha^*_{f+g,\max}) \subset H_{\omega^*}$.

We now show that $\dim_H H_{\omega^*} = 0$. Consider the set $H_{\mathbf{0}} := \{\omega \in \Omega : \mathbf{d}(\{i \in \mathbb{N} : \omega_i = 1\}) = 0\}$. Due to Example 3.1.1 we see that $\dim_H(H_{\mathbf{0}}) = 0$ as it equals $S_f(0)$ for f defined in that example. Given $\omega \in \Omega$ and $i \in \mathbb{N}$ we set $\nu(i, \omega) = \#\{j : \omega_j = 0,\ j \leq i\}$. We define a map $h : \Omega \to \Omega$ as follows: $h(\omega) = h_1 h_2 h_3 \ldots$, where

$$h_i := \begin{cases} \omega^*_{\nu(i,\omega)} & \text{if } \omega_i = 0 \\ 1 - \omega^*_{\nu(i,\omega)+1} & \text{if } \omega_i = 1. \end{cases}$$

It is easy to see that h is Lipschitz. One can also verify easily that $h(H_{\mathbf{0}}) \supset H_{\omega^*}$. Therefore, $0 \le \dim_H(H_{\omega^*}) \le \dim_H(h(H_{\mathbf{0}})) = 0$. $\square$

What remains from the proof of Theorem 3.2.5 is rather standard:

Proof of Theorem 3.2.5. It suffices to prove that a generic continuous function h has continuous spectrum at the points $\alpha^*_{h,\min}$ and $\alpha^*_{h,\max}$, and due to symmetry reasons, it suffices to prove the continuity in $\alpha^*_{h,\max}$ (if it holds in a residual set, the other also does in another residual set, and the intersection of these sets is still residual). We will prove in fact that the set

$$Z = \{h \in C(\Omega) : S_h \text{ is not continuous at } \alpha^*_{h,\max}\}$$

is meager. Note that we know that S_h is concave and achieves its maximum at $\int h\,d\lambda$, hence

$$Z = \bigcup_{n=1}^{\infty} Z_{\frac{1}{n}},$$

where

$$Z_\theta = \left\{h \in C(\Omega) : S_h(x) > \theta \text{ for all } x \in \left[\int h\,d\lambda, \alpha^*_{h,\max}\right]\right\}.$$

Now it suffices to prove that each Z_θ is nowhere dense, and clearly it is enough to consider small enough $\theta < 1$. To this end, take arbitrary $f \in \mathrm{PCC}^k(\Omega)$ for some k, and $\varepsilon > 0$. By Lemma 3.4.3, we can find $f+g$ in the ε-neighborhood of f such that it has continuous spectrum at $\alpha^*_{f+g,\max}$. Then $\alpha^*_{f+g,\max} > \int(f+g) \ge \alpha^*_{f+g,\min}$ necessarily holds, as $S_{f+g}(\int(f+g)d\lambda) = 1$. Now by continuity, we can take $x \in \left[\int h, \alpha^*_{f+g,\max}\right]$ such that $0 < S_{f+g}(x) < \frac{\theta}{2}$. By its concavity S_{f+g} is monotone decreasing on $[\int h, \alpha^*_{f+g,\max}]$ hence we can assume that

$$x - \alpha^*_{f,\min} \ge \alpha^*_{f,\max} - x.$$

Now apply Theorem 3.2.1 for $f+g$ with

$$\varepsilon' = \min\left\{\frac{\theta}{2}, \alpha^*_{f,\max} - x\right\}. \tag{3.4.25}$$

It guarantees that $0 < S_h(x) < \theta$ for any h with $\|h - (f+g)\| < \delta'$ for some $\delta' > 0$.

Moreover, if h and $f+g$ are close enough to each other, we also have that their integral cannot differ by much, hence we also have that $x \in \left[\int h, \alpha^*_{h,\max}\right]$. Consequently, if h is in a sufficiently small neighborhood of $f+g$ satisfying both this integral condition and what is given by (3.4.25), then h is not in Z_θ. It yields that Z_θ is nowhere dense, as $PCC(\Omega)$ is dense, and in the neighborhood of an arbitrary f belonging to this set we constructed an open ball which is disjoint from Z_θ. It concludes the proof. □

3.4.3 Supports of generic spectra are in $(\alpha_{f,\min}, \alpha_{f,\max})$

Proof of Theorem 3.2.7. It suffices to prove that each inequality in (3.2.1) holds in a dense open subset of $C(\Omega)$, and due to symmetry, it is sufficient to prove that $\alpha^*_{f,\min} < \alpha^*_{f,\max}$ and $\alpha^*_{f,\max} < \alpha_{f,\max}$ hold in dense open subsets, respectively. Given Remark 3.3.2, it immediately follows that each of these inequalities holds in an open subset, thus we only have to keep an eye on denseness.

Consider first $\alpha^*_{f,\min} < \alpha^*_{f,\max}$. By Theorem 3.2.5 we know that S_f is continuous for $f \in G_1$ with a dense subset $G_1 \subset C(\Omega)$. However, for $\alpha_\lambda = \int f d\lambda$ we have $S_f(\alpha_\lambda) = 1$, and $S_f(\alpha^*_{f,\min}) = S_f(\alpha^*_{f,\max}) = 0$, hence

$$\alpha^*_{f,\min} < \alpha^*_{f,\max}. \tag{3.4.26}$$

It yields that for any $f \in G_1$ we have $\alpha^*_{f,\min} < \alpha^*_{f,\max}$, thus this inequality holds in a dense subset indeed.

Let us consider now $\alpha^*_{f,\max} < \alpha_{f,\max}$. We know that functions $f \in \mathrm{PCC}(\Omega)$ are dense in $C(\Omega)$. Consider such a function f, we have $f \in \mathrm{PCC}^k(\Omega)$ for some $k > 0$. By Lemma 3.3.5 we know that there exists a periodic configuration ω_f with $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\sigma^n \omega_f) = \alpha^*_{f,\max}$. If $\alpha^*_{f,\max} < \alpha_{f,\max}$ then we are done. Hence we can suppose that $\alpha^*_{f,\max} = \alpha_{f,\max}$.

Assume first that ω_f can be chosen such that ω_f is neither identically 1^∞ nor 0^∞. Then we can choose a substring A of length k such that f is maximal on $[A]$ and A is neither $11\cdots1$ nor $00\cdots0$ (i.e. blocks of k many 1s or 0s, respectively), actually, by periodicity of ω_f any substring of A, different from $11\cdots1$ and $00\cdots0$ is suitable.

Now for given $\varepsilon > 0$ define $g \in \mathrm{PCC}^k(\Omega)$ such that $g = f + \varepsilon \mathbf{1}_{[A]}$. Select a periodic ω_g for which $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(\sigma^n \omega_g) = \alpha^*_{g,\max}$, the existence of ω_g is again guaranteed by Lemma 3.3.5. The relative frequency of the substring A in ω_g is strictly smaller than 1, as A contains both 0s and 1s, hence at least $1/k$ of the substrings start with a binary digit different from the first entry in A. Thus we can conclude

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(\sigma^n \omega_g) - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\sigma^n \omega_f) < \|g - f\| = \varepsilon,$$

hence

$$\alpha^*_{g,\max} - \alpha^*_{f,\max} < \varepsilon.$$

However,

$$\alpha_{g,\max} - \alpha_{f,\max} = \varepsilon$$

by definition. Hence we can find g arbitrarily close to f with $\alpha^*_{g,\max} < \alpha_{g,\max}$ in this case.

Assume now that the only possible choices for ω_f are amongst 1^∞ and 0^∞. If A can be chosen as in the first case, differing from the identically 1 and identically 0 strings of length k, then the previous argument might be repeated, thus it suffices to observe the cases when ω_f and A can only be identically 1 or identically 0. Clearly without loss of generality we can assume that the former one holds. In this case we perturb f as follows: let $A = 11\ldots10$, which is a block consisting of k-many 1's then followed by a 0. We define $g \in \mathrm{PCC}^{k+1}(\Omega)$ such that $g = f + \varepsilon \mathbf{1}_{[A]}$. Then $\alpha_{g,\max} - \alpha_{f,\max} = \varepsilon$ as previously. Moreover, if ω' is periodic then we compute that

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(\sigma^n \omega') &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\sigma^n \omega') + \lim_{N\to\infty} \frac{\varepsilon}{N} \sum_{n=1}^{N} \mathbf{1}_{[A]}(\sigma^n \omega') \\ &\le \alpha^*_{f,\max} + \varepsilon \cdot \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{[A]}(\sigma^n \omega').\end{aligned}$$

Note that $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{[A]}(\sigma^n \omega')$, the relative frequency of A in ω' is at most $\frac{1}{k+1}$ (which is obtained when $\omega' = A^\infty$). This implies that if ω_g is the maximal periodic configuration for g, then

$$\alpha^*_{g,\max} = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g(\sigma^n \omega_g) \leq \alpha^*_{f,\max} + \frac{\varepsilon}{k+1} < \alpha^*_{f,\max} + \varepsilon = \alpha_{g,\max}.$$

Thus in both of these two cases we showed that any $f \in \mathrm{PCC}^k(\Omega)$ can be approximated by functions satisfying $\alpha^*_{g,\max} < \alpha_{g,\max}$. It yields that such functions also form a dense set, which concludes the proof. $\square$

Remark 3.4.4. In ergodic optimization, a function $f \in C(\Omega)$ for which $\alpha^*_{f,\max} = \alpha_{f,\max}$ is called *revealed* (cf. [24, §5]). Theorem 3.2.7 tells us that the set of revealed functions in $C(\Omega)$ forms a nowhere dense set.

3.5 One-sided derivatives of the Birkhoff spectra at endpoints

In this section for functions with continuous spectrum we are interested in the one-sided derivatives of the spectrum at the endpoints of its support in the direction of the interior of the support.

3.5.1 One-sided derivatives at the endpoints of spectra for generic functions

The goal of this subsection to verify Theorem 3.2.8.

We start with a lemma which will be the building block for the proof of the above theorem.

Lemma 3.5.1. *Let $f_0 \in C(\Omega)$, $\varepsilon > 0$, and $\nu \in \mathbb{N}$ be given. Then there exists $f_2 \in C(\Omega)$ and $\delta > 0$ such that $\|f_0 - f_2\| < \varepsilon/2$, $\delta < \varepsilon/2$, and for any $f \in B(f_2,\delta) \subseteq B(f_0,\varepsilon)$ there exists $\alpha' < \alpha^*_{f,\max}$ such that*

$$\frac{S_f(\alpha') - S_f(\alpha^*_{f,\max})}{\alpha' - \alpha^*_{f,\max}} < -\nu. \tag{3.5.1}$$

Remark 3.5.2. As S_f is concave on the interval L_f, the inequality (3.5.1) in the lemma implies $\partial^- S_f(\alpha^*_{f,\max}) < -\nu$.

Proof. Using Theorem 3.2.3 choose $f_1 \in PCC(\Omega)$ with $\|f_0 - f_1\| < \varepsilon/4$ such that $\varepsilon_1 = S_{f_1}(\alpha^*_{f_1,\max}) > 0$.

Set $\varepsilon_2 = \min\{\varepsilon_1, \frac{\varepsilon}{2}, 1/2\}$.

Using Theorem 3.2.5 choose $f_2 \in C(\Omega)$ such that

$$\|f_1 - f_2\| < \frac{\varepsilon_2}{10\nu} \text{ and } S_{f_2}(\alpha^*_{f_2,\max}) = 0.$$

By Lemma 3.3.1 and Remark 3.3.2 applied to f_1 and f_2 we obtain that $\alpha^*_{f_2,\max} < \alpha^*_{f_1,\max} + \frac{\varepsilon_2}{10\nu}$ and there exists $\alpha' > \alpha^*_{f_1,\max} - \frac{\varepsilon_2}{10\nu}$ such that

$$S_{f_2}(\alpha') \geq S_{f_1}(\alpha^*_{f_1,\max}) = \varepsilon_1 \geq \varepsilon_2. \tag{3.5.2}$$

Then

$$\alpha^*_{f_2,\max} - \alpha' < 2 \cdot \frac{\varepsilon_2}{10\nu}. \tag{3.5.3}$$

Keep in mind that $S_{f_2}(\alpha^*_{f_2,\max}) = 0$ and choose $\delta_1 > 0$ such that

$$S_{f_2}(\alpha) < \frac{\varepsilon_2}{20} \text{ holds for } \alpha \in (\alpha^*_{f_2,\max} - \delta_1, \alpha^*_{f_2,\max}]. \tag{3.5.4}$$

Observe that from (3.5.2) it also follows that $\alpha^*_{f_2,\min} \leq \alpha' < \alpha^*_{f_2,\max} - \delta_1$. Now choose $\delta_2 > 0$ such that

$$\delta_2 < \min\left\{\frac{\alpha^*_{f_2,\max} - \alpha'}{10}, \frac{\delta_1}{5}, \frac{\varepsilon_2}{20\nu}\right\}. \tag{3.5.5}$$

Using this δ_2 as ε in Theorem 3.2.1 select $\delta \in (0, \delta_2)$ such that for $f \in B(f_2, \delta)$ we have

$$|S_f(\alpha) - S_{f_2}(\alpha)| < \delta_2 \text{ for } \alpha \in (\alpha^*_{f_2,\min} + \delta_2, \alpha^*_{f_2,\max} - \delta_2). \tag{3.5.6}$$

Suppose $f \in B(f_2, \delta)$. Then by Lemma 3.3.1, Remark 3.3.2, (3.5.3) and (3.5.5) we obtain

$$|\alpha^*_{f,\max} - \alpha^*_{f_2,\max}| < \delta_2 \text{ and hence } |\alpha' - \alpha^*_{f,\max}| < 1.1(\alpha^*_{f_2,\max} - \alpha') < 1.1 \cdot \frac{\varepsilon_2}{5\nu}.$$

By (3.5.4), $S_{f_2}(\alpha^*_{f_2,\max} - \delta_1/2) < \varepsilon_2/20$ and then by (3.5.6), $S_f(\alpha^*_{f_2,\max} - \delta_1/2) < \varepsilon_2/10 < 1$. By concavity of S_f and $S_f(\int f) = 1$ it is clear that S_f is monotone decreasing on $[\alpha^*_{f,\max} - \delta_1/2, \alpha^*_{f,\max}]$ and hence

$$S_f(\alpha^*_{f,\max}) < \frac{\varepsilon_2}{10}. \tag{3.5.7}$$

Using (3.5.2), (3.5.5) and (3.5.6) we infer

$$S_f(\alpha') > S_{f_2}(\alpha') - \delta_2 \geq 0.9\varepsilon_2.$$

By this, (3.5.7) and (3.5.3)

$$\frac{S_f(\alpha') - S_f(\alpha^*_{f,\max})}{\alpha' - \alpha^*_{f,\max}} < -\frac{0.8\varepsilon_2}{1.1 \cdot \frac{\varepsilon_2}{5\nu}} < -\nu.$$

□

Remark 3.5.3. We remark that due to symmetry reasons a version of Lemma 3.5.1 also holds at the other endpoint, $\alpha^*_{f,\min}$ of the spectrum yielding that for any $f \in B(f_2, \delta) \subseteq B(f_0, \varepsilon)$ there exists $\alpha' > \alpha^*_{f,\min}$ such that

$$\frac{S_f(\alpha') - S_f(\alpha^*_{f,\min})}{\alpha' - \alpha^*_{f,\min}} > \nu. \tag{3.5.8}$$

As we observed earlier in the one-dimensional case S_f is continuous on $[\alpha^*_{f,\min}, \alpha^*_{f,\max}]$ hence even in case of discontinuous spectra one can consider $\partial^- S_f(\alpha^*_{f,\max})$ and $\partial^+ S_f(\alpha^*_{f,\max})$, one might have a one-sided discontinuity only in the direction pointing towards the exterior of the support of the spectrum.

Lemma 3.5.1 easily implies Theorem 3.2.8:

Proof of Theorem 3.2.8. Consider an arbitrary $f_0 \in C(\Omega)$ and $\varepsilon > 0$. Fix $\nu \in \mathbb{N}$. We may apply Lemma 3.5.1 and Remark 3.5.2 to see that $B(f_0,\varepsilon)$ contains a smaller open set $B(f_2,\delta)$ of $C(\Omega)$ such that for any $f \in B(f_2,\delta)$ we have $\partial^- S_f(\alpha^*_{f,\max}) < -\nu$. It implies that the complement of

$$A_\nu = \{f \in C(\Omega) : \partial^- S_f(\alpha^*_{f,\max}) < -\nu\}$$

is nowhere dense for any ν. Hence $A = \bigcup_{\nu=1}^{\infty} A_\nu$ is a residual set of $C(\Omega)$, yielding that for the generic continuous function $f \in C(\Omega)$, we have $\partial^- S_f(\alpha^*_{f,\max}) = -\infty$.

However, by Remark 3.5.3 we may conclude the same way that for the generic continuous function $f \in C(\Omega)$, we have $\partial^+ S_f(\alpha^*_{f,\min}) = \infty$. Thus for the generic continuous function, we have both of these prescribed equalities, which concludes the proof. □

3.5.2 Finite one-sided derivatives at the endpoints of the spectrum

In this subsection, we will prove Theorem 3.2.9 and Theorem 3.2.10.

The first step towards the proof of Theorem 3.2.9 is the following lemma, in which we give upper bounds on a value of the spectrum for a suitably defined function. Since $S_f(\int f d\lambda) = 1$ if we have a function with continuous spectrum then by concavity of the spectrum $\partial^- S_f(\alpha^*_{f,\max}) \leq -\frac{1}{\alpha^*_{f,\max} - \int f d\lambda}$ and $\partial^+ S_f(\alpha^*_{f,\min}) \geq \frac{1}{\int f d\lambda - \alpha^*_{f,\min}}$.

In the next Lemma we define a PCC function with "very small" spectrum. This type of functions serve as building blocks in the proof of Theorem 3.2.9.

Lemma 3.5.4. *Let $b > a$, and let $f : \Omega \to \mathbb{R}$ be such that $f(\omega) = b$ if the first L coordinates of ω is 1, otherwise $f(\omega) = a$. Moreover, fix $\varepsilon > 0$ and $0 < \beta < 1$. Then if L is sufficiently large, then*

$$S_f(t) \leq \beta + \varepsilon \tag{3.5.9}$$

for $t = \beta a + (1-\beta)b$.

Remark 3.5.5. See Figure 3.1 for an illustration of this remark. Observe that in the above lemma if L is large then $\int f d\lambda = b \cdot 2^{-L} + a(1 - 2^{-L})$ and hence $S_f(b \cdot 2^{-L} +$

$a \cdot (1 - 2^{-L})) = 1$. The point $b \cdot 2^{-L} + a \cdot (1 - 2^{-L})$ is very close to $a = \alpha_{f,\min}$. It is also clear that $E_f(b) \neq \emptyset$, since 1^∞ belongs to it. By also considering 0^∞ we see that $[a, b] = [\alpha^*_{f,\min}, \alpha^*_{f,\max}]$. Hence the line segment connecting $(b \cdot 2^{-L} + a \cdot (1 - 2^{-L}), 1)$ to $(b, 0)$ should be under the graph of S_f on $[b \cdot 2^{-L} + a \cdot (1 - 2^{-L}), b]$. If β is small then t is very close to b and by concavity of the spectrum on $[b \cdot 2^{-L} + a \cdot (1 - 2^{-L}), t]$ the graph of S_f should be under the dashed line on the figure connecting $(t, \beta + \varepsilon) = (\beta a + (1 - \beta)b, \beta + \varepsilon)$ to $(b, 0)$. This implies that for small β and large L apart from a very short interval near the endpoint a the spectrum S_f is very close to the line segment connecting $(a, 1)$ to $(b, 0)$ and on $[a, b]$ approximates the upper part of the boundary (shown with dotted line on the figure) of the right angled triangle with vertices $(a, 0)$, $(a, 1)$ and $(b, 0)$.

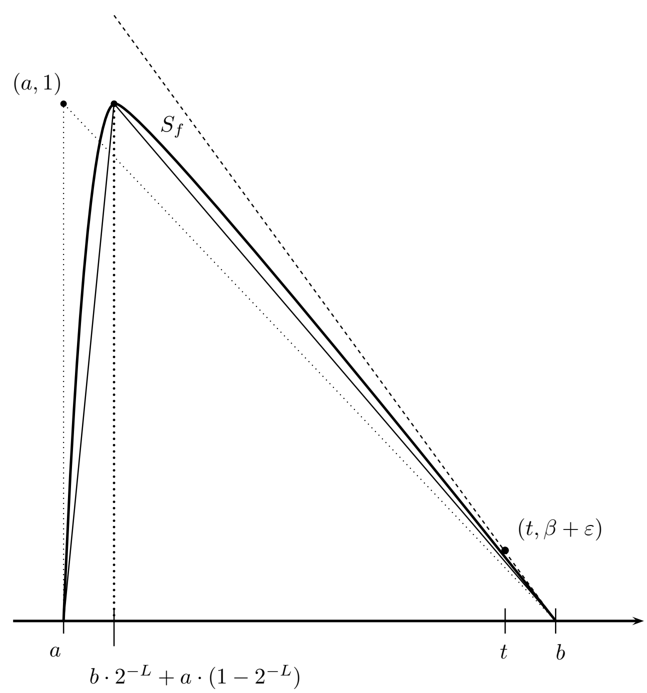

Figure 3.1: An illustration of Remark 3.5.5.

Proof. Let $t = \beta a + (1 - \beta)b$. Clearly it suffices to prove the statement of the lemma for small enough ε, thus we might assume that $\beta^* = \beta + \frac{\varepsilon}{2} < 1$. We would like to estimate the dimension of

$$E_f(t) = \Big\{ \omega : \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(\sigma^n \omega) = t \Big\}.$$

This set contains ω if and only if it contains $\sigma(\omega)$, thus we can shift the sum by one for technical convenience. Moreover, if we replace the lim by a lim inf, we can deduce that this set is contained by

$$\Big\{\omega : \liminf_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega) \geq t\Big\}.$$

If ω is in this set, then for large enough N the corresponding ergodic average exceeds $t^* = \beta^* a + (1-\beta^*)b < t$, that is

$$E_f(t) \subset \bigcup_{m=1}^{\infty}\bigcap_{N=m}^{\infty}\left\{\omega : \frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega) \geq t^*\right\}. \tag{3.5.10}$$

In the sequel we will use $\left\{\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega) \geq t^*\right\}$ instead of $\left\{\omega : \frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega) \geq t^*\right\}$ to ease the notation.

The union in (3.5.10) is the union of a growing sequence of sets, thus the dimension is simply the limit of $\dim_H A_m$, where

$$A_m = \bigcap_{N=m}^{\infty}\left\{\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega) \geq t^*\right\}.$$

Now we focus on estimating the dimension of this set. To this end, we would like to count the cylinder sets of length $N+L-1$ which intersect $\left\{\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega) \geq t^*\right\}$ for large N, as they give a cover of A_m for any $N \geq m$. (We are concerned with cylinders of length $N+L-1$ instead of the ones with length N as the first $N+L-1$ coordinates affect $\sum_{n=0}^{N-1} f(\sigma^n\omega)$.) For our purposes it suffices to choose N such that $L|N+L-1$, as we can diverge to infinity with N even under this restriction and we need an upper estimate of the dimension.

The number of blocks consisting of at least L consecutive 1s is at most $\frac{N+L-1}{L}$. If $L \geq 2$, and there are i such blocks, the number of ways to place them among the $N+L-1$ coordinates is at most $\binom{N+L-1}{2i}$, since the placement of each block can be uniquely specified by the coordinates for which the first and the last coordinates of the block occupy. (We note that it is indeed an upper estimate: this expression does not deal with the length of the blocks, neither with the fact that blocks are separated from each other with at least one intermediate coordinate.) Moreover, if $L \geq 5$, then for the largest possible value of i, that is for $i = \frac{N+L-1}{L}$ we still have

$$2i = 2 \cdot \frac{N+L-1}{L} < \frac{N+L-1}{2}.$$

Thus the number of ways we can arrange the blocks of at least L consecutive 1s is at most

$$\sum_{i=0}^{\frac{N+L-1}{L}} \binom{N+L-1}{2i} \leq \left(\frac{N+L-1}{L}+1\right) \cdot \binom{N+L-1}{2 \cdot \frac{N+L-1}{L}} \tag{3.5.11}$$
$$\leq (N+L-1) \cdot \binom{N+L-1}{2 \cdot \frac{N+L-1}{L}},$$

as the binomial coefficients are increasing until the middle ones.

We should also give a bound on the number of ways we can choose the other coordinates. Since $\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega) \geq t^*$, we know that most of the coordinates belong to one of the above blocks. More specifically, in the first N coordinates there are at most $\beta^* N$ not covered by them, as otherwise the number of terms in $\sum_{n=0}^{N-1} f(\sigma^n\omega)$ with $f(\sigma^n\omega) = a$ exceeds $\beta^* N$, which yields that

$$\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega) < \beta^* a + (1-\beta^*)b = t^*.$$

Thus a raw upper estimate for the number of the ways we can choose the remaining coordinates in order to have an $N+L-1$-cylinder intersecting

$$\left\{\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega) \geq t^*\right\}$$

is $2^{\beta^* N} \cdot 2^{L-1}$, where the last factor is simply the number of ways we can choose the last $L-1$ coordinates.

Combining the results of the preceding two paragraphs yields that

$$\left\{ \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t^* \right\}$$

is covered by at most

$$(N+L-1) \cdot \binom{N+L-1}{2 \cdot \frac{N+L-1}{L}} \cdot 2^{\beta^* N+L-1}$$

many cylinders of diameter $2^{-(N+L-1)}$. By using the standard $\binom{a}{b} \leq \left(\frac{ae}{b}\right)^b$ bound on the binomial coefficients, we can relax this upper bound to

$$(N+L-1) \cdot \left(\frac{eL}{2}\right)^{2 \cdot \frac{N+L-1}{L}} \cdot 2^{\beta^* N+L-1} = k \cdot \left(\frac{eL}{2}\right)^{\frac{2k}{L}} \cdot 2^{\beta^* k} \cdot 2^{(1-\beta^*)(L-1)}, \qquad (3.5.12)$$

where $k = N + L - 1$. Notice that for large enough L (and consequently, large enough k) we have

$$2^{\frac{\varepsilon}{2}} > \sqrt[k]{k} \left(\frac{eL}{2}\right)^{\frac{2}{L}},$$

as both factors on the right tend to 1. Fix L to be sufficiently large in order to guarantee this. Consequently, (3.5.12) can be estimated from above by

$$2^{(\beta^* + \frac{\varepsilon}{2})k} \cdot 2^{(1-\beta^*)(L-1)}. \qquad (3.5.13)$$

Hence

$$\left\{ \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t^* \right\}$$

can be covered by at most $2^{(\beta^* + \frac{\varepsilon}{2})k} \cdot 2^{(1-\beta^*)(L-1)}$ many cylinders of diameter 2^{-k} for any k with $L|k$. It immediately yields

$$\mathcal{H}_{2^{-k}}^{\beta^* + \frac{\varepsilon}{2}} \left(\left\{ \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t^* \right\} \right) \leq 2^{(1-\beta^*)(L-1)}$$

where $N = k - L + 1$ as before. However, this set contains A_m for large enough k, N, thus

$$\mathcal{H}_{2^{-k}}^{\beta^*+\frac{\varepsilon}{2}}(A_m) \leq 2^{(1-\beta^*)(L-1)}.$$

As k, N can be arbitrarily large, it shows that in fact

$$\mathcal{H}^{\beta^*+\frac{\varepsilon}{2}}(A_m) \leq 2^{(1-\beta^*)(L-1)}$$

and consequently,

$$\dim_H(A_m) \leq \beta^* + \frac{\varepsilon}{2} = \beta + \varepsilon.$$

Consequently, by our initial observations

$$S_f(t) \leq \beta + \varepsilon,$$

as stated. □

Proof of Theorem 3.2.9. We define f to be a more elaborate variant of the function appearing in Lemma 3.5.4. Set $t_j = 1 - 2^{-j}$. Then $t_j \in (0, 1)$ and $t_j \to 1$. We will define a strictly increasing sequence (L_j) of positive integers, to be fixed later and chosen recursively. We can suppose that $L_1 > 5$.

Now we let $f(\omega) = t_j$ if ω starts with a block of 1s of length at least L_j, but less than L_{j+1}. Moreover, $f(\omega) = -t_j$ if ω starts with a block of 0s of length at least L_j, but less than L_{j+1}. Finally, let $f(1^\infty) = 1$ and $f(0^\infty) = -1$ for the constant sequences, and let $f(\omega) = 0$ for any remaining ω. Due to symmetry, it is clear that $\int f = 0$, and it is straightforward to check continuity. It remains to prove that the relevant derivatives are finite. By symmetry again, it suffices to verify $\partial^- S_f(\alpha^*_{f,\max}) > -\infty$. To this end, we will use an argument similar to the one seen in the proof of Lemma 3.5.4.

As in (3.5.10), we can deduce

$$E_f(t_{j+1}) \subset \bigcup_{m=1}^{\infty} \bigcap_{N=m}^{\infty} \left\{ \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t_j \right\}.$$

This union is the union of a growing sequence of sets, thus the dimension is simply the limit of $\dim_H A_m$, where

$$A_m = \bigcap_{N=m}^{\infty} \left\{ \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t_j \right\}.$$

In order to estimate this dimension, we first introduce an auxiliary function, which is easier to examine. Explicitly, we let $f_j = 0$, if $f \leq 0$, and we let $f_j = 1$ if $f \geq t_j$. In any other case we let $f_j = f$. Then $f_j \geq f$, consequently

$$A_{m,j} = \bigcap_{N=m}^{\infty} \left\{ \frac{1}{N} \sum_{n=0}^{N-1} f_j(\sigma^n \omega) \geq t_j \right\}$$

contains A_m. Thus it suffices to estimate the dimension of $A_{m,j}$. The argument is similar to the one in the proof of Lemma 3.5.4. We would like to count the cylinder sets of length $N + L_j - 1$ which intersect $\left\{ \frac{1}{N} \sum_{n=0}^{N-1} f_j(\sigma^n \omega) \geq t_j \right\}$ for large N, as they give a cover of $A_{m,j}$ for any $N \geq m$. In order to avoid the inconvenience caused by integer parts, we will only consider Ns with certain divisibility properties, as before.

First of all, the number of blocks consisting of at least L_j consecutive 1s is at most $\frac{N+L_j-1}{L_j}$, which is an integer for infinitely many N. Thus the number of ways we can arrange the blocks of at least L_j consecutive 1s is at most

$$\sum_{i=0}^{\frac{N+L_j-1}{L_j}} \binom{N + L_j - 1}{2i} \leq \left(\frac{N + L_j - 1}{L_j} + 1 \right) \cdot \binom{N + L_j - 1}{2 \cdot \frac{N+L_j-1}{L_j}} \tag{3.5.14}$$

$$\leq (N + L_j - 1) \cdot \binom{N + L_j - 1}{2 \cdot \frac{N+L_j-1}{L_j}},$$

using $L_j \geq L_1 > 5$, as in (3.5.11). We call these blocks j-blocks.

The novelty of cylinder counting in this proof compared to the previous one is that we have to take into account the blocks responsible for the values of f_j between 0 and t_{j-1}. As $\frac{1}{N} \sum_{n=0}^{N-1} f_j(\sigma^n \omega) \geq t_j$, in the first N coordinates there are at most $\frac{1-t_j}{1-t_{j-1}} N = \frac{N}{2}$ not covered by the j-blocks, as otherwise the number of terms in $\sum_{n=0}^{N-1} f(\sigma^n \omega)$ with $f(\sigma^n \omega) \leq t_{j-1}$ is too large and we have $\frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega) < t_j$. Thus beside the already placed j-blocks, there are at most $\frac{1-t_j}{1-t_{j-1}}N + L_j - 1 = \frac{N}{2} + L_j - 1$ coordinates remaining, which might contain some $(j-1)$-blocks of at least L_{j-1} consecutive 1s. By a similar estimate to (3.5.14) we find that the number of possible arrangements of these $(j-1)$-blocks is at most

$$\sum_{i=0}^{\frac{\frac{N}{2}+L_j-1}{L_{j-1}}} \binom{\frac{N}{2}+L_j-1}{2i} \leq \left(\frac{\frac{N}{2}+L_j-1}{L_{j-1}}+1\right) \cdot \binom{\frac{N}{2}+L_j-1}{2\cdot\frac{\frac{N}{2}+L_j-1}{L_{j-1}}} \tag{3.5.15}$$

$$\leq \left(\frac{N}{2}+L_j-1\right) \cdot \binom{\frac{N}{2}+L_j-1}{2\cdot\frac{\frac{N}{2}+L_j-1}{L_{j-1}}},$$

using $L_{j-1} \geq L_1 > 5$.

Suppose that $j_0 \in \{0, ..., j-1\}$. Proceeding recursively, by the same argument we can conclude that the union of the $(j-i)$-blocks taken for $i = 0, 1, ..., j_0 - 1$ cover all but at most $\frac{1-t_j}{1-t_{j_0}}N = \frac{N}{2^{j_0}}$ of the first N coordinates. Thus beside these blocks there are at most $\frac{N}{2^{j_0}} + L_j - 1$ coordinates remaining, which yields similarly to (3.5.15) that the number of possible arrangements of the $(j-j_0)$-blocks is at most

$$\left(\frac{N}{2^{j_0}}+L_j-1\right) \cdot \binom{\frac{N}{2^{j_0}}+L_j-1}{2\cdot\frac{\frac{N}{2^{j_0}}+L_j-1}{L_{j-j_0}}} < (N+L_j-1) \cdot \binom{\frac{N}{2^{j_0}}+L_j-1}{2\cdot\frac{\frac{N}{2^{j_0}}+L_j-1}{L_{j-j_0}}}. \tag{3.5.16}$$

We can use this bound for $j_0 = 0, 1, ..., j-1$. (We note that for infinitely many values of N each number appearing in the above binomial coefficients is an integer.) Finally, there can be coordinates which are not contained by any such block. At most $(1-t_j)N$ of them in the first N coordinates, and arbitrarily many of them in the last $L_j - 1$ coordinates. Thus they can be chosen at most $2^{(1-t_j)N+L_j-1}$ different ways. Hence the number of cylinders which intersect $\left\{\frac{1}{N}\sum_{n=0}^{N-1} f_j(\sigma^n\omega) \geq t_j\right\}$ can be bounded by taking the product of the estimates in (3.5.16), and multiplying it by $2^{(1-t_j)N+L_j-1}$. Hence $\left\{\frac{1}{N}\sum_{n=0}^{N-1} f_j(\sigma^n\omega) \geq t_j\right\}$ can be covered by at most

$$(N+L_j-1)^j \cdot 2^{(1-t_j)N+L_j-1} \cdot \prod_{j_0=0}^{j-1} \binom{\frac{N}{2^{j_0}}+L_j-1}{2\cdot\frac{\frac{N}{2^{j_0}}+L_j-1}{L_{j-j_0}}} \tag{3.5.17}$$

many cylinders of diameter $2^{-(N+L_j-1)}$. Observe that the $j_0 = 0$ case in (3.5.17) includes the estimate (3.5.14). By the standard estimate of binomial coefficients we can estimate it further from above by

$$(N+L_j-1)^j \cdot 2^{(1-t_j)N+L_j-1} \prod_{j_0=0}^{j-1} \left(\frac{eL_{j-j_0}}{2}\right)^{2\cdot\frac{\frac{N}{2^{j_0}}+L_j-1}{L_{j-j_0}}}. \tag{3.5.18}$$

Introduce the notation $k = N + L_j - 1$ again. By factoring out constants depending on $L_1, ..., L_j$ into a constant denoted by $C(L_1, ..., L_j)$, and rearranging (3.5.18) one can obtain that it equals

$$C(L_1,...,L_j) \cdot k^j \cdot 2^{(1-t_j)k} \prod_{j_0=0}^{j-1} \left(\frac{eL_{j-j_0}}{2}\right)^{\frac{2k}{2^{j_0}L_{j-j_0}}}. \tag{3.5.19}$$

This formulation leads us to a suitable choice of L_n: for an arbitrary fixed $\tau > 0$, define L_n large enough to guarantee that

$$\left(\frac{eL_n}{2}\right)^{\frac{2}{L_n}} < 2^{\frac{\tau}{2^{2n}}}. \tag{3.5.20}$$

With this choice, (3.5.19) can be estimated by

$$C(L_1,...,L_j) \cdot k^j \cdot 2^{(1-t_j)k} \prod_{j_0=0}^{j-1} 2^{\frac{\tau k}{2^{2j-j_0}}} \le C(L_1,...,L_j) \cdot k^j \cdot 2^{(1-t_j+\frac{\tau}{2^j})k} \tag{3.5.21}$$

$$\le C(L_1,...,L_j) \cdot 2^{(1-t_j+\frac{2\tau}{2^j})k},$$

where the last inequality holds for large enough N, k. It immediately yields

$$\mathcal{H}_{2^{-k}}^{1-t_j+\frac{2\tau}{2^j}}\left(\left\{\frac{1}{N}\sum_{n=0}^{N-1} f_j(\sigma^n\omega) \ge t_j\right\}\right) \le C(L_1,...,L_j)$$

where $N = k - L_j + 1$ as before. However, this set contains $A_{m,j}$ for large enough k, N, thus

$$\mathcal{H}_{2^{-k}}^{1-t_j+\frac{2\tau}{2^j}}(A_m) \le C(L_1,...,L_j).$$

As k, N can be arbitrarily large, it shows that in fact

$$\mathcal{H}^{1-t_j+\frac{2\tau}{2^j}}(A_m) \leq C(L_1, ..., L_j)$$

and consequently,

$$\dim_H(A_{m,j}) \leq 1 - t_j + \frac{2\tau}{2^j}.$$

Consequently, by our initial observations

$$S_f(t_j) \leq 1 - t_j + \frac{2\tau}{2^j},$$

that is, using $t_j = 1 - 2^{-j}$ we have

$$S_f(1 - 2^{-j}) \leq \frac{1 + 2\tau}{2^j}.$$

Thus if we calculate the left derivative of S_f at 1 by going along the sequence t_j, we find that it is at most $-(1 + 2\tau) > -\infty$, which concludes the proof. □

Remark 3.5.6. We note that as the spectrum is concave, for any function $f \in C_0(\Omega)$ such that $\alpha^*_{f,\min} = -1$ and $\alpha^*_{f,\max} = 1$ we have that the graph of S_f is above the triangle graph with vertices $(-1, 0), (0, 1), (1, 0)$. On the other hand, it must be below the constant 1 function in the interval $[-1, 1]$. It is natural to ask whether these extremes can be attained/approximated. We do not give the complete answer for these questions, but make a few observations.

First of all, Theorem 3.2.9 easily yields that S_f can be arbitrarily close to the triangle graph: notably for the function f constructed in the previous proof, S_f is contained by the triangle with vertices $(-1, 0), (0, 1 + 2\tau), (1, 0)$ due to concavity. Thus the theoretic minimum can be approximated.

On the other hand, if we would like to construct some f such that S_f is considerably large, we can consider a function similar to the one in Example 3.1.4. More explicitly, let $f \in \mathrm{PCC}^{2k+1}(\Omega)$ be such that it takes the value -1 on cylinders which contain more 0s than 1s in their first $2k + 1$ coordinates, and $f(\omega) = 1$ otherwise. As in the proof of Example 3.1.4, we can show by Hutchinson's theorem that $S_f(-1) = S_f(1)$ is at least $\frac{k}{2k+1}$. Thus the piecewise linear graph determined by the vertices (-1,1/2), (0,1), (1,1/2) can be arbitrarily close to a lower estimate of the spectrum, which means that S_f is considerably large, even though it is far from what we strove for.

We also provide another example, which displays that $S_f(\alpha^*_{f,\max})$ can be arbitrarily close to 1 even for nonconstant functions, if we drop the condition that $\alpha^*_{f,\max} = 1$. Notably, let $f \in \mathrm{PCC}^k(\Omega)$ such that it takes the value -1 if the first k coordinates equal 0, while it takes the value $\frac{1}{2^k-1}$ if these coordinates contain at least one 1. Then similarly to the previous argument we have that $S_f\left(\frac{1}{2^k-1}\right) \geq \frac{k-1}{k}$. It would be interesting to see how large $S_f(\alpha^*_{f,\max})$ can be if $f \in C_0(\Omega)$ such that $\alpha^*_{f,\min} = -1$ and $\alpha^*_{f,\max} = 1$.

Proof of 3.2.10. Choose k such that $f \in \mathrm{PCC}^k(\Omega)$. By symmetry, it clearly suffices to prove $\partial^- S_f(\alpha^*_{f,\max}) = -\infty$. Consider the directed graph $G = (V, E)$ defined in the proof of Lemma 3.3.5, and the set $\mathcal{C}$ of its cycles. By that reasoning it is clear that there exist cycles with distinct weight averages as otherwise for any infinite path Γ we would get the same weight average in limit, which means that the ergodic averages have the same limit for all configurations, hence S_f cannot be continuous. Moreover, as G is connected as a directed graph, the graph of cycles $G_{\mathcal{C}}$ is also connected, in which the vertices are the elements of $\mathcal{C}$, and two of them are connected if they have a common vertex. This, together with our previous observation implies that we can choose cycles C and C' such that they have a common vertex v, the cycle C has maximal weight average amongst the elements of $\mathcal{C}$, while C' does not. Now consider the set of infinite paths in G denoted by H_β which consists of the paths which start from v, and can be partitioned into finite pieces $\Gamma_1, \Gamma_2, \ldots$ such that each Γ_i equals either C or C', and the density $\mathbf{d}\left(\{i : \Gamma_i = C\}\right) = \beta$. Then it is obvious to see that the weight average along any $\Gamma \in H_\beta$ tends to

$$\beta \cdot \frac{1}{|C|} \sum_{e \in C} f(e) + (1-\beta) \cdot \frac{1}{|C'|} \sum_{e \in C'} f(e) = \beta \alpha^*_{f,\max} + (1-\beta)\alpha',$$

where $\alpha' < \alpha^*_{f,\max}$ by the choice of C'. Thus if we take the corresponding configuration

$\omega(\Gamma)$, and in the ergodic averages we shift the indexing again by one, we see that

$$\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega(\Gamma)) \to \beta\alpha^*_{f,\max} + (1-\beta)\alpha'.$$

That is, if Ω_β denotes the set of $\omega(\Gamma)$s for which $\Gamma \in H_\beta$, we have

$$\Omega_\beta \subseteq E_f(\beta\alpha^*_{f,\max} + (1-\beta)\alpha'). \tag{3.5.22}$$

However, the dimension of Ω_β is easy to estimate from below using the following mapping: for $\omega(\Gamma) \in \Omega_\beta$ define $h(\omega(\Gamma)) = h_1h_2...$ by

$$h_i := \begin{cases} 1 & \text{if } \Gamma_i = C \\ 0 & \text{if } \Gamma_i = C'. \end{cases}$$

Now h is a Hölder-mapping. Note that the starting point of Γ determines the first k coordinates of $\omega(\Gamma)$, and then going along C (resp. C') determines the next $|C|$ (resp. $|C'|$) coordinates. By reversing this argument, if $K = \max\{|C|, |C'|\}$, the first $k + mK$ coordinates of $\omega(\Gamma)$ uniquely determine the cycles $\Gamma_1, ..., \Gamma_m$ in the decomposition of Γ. In other words, the first m coordinates of $h(\omega(\Gamma))$ are uniquely determined by the first $k + mK$ coordinates of $\omega(\Gamma)$. From this, one easily obtains that h is a Hölder-$1/K$ mapping.

Moreover, by the definition of H_β and Ω_β, it is clear that $h(\Omega_\beta)$ equals the set of configurations in which the density of 1s equals β. Thus by Example 3.1.1, we can deduce that

$$\dim_H(h(\Omega_\beta)) = -\frac{\beta\log(\beta) + (1-\beta)\log(1-\beta)}{\log 2}.$$

Hence as h was Hölder-$1/K$:

$$\dim_H(\Omega_\beta) \geq -\frac{\beta\log(\beta) + (1-\beta)\log(1-\beta)}{K\log 2}.$$

Thus by (3.5.22):

$$S_f(\beta\alpha^*_{f,\max} + (1-\beta)\alpha') \geq -\frac{\beta\log(\beta) + (1-\beta)\log(1-\beta)}{K\log 2}.$$

Consequently, also using that by continuity of S_f we have $S_f(\alpha^*_{f,\max}) = 0$ we infer

$$\frac{S_f(\alpha^*_{f,\max}) - S_f(\beta\alpha^*_{f,\max} + (1-\beta)\alpha')}{\alpha^*_{f,\max} - (\beta\alpha^*_{f,\max} + (1-\beta)\alpha')} \leq \frac{\beta\log(\beta) + (1-\beta)\log(1-\beta)}{(1-\beta)(\alpha^*_{f,\max} - \alpha')K\log 2}$$

However, the right hand side can be estimated from above by omitting the negative first term, and after simplifying by $1-\beta$ we see that it tends to $-\infty$ as $\beta \to 1$. Hence the same holds for the left hand side, showing that $\partial^- S_f(\alpha^*_{f,\max}) = -\infty$. $\square$

Parallel research

The author of this dissertation have produced a number of further papers, most prominently as a member of a research group consisting of Zoltán Buczolich, Bruce Hanson, Gáspár Vértesy, and himself. Out of these papers three were used in this dissertation: [12], [11], and [13]. (In the list below, these are items [VIII], [VII], [X], respectively.)

Since G. Vértesy has also been a graduate student during our collaboration, we had to be careful to avoid any collisions between our dissertations. Thus G. Vértesy exclusively included other papers produced by our research group, in a topic which does not overlap with the content of this thesis.

On all of my papers, like it is customary in pure mathematics, author's names are listed in alphabetical order, first/last listed author has no specific role in the production of the paper. Contribution to the papers is considered to be equal by all coauthors. Although in the dissertations the two Ph.D. students are mutually "giving up" their share of the articles for the benefit of the other student.

List of all papers to which the author of this thesis contributed:

[I] Z. Buczolich, B. Hanson, B. Maga and G. Vértesy, *Random constructions for translates of non-negative functions*, Journal of Mathematical Analysis and Applications, **468** (2018), no. 1, 491–505.

[II] Z. Buczolich, B. Hanson, B. Maga and G. Vértesy, *Type 1 and 2 sets for series of translates of functions*, Acta Mathematica Hungarica, **158** (2019), 271–293.

[III] Z. Buczolich, B. Maga and G. Vértesy, *On series of translates of positive functions. III*, Analysis Mathematica, **44** (2018), no. 2, 185–205.

[IV] Z. Buczolich, B. Hanson, B. Maga and G. Vértesy, *Big and little Lipschitz one sets*, Eur. J. Math. **7** (2021), no. 2, 464—488.

[V] Z. Buczolich, B. Hanson, B. Maga and G. Vértesy, *Characterization of lip sets*, J. Math. Anal. Appl., **489** (2020), no. 2.

[VI] Z. Buczolich, B. Hanson, B. Maga and G. Vértesy, *Lipschitz one sets modulo sets of measure zero*, Math. Slovaca, **70** (2020), no. 3, 567—584.

[VII] Z. Buczolich, B. Maga and G. Vértesy, *Generic Hölder level sets and fractal conductivity*, Chaos, Solitons and Fractals, **164** (2022).

[VIII] Z. Buczolich, B. Maga and G. Vértesy, *Generic Hölder level sets on Fractals*, J. Math. Anal. Appl., **519** (2022), no. 2.

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Summary

This thesis synthesizes two research projects.

The first of these topics concerns with the Hausdorff dimension of level sets of a generic Hölder function defined on various fractals, while the second one deals with generic Birkhoff spectra, defined by the Hausdorff dimension of level sets of Birkhoff averages. The motivation and historical background of these topics are explained in Chapter 1.

Chapter 2 contains our contribution to the first of these topics. The necessary notation and preliminaries are introduced in Section 2.1, while Section 2.2 enumerates our main results. The main object of our interest is $D_*(\alpha, F)$ for $F \subseteq \mathbb{R}^p$, which is the essential supremum of the Hausdorff dimension of level sets for the generic 1-Hölder-α function, defined on F.

In Sections 2.3-2.4, we verify the existence of this generic value and prove other qualitative results concerning certain families of fractals, such as the monotonicity and various estimates of $D_*(\alpha, F)$. In Section 2.5, we investigate the phenomenon of phase transition. Section 2.6 concludes the chapter with quantitative results, giving lower and upper bounds for $D_*(\alpha, F)$ on the Sierpiński triangle.

Chapter 3 is dedicated to the second topic. The necessary notation and preliminaries are introduced in Section 3.1, while Section 3.2 enumerates our main results. The main object of our interest is the Birkhoff spectrum $S_f(\alpha)$ for continuous f on $\{0,1\}^{\mathbb{N}}$, which is a concave, continuous function on its support interval L_f, and 0 outside of it.

Our contribution deals with the continuity and differentiability properties of S_f at the endpoints of L_f. After introducing some vital tools in Section 3.3, in Section 3.4, we prove that while the generic Birkhoff spectrum is continuous, discontinuous spectra also occur densely.

In Section 3.5, we discuss the nontrivial one-sided derivatives of S_f at the endpoints of L_f. First, we prove that generically, these derivatives are infinite. On the other hand, we construct an exceptional continuous function f, for which one of these derivatives is finite.

Összefoglaló

Tézisemben két kutatásomat mutatom be.

Az előbbi ezen kutatások közül fraktálokon definiált generikus Hölder-függvények szinthalmazainak Hausdorff-dimenziójának vizsgálata, míg az utóbbi a generikus Birkhoff spektrum vizsgálata, mely spektrumot Birkhoff-átlagok szinthalmazainak Hausdorff-dimenziójából eredeztetjük. Ezen témák motivációját és történeti hátterét mutatja be a 1. fejezet.

A 2. fejezetben mutatom be hozzájárulásunkat az előbbi témához. A szükséges jelöléseket és előismereteket a 2.1. szekció tartalmazza, míg a 2.2. szekcióban a főbb eredmények kerülnek felsorolásra. Vizsgálódásunk elsődleges tárgya $F \subseteq \mathbb{R}^p$ mellett $D_*(\alpha, F)$, mely az F-en definiált generikus 1-Hölder-α függvényhez tartozó szinthalmazok Hausdorff-dimenziójának lényeges szuprémuma.

A 2.3-2.4. szekciókban igazoljuk, hogy ez a generikus érték valóban létezik, s további kvalitatív eredményeket igazolunk különböző fraktálcsaládokon, például $D_*(\alpha, F)$ monotonitását, illetve általános érvényű becsléseket. A 2.5. szekcióban a fázisátalakulás jelenségét vizsgáljuk, míg a fejezetet záró 2.6. szekcióban kvantitatívabb jellegű tételeket bizonyítunk, ezekkel alsó és felső becslést adva $D_*(\alpha, F)$-re a Sierpiński-háromszögön.

A 3. fejezetben a második témával foglalkozunk. A szükséges jelöléseket és előismereteket a 3.1. szekció tartalmazza, míg a 3.2. szekcióban a főbb eredmények kerülnek felsorolásra. Vizsgálódásunk elsődleges tárgya $\{0,1\}^{\mathbb{N}}$-en definiált folytonos f függvény $S_f(\alpha)$ Birkhoff-spektruma, ami egy konkáv, folytonos függvény az L_f-fel jelölt tartóintervallumán, s 0 azon kívül.

Hozzájárulásunk S_f folytonossági és differenciálhatósági tulajdonságaival kapcsolatos L_f végpontjaiban. Miután a 3.3. szekcióban bevezetünk több fontos eszközt, a 3.4. szekcióban igazoljuk, hogy míg a generikus Birkhoff spektrum folytonos, a nemfolytonos Birkhoff spektrumok is sűrűn fordulnak elő.

A 3.5. szekcióban S_f nemtriviális féloldali deriváltjaival foglalkozunk L_f végpontjaiban. Először belátjuk, hogy generikusan ezek a deriváltak végtelenek, majd példával igazoljuk, hogy ez a derivált lehet véges is.