CHARACTERIZATIONS AND PROPERTIES OF GRAPHS OF BAIRE FUNCTIONS

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ABSTRACT. Let X be a paracompact topological space and Y be a Banach space. In this paper, we will characterize the Baire-1 functions $f : X \to Y$ by their graph: namely, we will show that f is a Baire-1 function if and only if its graph $gr(f)$ is the intersection of a sequence $(G_n)_{n=1}^{\infty}$ of open sets in $X \times Y$ such that for all $x \in X$ and $n \in \mathbb{N}$ the vertical section of G_n is a convex set, whose diameter tends to 0 as $n \to \infty$. Afterwards, we will discuss a similar question concerning functions of higher Baire classes and formulate some generalized results in slightly different settings: for example we require the domain to be a metrized Suslin space, while the codomain is a separable Fréchet space. Finally, we will characterize the accumulation set of graphs of Baire-2 functions between certain spaces.

1. INTRODUCTION

In [1], S. J. Agronsky, J. G. Ceder and T. L. Pearson gave an equivalent definition of the real valued Baire class 1 functions defined on a metric space X by characterizing their graph, which we will denote throughout this paper by $gr(f)$. In their article, Theorem 2.2 stated the following:

Proposition 1.1. *Let X be a metric space. Let us call an open set $G \subseteq X \times \mathbb{R}$ an open strip if for each $x \in X$ the intersection of G_n and $\{x\} \times \mathbb{R}$ is an interval. Let $f : X \to \mathbb{R}$ be a function. It is Baire-1 if and only if there is a sequence $(G_n)_{n=1}^{\infty}$ of open strips such that $\cap_{n=1}^{\infty} G_n = gr(f)$.*

In the case $X = [0, 1]$, they gave a somewhat elementary proof. However, for the case when X was an arbitrary metric space, they used Michael's Selection Theorem (see [2]). This fact might lead us to the idea to regard the Baire-1 functions defined on a paracompact topological space X with values from a Banach space Y instead of $\mathbb{R}$ as this selection theorem holds in this more general situation. Furthermore, if Y is a Banach space, we can easily find a natural counterpart of the notion of open strips in $X \times Y$:

Definition 1.2. *We say that an open set $G \subseteq X \times Y$ is an open strip if the vertical section $G(x) = G \cap (\{x\} \times Y)$ is convex for all $x \in X$.*

Using this definition, the following holds:

Theorem 1.3. *Let $f : X \to Y$ be a function where X is a paracompact topological space and Y is a Banach space. Then f is Baire-1 if and only if there is a sequence $(G_n)_{n=1}^{\infty}$ of open strips such that $\bigcap_{n=1}^{\infty} G_n = gr(f)$ and $diam(G_n(x)) \to 0$ for each $x \in X$ as n tends to infinity.*

Concerning the graphs of Baire-α functions for $\alpha > 1$ countable ordinals, we did not manage to prove a theorem in the setting of Theorem 1.3. Analogous results to the one achieved in the direction in which we assume f is Baire-α can be obtained though rather easily in even more general settings, as it will be shown in Theorem 1.6. Namely, we can find some nice properties of the graph of a Baire-α function $f : X \to Y$ where X is a topological space and Y is a metric space, and then maybe we can achieve more specific ones if we require some conditions on Y: for example in Theorem 1.3 we restricted Y to be a Banach space and we managed to prove a property of $gr(f)$ which does not even make sense if Y is an arbitrary metric space. Following this line of thought, we can have a generalization of this direction of Theorem 1.3 in more general settings for any Baire class, but before we would formulate it, we would like to recall the following notation for higher Borel classes:

Definition 1.4. *A set A is of additive class 1, $(A \in \Sigma_1)$, if and only if it is open. For any countable ordinal greater than zero, A is of multiplicative class α, $(A \in \Pi_\alpha)$, if and only if its complement is in Σ_α. Finally, A is of additive class α, $(A \in \Sigma_\alpha)$, for $\alpha > 1$ if and only if there is a sequence of sets $A_1, A_2, \ldots$ such that each A_i is in Π_{α_i} for some $\alpha_i < \alpha$ and $\bigcup_{i=1}^{\infty} A_i = A$.*

It is useful to remark that the behaviour of the Borel hierarchy can be a bit chaotic in general topological spaces. To be more precise, we prefer if the higher Borel classes contain the lower ones, that is for $0 < \beta < \alpha < \omega_1$, every set in Π_β or Σ_β is also in Π_α and Σ_α. However, this property does not hold necessarily: for example if we regard the cofinite topology over any uncountable set, we can immediately see that none of the nontrivial open sets is in Σ_2. The following result is well-known and can be easily obtained by transfinite induction: if X has the property that any open set is in Π_2 (or equivalently, any closed set is in Π_2) then every set in Π_β or Σ_β is also in Π_α and Σ_α for any $0 < \beta < \alpha < \omega_1$. The spaces satisfying this requirement are called G_δ or perfect spaces and their defining property can be regarded as a separation axiom: the closed sets can be separated from their complements using only countably many open sets. It can be easily checked that all the metrizable spaces are perfect spaces, which is a fact we will use in this paper.

Let us recall that a topological vector space is a Fréchet space if it is locally convex and complete with a translation invariant metric. As the concept of convex sets exists in Fréchet spaces, we can similarly define open strips of $X \times Y$ if Y is a Fréchet space instead of being a Banach space. Furthermore, we can generalize this definition to higher Borel classes:

Definition 1.5. *Let X be a topological space and Y be a Fréchet space. We say that a Σ_α set $S \subseteq X \times Y$ is a Σ_α-strip if the vertical section $S(x) = S \cap (\{x\} \times Y)$ is convex for all $x \in X$.*

Theorem 1.6. *Let $f : X \to Y$ be a Baire-α function where X is a topological space, Y is a metric space and let α be a countable ordinal. Then there exists a sequence $(G_n)_{n=1}^\infty$ of Σ_α sets in $X \times Y$ such that $\cap_{n=1}^\infty G_n = gr(f)$ and $diam(G_n(x)) \to 0$ for each $x \in X$ as n tends to infinity. Furthermore, if Y is a Fréchet space, these Σ_α sets can be chosen to be Σ_α strips.*

Proving the converse of this theorem in this rather general setting appeared to be much more difficult. The idea we may follow is similar to the one we will use in the proof of Theorem 1.3: we construct Baire-α_n functions for $\alpha_n < \alpha$ through Σ_α sets satisfying the conditions we found during the proof of the other direction, and with the help of these conditions we attempt to show that f is the pointwise limit of these Baire-α_n functions. The essence of this concept is the construction of these functions which reduces our question to a selection problem concerning Baire functions of a given class (e.g. in Theorem 1.3 to the problem continuous selections). With the help of the Kuratowski–Ryll-Nardzewski Theorem about measurable selections (see [5] and Proposition 2.2 in the next section) we will prove the following for functions defined on Suslin spaces, which are the continuous images of Polish spaces:

Theorem 1.7. *Let $f : X \to Y$ be a function where X is a metrized Suslin space, Y is a separable Fréchet space. Then f is Baire-α for some successor countable ordinal α if and only if there is a nested sequence $(G_n)_{n=1}^\infty$ of Σ_α strips in $X \times Y$ such that*

- *$\cap_{n=1}^\infty G_n = gr(f)$,*
- *$diam(G_n(x)) \to 0$ for each $x \in X$ as n tends to infinity,*
- *the projection of $(X \times U) \cap G_n$ to X is in Σ_α for each open subset U of Y.*

Finally, we focus on generalizing some of the results of [3], in which we characterized the accumulation sets of graphs of Baire-2 functions $f : [0,1] \to \mathbb{R}$. In that paper, Theorem 4.1 and 4.2 stated the followings:

Proposition 1.8. *Suppose $T \subseteq [0,1] \times \mathbb{R}$ and let us denote the set of accumulation points of some $gr(f)$ by L_f. There exists a bounded Baire-2 function satisfying $L_f = T$ if and only if T is closed and $T \cap (\{x\} \times \mathbb{R}) \neq \emptyset$ for each $x \in [0,1]$.*

Furthermore, there exists a not necessarily bounded Baire-2 function satisfying $L_f = T$ if and only if T is closed and there is a countable set $D \subseteq [0,1]$ such that $T \cap (\{x\} \times \mathbb{R}) \neq \emptyset$ for each $x \in [0,1] \setminus D$.

The proofs appearing in that article are a bit complicated, however elementary. We will see that these problems also can be handled in a much more general setting using stronger tools as certain selection theorems. We recall that a space is σ-compact if it can be expressed as the countable union of compact sets.

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Theorem 1.9. *Let $T \subseteq X \times Y$, where X is a σ-compact metrizable Suslin space with no isolated points, and Y is a compact Fréchet space. There exists a Baire-2 function f satisfying $L_f = T$ if and only if T is closed and $T \cap (\{x\} \times Y) \neq \emptyset$ for each $x \in X$.*

Furthermore, if Y is σ-compact, but not compact, there exists a Baire-2 function f satisfying $L_f = T$ if and only if T is closed and there is a countable set $D \subseteq X$ such that $T \cap (\{x\} \times Y) \neq \emptyset$ for each $x \in X \setminus D$.

2. PRELIMINARIES

In order to prove our theorems, we need to recall a classical result about the relationship of Baire classes and Borel classes (see [4]). As it is short and useful to prove, we will not omit the proof and formulate it as a proposition:

Proposition 2.1. *Let $f : X \to Y$ be a Baire-α function where X is a topological space, Y is a metric space, and α is a countable ordinal. Then for any open set $G \subseteq Y$ the set $f^{-1}(G) \subseteq X$ is a $\Sigma_{\alpha+1}$ set, or in other words, f is a Borel-$(\alpha+1)$ mapping.*

Proof. We proceed by transfinite induction. For $\alpha = 0$ the proposition states that for continuous functions the inverse image of an open set is open which is true by definition. What remains to discuss is the inductive step. Let us assume $\alpha \geq 1$ and we already know the statement for smaller ordinals, and let $(f_k)_{k=1}^{\infty}$ be a sequence of functions from lower Baire classes whose pointwise limit is f, namely let f_k be Baire-α_k where $\alpha_k < \alpha$. If α is a successor ordinal, we might assume $\alpha_k = \alpha - 1$. Let us denote the neighborhood of radius $\varepsilon > 0$ of a closed set F by $B(F, \varepsilon)$, which is clearly an open set. Then we may construct the following decomposition of G into closed sets $(F_n)_{n=1}^{\infty}$:

$$G = \bigcup_{n=1}^{\infty} Y \setminus B\left(Y \setminus G, \frac{1}{n}\right) = \bigcup_{n=1}^{\infty} F_n.$$

One can easily check that our decomposition implies that $f(x) = \lim_{k \to \infty} f_k(x) \in G$ holds if and only if there is an n such that $f_k(x) \in F_n$ for all large enough k. Indeed, as F_n is closed, if there is such an n, then the sequence $(f_k(x))$ cannot converge out of $F_n \subseteq G$ hence $f(x) \in G$. Conversely, if $f(x) \in G$, it has a neighborhood of radius ε for suitable positive ε in G. By convergence, for large enough k the point $f_k(x)$ is in the neighborhood of $f(x)$ of radius $\frac{\varepsilon}{2}$, thus $f_k(x) \in B\left(Y \setminus G, \frac{\varepsilon}{2}\right)$ for large enough k. Choosing n such that $\frac{1}{n} < \frac{\varepsilon}{2}$ gives us a suitable n in our statement, thus it proves the other direction of our equivalence.

This equivalence yields the following equation:

$$f^{-1}(G) = \{x : f(x) \in G\} = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \{x : f_k(x) \in F_n\}.$$

Now, for any set $\{x : f_k(x) \in F_n\}$ the inductive hypothesis can be used: $f_k(x)$ is Baire-α_k thus the inverse image of any open set is Σ_{α_k+1}, hence the inverse image of the closed set F_n is in Π_{α_k+1}. Indeed, the inverse image of the complement is the complement

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of the inverse image, and the complement of F_n is open, while the complement of its inverse image is in Σ_{α_k+1} whose complement is in Π_{α_k+1}. Now if α is a successor ordinal, these sets in Π_{α_k+1} are in Π_α as $\alpha_k + 1 = \alpha$. Otherwise, if α is a limit ordinal the sets in Π_{α_k+1} are in Π_α by definition: the same unions can be regarded. Hence if we take the intersection of sets of these type, for all $k \geq m$, we will still have a Π_α set for any ordinal. Finally if we take the countable union of such sets (that is, for all n and m) we will obtain a $\Sigma_{\alpha+1}$ set as the inverse image of the open set G. $\qquad\square$

The proofs of Theorem 1.7 and 1.9 rests on the following corollary of the Kuratowski–Ryll-Nardzewski Theorem about measurable selections, which we already mentioned in the introduction:

Proposition 2.2. *Let X be a metric space and let Y be a separable complete metric space. Assume $\alpha \geq 1$ is a countable ordinal and let $\Psi : X \to 2^Y$ be a multifunction with nonempty closed values such that $\Psi^{-1}(G)$ is in Σ_α for each open subset G of Y. Then Ψ admits a Borel class α selection, that is a mapping $f : X \to Y$ such that the inverse image of any open set of Y is in Σ_α.*

As we already remarked, we are interested in Baire selections. However, Baire classes of functions and Borel classes of mappings have a strong relationship. For example, as we have seen in Proposition 2.1, any Baire-α function defined on a topological space with values from a metric space is a Borel-$(\alpha + 1)$ mapping. Nevertheless we have to be cautious since the converse does not hold in general: for instance if X is a connected topological space with at least two points, and Y is the two point discrete space $\{0, 1\}$, then the characteristic function of a single point of X is Borel-2, but not Baire-1, as all the continuous functions from X to Y are constants. As our aim is to use Proposition 2.2 in as general setting as it is possible, it would be beneficial to know some results concerning conditions yielding the equivalence of Baire-α functions and Borel-$(\alpha + 1)$ mappings. We can recall a special form of Theorem 8 of [6] (in that paper, every space is assumed to be perfect):

Proposition 2.3. *Let X be a perfect Suslin space and Y be a metric space. If X is metrizable and has topological dimension zero, or Y is a locally convex topological linear space then the family of Baire-α functions coincides with the family of Borel-$(\alpha + 1)$ mappings.*

Remark 2.4. In [4], [5] and [6], and in several further articles and books other types of notation are used for Borel classes, causing a subtle ambiguity with our recent paper. In particular, in many papers the elements of Σ_0 are the open sets instead of the elements of Σ_1, and the higher Borel classes are defined from this starting point the same way we did in Definition 1.4. It is worth mentioning that this translation of the indices only leads to a difference in the case of finite ordinals as in the definition of Σ_ω we consider the same unions.

We wish to use Propositions 2.2 and 2.3 simultaneously, thus we have to restrict our observations to spaces satisfying the conditions of both. This explains why we stated Theorem 1.7 in the setting we did.

For the proof of Theorem 1.9, we first generalize Lemma 3.1 of [3]. We can formulate an almost identical proposition and the proof is also verbatim:

Lemma 2.5. *Let X and Y be σ-compact metric spaces such that X has no isolated points. For a given closed set $T \subseteq X \times Y$ there exists a countable set $A \subseteq X$ such that there is a function $f : A \to Y$ satisfying $L_f = T$.*

Proof. The product space $X \times Y$ is also σ-compact, hence there is an increasing sequence of compact sets $(C_n)_{n=1}^\infty$ with limit $X \times Y$. Then $T_n = T \cap C_n$ is also a compact set. We will construct A and f by induction. Let us consider an open ball of radius one around each point of T_1. These balls give an open cover of the compact set T_1 hence it is possible to choose a finite cover. Let us take a point in each ball of the finite cover such that the x coordinates of these points are pairwise different. As none of the points of X is isolated, it is clearly possible. Denote the set of these points by F_1, and the set of their x coordinates by A_1. In the following step, let us take open balls of radii $\frac{1}{2}$ around each point of T_2, choose a finite cover, and take points in each of these balls with pairwise different x coordinates, which are also distinct from the points in A_1. Let us define A_2 and F_2 analogously, and continue this procedure: in the nth step regard the $\frac{1}{n}$-neighborhoods of the points of T_n, and define the finite sets F_n and A_n using these open balls. Now if we let $A = \bigcup_{n=1}^\infty A_n$ and $F = \cup_{n=1}^\infty F_n$, these are countable sets, and we may define f to be the function that assigns to every $x \in A$ the y coordinate of the chosen point in F above x. The equality $L_f = T$ can easily be checked, as in [3]. $\qquad\square$

3. Baire-1 functions

Before we would start the proof of Theorem 1.3, a short remark should be mentioned. Theorem 1.3 gives almost the same characterization for the graphs of Baire-1 functions from X to Y as the one given in [1] for the graphs of Baire-1 functions from X to $\mathbb{R}$: in our statement, we can find the reasonable counterpart of the open strip condition of the real-valued case. However, besides that we drew up an additional limit condition concerning the diameter of the vertical sections, that has importance during the proof of the direction in which we show that if the graph has the given properties, then it is the graph of a Baire-1 function, and we work with the closure of the convex sections. What we would like to emphasize that this limit condition is vital and can be found implicitly in the more specific form of the theorem, too. We formulate the relevant fact as a proposition, since the author of this paper firmly believes this result has been published already but has yet to see a source:

Proposition 3.1. *Let E be a finite dimensional Banach space and $(C_n)_{n=1}^\infty$ is a nested sequence of closed convex sets such that $\bigcap_{n=1}^\infty C_n$ equals a point p. Then $diam(C_n) \to 0$.*

Proof. Proceeding towards a contradiction, let us assume $diam(C_n) > d > 0$ for all $n \in \mathbb{N}$. Then C_n must contain a point p_n such that $\|p - p_n\| > \frac{d}{2}$ as it easily follows from the triangle inequality. As C_n is convex and it contains p and p_n, it also contains the $[p, p_n]$ segment, and on this segment a point x_n satisfying $\|p - x_n\| = \frac{d}{2}$. Now the points x_n all lie on the boundary of the ball with centre p and radius $\frac{d}{2}$. By a consequence of Riesz's lemma, in our finite dimensional space this set is compact, hence the sequence $(x_n)_{n=1}^{\infty}$ has an accumulation point on this boundary. Let us denote it by x. As C_n is closed for each $n \in \mathbb{N}$ and their sequence is nested, it implies $x \in C_n$. As a consequence, $x \in \bigcap_{n=1}^{\infty} C_n = \{p\}$, which is clearly a contradiction as the distance of x and p is $\frac{d}{2}$. $\quad\square$

This proposition implies that if we work with a finite dimensional Banach space, it is unnecessary to have the additional limit condition concerning the diameters of the vertical sections. However, one can easily construct counterexamples to Proposition 3.1 if we permit infinite dimensional spaces, for instance we can take the subspace spanned by a countable set of linearly independent vectors as C_1, and then reduce this subspace by removing the generators one by one.

Proof of Theorem 1.3. The proof is similar to the one given in [1] for the more specific case, with some suitable modifications. First, let us assume that f is Baire-1, hence there is a sequence of continuous functions $(f_n)_{n=1}^{\infty}$ with pointwise limit f. Let us notice that the set $\{x : \|f_n(x) - f(x)\| < \frac{1}{k}\}$ is in Σ_2. Indeed, if we let $g_n(x) = f_n(x) - f(x)$, it is also a Baire-1 function, and the set we are interested in is $g_n^{-1}\{y : \|y\| < \frac{1}{k}\}$, which is the inverse image of an open ball. Applying Proposition 2.1 yields that $\{x : \|f_n(x) - f(x)\| < \frac{1}{k}\}$ is in Σ_2 as we stated. As a consequence, it can be written as the countable union of closed sets $A(n,k,i) \subseteq X$:

$$\left\{ x : \|f_n(x) - f(x)\| < \frac{1}{k} \right\} = \bigcup_{i=1}^{\infty} A(n,k,i).$$

We will define the subsets $H(n,k,i)$ of $X \times Y$ as follows:

$$H(n,k,i) = \left\{ (x,y) : x \in A(n,k,i), \|y - f_n(x)\| \geq \frac{1}{k} \right\}.$$

We show that $H(n,k,i)$ is closed. In order to prove it, let us write it as an intersection of two sets which are easier to handle:

$$H(n,k,i) = [A(n,k,i) \times Y] \cap \left\{ (x,y) : \|y - f_n(x)\| \geq \frac{1}{k} \right\}.$$

The first one of these sets on the right hand side is clearly closed in $X \times Y$ as $A(n,k,i)$ was closed in X, hence it suffices to prove that the second set on the right hand side is also closed. Let us define the following function $h_n : X \times Y \to \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the nonnegative halfline:

$$h_n(x,y) = \|y - f_n(x)\|.$$

Our claim is that the continuity of f_n implies the continuity of h_n. To prove this, we need to show that the inverse image of an open set $G \subseteq \mathbb{R}_+$ under h_n is open in $X \times Y$. Thus let us assume $h_n(x_0, y_0) \in G$ for some $(x_0, y_0) \in X \times Y$, which yields for some $\varepsilon > 0$ its neighborhood of radius ε is the subset of G, that is $B(h_n(x_0, y_0), \varepsilon) \subseteq G$. We need that $h_n(x, y)$ is also in G if (x, y) is an element of a suitable neighborhood U of (x_0, y_0). We state this holds if we regard the following neighborhood:

$$U = f_n^{-1} \left(B \left(f_n(x_0), \frac{\varepsilon}{2} \right) \right) \times B \left(y_0, \frac{\varepsilon}{2} \right)$$

By the continuity of f_n it is indeed a neighborhood of (x_0, y_0) as $f_n^{-1} \left(B \left(f_n(x_0), \frac{\varepsilon}{2} \right) \right)$ is an open subset of X. Furthermore, if $(x, y) \in U$, by the triangle inequality we have

$$\|y - f_n(x)\| \leq \|y - y_0\| + \|y_0 - f_n(x_0)\| + \|f_n(x_0) - f_n(x)\| < \varepsilon + \|y_0 - f_n(x_0)\|,$$

and

$$\|y - f_n(x)\| \geq -\|y - y_0\| + \|y_0 - f_n(x_0)\| - \|f_n(x_0) - f_n(x)\| > -\varepsilon + \|y_0 - f_n(x_0)\|,$$

which implies $h_n(x, y) \in B(h_n(x_0, y_0), \varepsilon) \subseteq G$.

Thus h_n is continuous indeed, yielding $\left\{ (x, y) : \|y - f_n(x)\| \geq \frac{1}{k} \right\} = h_n^{-1} \left(\left[\frac{1}{k}, \infty \right) \right)$ is a closed set of $X \times Y$. By our previous remarks it implies that $H(n, k, i)$ is also closed.

The set of such sets $H(n, k, i)$ is countable thus we can take an enumeration H_1, H_2, ..., of them. Let us denote by G_j^* the complement of H_j in $X \times Y$, that is an open set. Furthermore, one can easily check that G_j^* is an open strip, that is the $G_j^*(x)$ vertical section is convex for each $j \in \mathbb{N}$ and $x \in X$. Indeed, by the construction of G_j^*, this vertical section is either the complete space Y or the ball of radius $\frac{1}{k}$ centered at $f_n(x)$ for some $k \in \mathbb{N}$ and $x \in X$. However, balls are convex in Banach spaces, hence G_j^* is an open strip. It implies $G_j = \bigcap_{l=1}^{j} G_l^*$ is also an open strip. Furthermore, the sequence $(G_j)_{j=1}^{\infty}$ is nested and $diam(G_j(x))$ tends to 0 for each $x \in X$. Indeed, when we constructed G_j, we took the intersection of the complements of some sets $H(n, k, i)$. A vertical section of this complement is either the entire Y or a ball with diameter $\frac{2}{k}$. But as all $x \in X$ appears in $A(n, k, i)$ for any k, for some i and large enough n, this implies that $diam(G_j(x)) \leq \frac{2}{k}$ for large enough j. As a consequence, $diam(G_j(x)) \to 0$. Hence if we could verify that the intersection of the open strips $(G_j^*)_{j=1}^{\infty}$ equals $gr(f)$, that would conclude the proof. But the proof of this fact is quite straightforward, we can check two inclusions. First, $(x, f(x)) \in G_j^*$ for any $j \in \mathbb{N}$ and $x \in X$, implying $gr(f) \subseteq \bigcap_{j=1}^{\infty} G_j$. In order to show this, let us recall that the complement of G_j^* is $H_j = H(n, k, i)$ for some $n, k, i \in \mathbb{N}$. We need $(x, f(x)) \notin H(n, k, i)$. Proceeding towards a contradiction, let us assume $(x, f(x)) \in H(n, k, i)$, yielding $x \in A(n, k, i)$. Then by the definition of $A(n, k, i)$, the inequality $\|f_n(x) - f(x)\| < \frac{1}{k}$ holds. However, $\|f(x) - f_n(x)\| \geq \frac{1}{k}$ by the definition of $H(n, k, i)$, a contradiction. Thus $gr(f) \subseteq \bigcap_{j=1}^{\infty} G_j$. For the other inclusion, it suffices to prove that for any $x \in X$ and $y \in Y$ distinct from $f(x)$, we have $(x, y) \in H(n, k, i)$ for suitable $n, k, i \in \mathbb{N}$. In order to verify this, choose k such that $\frac{1}{k} < \frac{\|y - f(x)\|}{2}$ and n

such that $\|f_n(x) - f(x)\| < \frac{1}{k}$. As f is the pointwise limit of $(f_n)_{n=1}^\infty$, it is possible. Then by definition there exists i such that $x \in A(n,k,i)$. Furthermore, $\|y - f_n(x)\| \geq \|y - f(x)\| - \|f_n(x) - f(x)\| > \frac{1}{k}$ by the triangle-inequality, implying $(x,y) \in H(n,k,i)$, which concludes the proof. Thus $\bigcap_{j=1}^\infty (G_j) = gr(f)$, we finished the proof of this direction.

For the other direction, let us assume $gr(f) = \bigcap_{j=1}^\infty G_j$ where for each j the set G_j is an open strip. We can also assume that their sequence is nested as the finite intersection of open sets is open and any intersection of convex sets is convex. Thus $G_{j+1} \subseteq G_j$ for any j. Let us define F_j as it follows:

$$F_j = \bigcup_{x \in X} \overline{G_j(x)},$$

where the overline means the closure. Hence F_j stands for the closure by coordinates. Regard it as a multivalued function defined on X with range 2^Y, whose values are naturally the vertical sections of the set. Then this multivalued function has nonempty closed, convex values. Furthermore, we can easily show that F_j is lower hemicontinuous: let us assume $V \cap F_j(x)$ is nonempty for some open set V of Y and $x \in X$. Since $F_j(x)$ is the closure of the open set $G_j(x)$, we have $V \cap G_j(x)$ is nonempty. Let $y \in Y$ be one of its elements. As G_j is open, it contains a neighborhood of (x,y). This neighborhood intersects $X \times \{y\}$ in a set whose projection to X is open and suitable for us in the definition of lower hemicontinuity as one can easily check. Thus $F_j : X \to 2^Y$ is a lower hemicontinuous function with nonempty closed, convex values. By the Michael selection theorem there exists a continuous selection $f_j : X \to Y$ in F_j. Furthermore, as the intersection of the sets $F_j(x)$ is only $f(x)$ and their diameter tends to 0, we obtain $f_j(x) \to f(x)$. Hence f is the pointwise limit of continuous functions, meaning f is Baire-1. $\qquad \square$

4. HIGHER BAIRE CLASSES

Proof of Theorem 1.6. The proof has a similar structure to the proof of Theorem 1.3, we just have to be more careful with the sets in higher Borel classes and make some slight, but necessary changes. As f is Baire-α, there is a sequence of functions $(f_n)_{n=1}^\infty$ with pointwise limit f, where f_n is Baire-α_n for some $\alpha_n < \alpha$, and if α is a successor ordinal, we can assume $\alpha_n = \alpha - 1$. Proposition 2.1 easily yields that the set $\{x : d_Y(f_n(x), f(x)) < \frac{1}{k}\}$ is in $\Sigma_{\alpha+1}$. To verify this, we show that if the functions $g_1, g_2 : X \to Y$ are Baire-α, then the function $\rho_{g_1,g_2} : X \to \mathbb{R}_+$ defined by $\rho_{g_1,g_2}(x) = d_Y(g_1(x), g_2(x))$ is also Baire-α. We proceed by transfinite induction: if $\alpha = 0$, that is our functions are continuous, then our claim can be proven as the similar statement in the proof of Theorem 1.3. Furthermore, if we have $\alpha > 0$, then g_1 is the pointwise limit of the functions $(g_{1,n})_{n=1}^\infty$ and g_2 is the pointwise limit of the functions $(g_{2,n})_{n=1}^\infty$, such that these functions are in lower Baire classes. Thus by the continuity of the metric d_Y,

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we have

$$\rho_{g_1,g_2}(x) = \lim_{n\to\infty} d_Y\left(g_{1,n}(x), g_{2,n}(x)\right) = \lim_{n\to\infty} \rho_{g_1,g_2,n}(x).$$

However, the induction hypothesis easily yields that each of the functions $\rho_{g_1,g_2,n}$ are in lower Baire classes than Baire-α. Thus ρ_{g_1,g_2} is a Baire-α function, as we stated. As a consequence,

$$\left\{x : d_Y\left(f_n(x), f(x)\right) < \frac{1}{k}\right\} = \rho_{g_1,g_2}^{-1}\left(\left[0, \frac{1}{k}\right)\right)$$

is in $\Sigma_{\alpha+1}$ by Proposition 2.1, as we consider the inverse image of an open set in $\mathbb{R}_+$ under a Baire-α function. Thus it can be written as the countable union of Π_α sets $A(n,k,i) \subseteq Y$:

$$\left\{x : d_Y\left(f_n(x), f(x)\right) < \frac{1}{k}\right\} = \bigcup_{i=1}^{\infty} A(n,k,i).$$

We define the subsets $H(n,k,i)$ of $X \times Y$ as follows:

$$H(n,k,i) = \left\{(x,y) : x \in A(n,k,i), d_Y\left(y, f_n(x)\right) \geq \frac{1}{k}\right\}.$$

We state $H(n,k,i)$ is in Π_α. The proof of this claim starts with the same reformulation, that is we write $H(n,k,i)$ as the intersection of two simpler sets:

$$H(n,k,i) = [A(n,k,i) \times Y] \cap \left\{(x,y) : d_Y\left(y, f_n(x)\right) \geq \frac{1}{k}\right\}.$$

The first one of these sets on the right hand side is clearly in Π_α in $X \times Y$ as $A(n,k,i)$ was in Π_α in X, hence it suffices to prove that the second set on the right hand side is also in Π_α. Let us define the following function $h_n : X \times Y \to \mathbb{R}_+$:

$$h_n(x,y) = d_Y\left(y, f_n(x)\right).$$

One can easily prove by transfinite induction on α_n that if f_n is Baire-α_n then h_n is also Baire-α_n: the base case $\alpha_n = 0$, where f_n is continuous, can be verified exactly as we did it in the proof of Theorem 1.3, we only have to replace the norms of the differences in the inequalities with the respective distances. Now if f_n is the pointwise limit of a sequence of functions $(\phi_{n,m})_{m=1}^{\infty}$ from lower Baire classes, then h_n is the pointwise limit of the sequence of functions $d_Y\left(y, \phi_{n,m}(x)\right)_{m=1}^{\infty}$, and for these functions the inductive hypothesis can be used. Hence h_n is Baire-α_n, yielding $\left\{(x,y) : d_Y\left(y, f_n(x)\right) \geq \frac{1}{k}\right\}$ is in Π_{α_n+1}, and as a consequence, it is also in Π_α, as we can separate the cases of successor and limit ordinals as in the proof of Proposition 2.1. Thus $H(n,k,i)$ is in Π_α.

At this point, we can proceed exactly as we did in the proof of Theorem 1.3. We can take an enumeration $H_1, H_2, \ldots$ of the sets $H(n,k,i)$ and define G_j as $(X \times Y) \setminus \bigcup_{l=1}^{j} H_l$. Then these sets are in Σ_α and their intersection is $gr(f)$. Furthermore, if Y is a Fréchet space, these sets are also Σ_α strips as balls are convex sets in Fréchet spaces. $\qquad\square$

Proof of Theorem 1.7. For the direction in which we assume that f is Baire-α, we can refer to the proof of Theorem 1.6, the only detail we have to check that is the third condition is also satisfied. In general, let us denote the projection of a set $C \subseteq X \times Y$ to X by $\pi(C)$, and let us denote the projection of $(X \times U) \cap G_n$ to X for the sake of simplicity by $\pi_n^*(U)$. Let us define the sets $A(n,k,i)$ and $H(n,k,i)$, and then the sequences $(H_j)_{j=1}^\infty$ and $(G_j)_{j=1}^\infty$ as we did in that proof. Namely, if $(f_n)_{n=1}^\infty$ is the sequence of functions from lower Baire classes with pointwise limit f, then

$$\left\{ x : d_Y(f_n(x), f(x)) < \frac{1}{k} \right\} = \bigcup_{i=1}^\infty A(n,k,i), \text{where } A(n,k,i) \in \Pi_\alpha,$$

$$H(n,k,i) = [A(n,k,i) \times Y] \cap \left\{ (x,y) : d_Y(y, f_n(x)) \geq \frac{1}{k} \right\},$$

$(H_j)_{j=1}^\infty$ is the enumeration of these sets $H(n,k,i)$, and

$$G_j = (X \times Y) \setminus \bigcup_{l=1}^j H_l = \bigcap_{l=1}^j (X \times Y) \setminus H_l.$$

Our goal is to prove that $\pi_j^*(U)$ is in Σ_α for each open subset U of Y. Assume that G_j can be decomposed as the following:

$$G_j = (X \times Y) \setminus \bigcup_{l=1}^j H(n_l, k_l, i_l) = \bigcap_{l=1}^j (X \times Y) \setminus H(n_l, k_l, i_l).$$

Now we can divide each $(X \times Y) \setminus H(n_l, k_l, i_l)$ into two parts with disjoint projections to X:

$$(X \times Y) \setminus H(n_l, k_l, i_l) = [(X \setminus A(n_l, k_l, i_l)) \times Y] \cup [(A(n_l, k_l, i_l) \times Y) \setminus H(n_l, k_l, i_l)] = V_{l,1} \cup V_{l,2},$$

yielding

$$G_j = \bigcap_{l=1}^j (X \times Y) \setminus H(n_l, k_l, i_l) = \bigcap_{l=1}^j (V_{l,1} \cup V_{l,2}).$$

By distributivity, we can replace this intersection of unions by a union of intersections:

$$\bigcap_{l=1}^j (V_{l,1} \cup V_{l,2}) = \bigcup_{(\theta_1,\dots\theta_j)\in\{1,2\}^j} \bigcap_{l=1}^j V_{l,\theta_l}.$$

What is intriguing about this expression, that is the projections of the sets $\bigcap_{l=1}^j V_{l,\theta_l}$ to X are clearly disjoint as two such intersection differs in at least one θ-coordinate, and the projections $\pi(V_{l,1})$ and $\pi(V_{l,2})$ are disjoint. As a consequence, the projection of the union

$$\bigcup_{(\theta_1,\dots\theta_j)\in\{1,2\}^j} \bigcap_{l=1}^j V_{l,\theta_l}$$

to X equals the union of the projections, hence

$$\pi_j^*(U) = \pi((X \times U) \cap G_j) = \bigcup_{(\theta_1,\ldots\theta_j) \in \{1,2\}^j} \pi\left((X \times U) \cap \bigcap_{l=1}^{j} V_{l,\theta_l}\right). \tag{1}$$

We would like to show that this set is in Σ_α. Let us consider one of these sets

$$\pi\left((X \times U) \cap \bigcap_{l=1}^{j} V_{l,\theta_l}\right)$$

and take a closer look at $\bigcap_{l=1}^{j} V_{l,\theta_l}$. Amongst these sets, certain ones are of the type $V_{l,1}$, others are of the type $V_{l,2}$. Let us denote the set of indices belonging to the first type by J_1, and the set of indices belonging to the second type by J_2, yielding

$$\pi\left((X \times U) \cap \bigcap_{l=1}^{j} V_{l,\theta_l}\right) = \pi\left((X \times U) \cap \bigcap_{l \in J_1} V_{l,1} \cap \bigcap_{l \in J_2} V_{l,2}\right).$$

In this expression, $V_{l,1} = [(X \setminus A(n_l,k_l,i_l)) \times Y]$ for $l \in J_1$, meaning $V_{l,1}$ contains the whole space Y above $X \setminus A(n_l,k_l,i_l)$. As a consequence, one can easily verify that

$$\pi\left((X \times U) \cap \bigcap_{l \in J_1} V_{l,1} \cap \bigcap_{l \in J_2} V_{l,2}\right) = \bigcap_{l \in J_1}(X \setminus A(n_l,k_l,i_l)) \cap \pi\left((X \times U) \cap \bigcap_{l \in J_2} V_{l,2}\right). \tag{2}$$

Let us recall the definiton of $V_{l,2}$:

$$\pi\left((X \times U) \cap \bigcap_{l \in J_2} V_{l,2}\right) = \left\{x : x \in \bigcap_{l \in J_2} A(n_l,k_l,i_l), U \cap \bigcap_{l \in J_2} B_Y\left(f_{n_l}(x), \frac{1}{k_l}\right) \neq \emptyset\right\}.$$

Using these identities, we can reformulate (1), yielding $\pi_j^*(U)$ equals the following:

$$\bigcup_{J_1,J_2}\left(\bigcap_{l \in J_1}(X \setminus A(n_l,k_l,i_l)) \cap \left\{x : x \in \bigcap_{l \in J_2} A(n_l,k_l,i_l), U \cap \bigcap_{l \in J_2} B_Y\left(f_{n_l}(x), \frac{1}{k_l}\right) \neq \emptyset\right\}\right). \tag{3}$$

Now we will show that the condition $x \in \bigcap_{l \in J_2} A(n_l,k_l,i_l)$ might be omitted from this expression for each J_1,J_2 without changing the union. This omission extends each of the sets

$$\bigcap_{l \in J_1}(X \setminus A(n_l,k_l,i_l)) \cap \left\{x : x \in \bigcap_{l \in J_2} A(n_l,k_l,i_l), U \cap \bigcap_{l \in J_2} B_Y\left(f_{n_l}(x), \frac{1}{k_l}\right) \neq \emptyset\right\}$$

to

$$\bigcap_{l \in J_1}(X \setminus A(n_l,k_l,i_l)) \cap \left\{x : U \cap \bigcap_{l \in J_2} B_Y\left(f_{n_l}(x), \frac{1}{k_l}\right) \neq \emptyset\right\}, \tag{4}$$

however, as we will show the increment is contained by other sets of the union in (3), yielding this union remains the same. Indeed, as J runs over the subsets of J_2, the sets

$$\bigcap_{l \in J} (X \setminus A(n_l, k_l, i_l)) \cap \bigcap_{l \in J_2 \setminus J} A(n_l, k_l, i_l)$$

give a natural partition of X. As a consequence, the set in (4) can be expressed as it follows, by taking the intersection with each of the elements of this partition and then forming their union:

$$\bigcup_{J \subseteq J_2} \left(\bigcap_{l \in J \cup J_1} (X \setminus A(n_l, k_l, i_l)) \cap \bigcap_{l \in J_2 \setminus J} A(n_l, k_l, i_l) \cap \left\{ x : U \cap \bigcap_{l \in J_2} B_Y \left(f_{n_l}(x), \frac{1}{k_l} \right) \neq \emptyset \right\} \right).$$
(5)

Taking the intersection of the sets $B_Y \left(f_{n_l}(x), \frac{1}{k_l} \right)$ only for $J_2 \setminus J$ clearly extends this set, and $\bigcap_{l \in J_2 \setminus J} A(n_l, k_l, i_l)$ can be moved inside $\left\{ x : U \cap \bigcap_{l \in J_2} B_Y \left(f_{n_l}(x), \frac{1}{k_l} \right) \neq \emptyset \right\}$, yielding the set in (4) is contained by

$$\bigcup_{J \subseteq J_2} \left(\bigcap_{l \in J_1 \cup J} (X \setminus A(n_l, k_l, i_l)) \cap \left\{ x : x \in \bigcap_{l \in J_2 \setminus J} A(n_l, k_l, i_l), U \cap \bigcap_{l \in J_2 \setminus J} B_Y \left(f_{n_l}(x), \frac{1}{k_l} \right) \neq \emptyset \right\} \right).$$

Now we may notice that each of the unioned sets in this expression appears in the union in (3), which verifies our statement: we can make the omissions for any J_1 and J_2 without changing the union there. In other words, $\pi_j^*(U)$ is also the union of these modificated sets, that is

$$\pi_j^*(U) = \bigcup_{J_1, J_2} \left(\bigcap_{l \in J_1} (X \setminus A(n_l, k_l, i_l)) \cap \left\{ x : U \cap \bigcap_{l \in J_2} B_Y \left(f_{n_l}(x), \frac{1}{k_l} \right) \neq \emptyset \right\} \right).$$

As it is a finite union, it suffices to prove about each of the unioned sets that they are in Σ_α, that is

$$\bigcap_{l \in J_1} (X \setminus A(n_l, k_l, i_l)) \cap \left\{ x : U \cap \bigcap_{l \in J_2} B_Y \left(f_{n_l}(x), \frac{1}{k_l} \right) \neq \emptyset \right\} \in \Sigma_\alpha.$$

The sets $X \setminus A(n_l, k_l, i_l)$ are also in Σ_α, therefore it would be sufficient to prove the same about $\left\{ x : U \cap \bigcap_{l \in J_2} B_Y \left(f_{n_l}(x), \frac{1}{k_l} \right) \neq \emptyset \right\}$. The intersection which we regard in this set is the intersection of a finite collection of open sets, hence it is also open. Furthermore, U is separable as a subspace of the separable space Y. Thus it contains a countable dense set $\{u_1, u_2, ...\}$. As a consequence, $U \cap \bigcap_{l \in J_2} B_Y \left(f_{n_l}(x), \frac{1}{k_l} \right) \neq \emptyset$ holds if and

only if there exists some u_t for $t \in \mathbb{N}$ such that $u_t \in \bigcap_{l \in J_2} B_Y \left(f_{n_l}(x), \frac{1}{k_l} \right)$, thus

$$\left\{ x : U \cap \bigcap_{l \in J_2} B_Y \left(f_{n_l}(x), \frac{1}{k_l} \right) \neq \emptyset \right\} = \bigcup_{t=1}^{\infty} \left\{ x : u_t \in \bigcap_{l \in J_2} B_Y \left(f_{n_l}(x), \frac{1}{k_l} \right) \right\}.$$

For some $x \in X$, the relation $u_t \in B_Y \left(f_{n_l}(x), \frac{1}{k_l} \right)$ holds if and only if $f_{n_l}(x) \in B_Y \left(u_t, \frac{1}{k_l} \right)$ by symmetry. Hence

$$\bigcup_{t=1}^{\infty} \left\{ x : u_t \in \bigcap_{l \in J_2} B_Y \left(f_{n_l}(x), \frac{1}{k_l} \right) \right\} = \bigcup_{t=1}^{\infty} \bigcap_{l \in J_2} \left\{ x : f_{n_l}(x) \in B_Y \left(u_t, \frac{1}{k_l} \right) \right\} = \bigcup_{t=1}^{\infty} \bigcap_{l \in J_2} S(t, l).$$

On the right hand side, each set $S(t, l)$ is the inverse image of an open set under f_{n_l} which is Baire-α_{n_l}, where $\alpha_{n_l} < \alpha$. Thus each $S(t, l)$ is in Σ_α by Proposition 2.1 as $X \times Y$ is metrizable, yielding that it is perfect. Hence if we take the finite intersection for $l \in J_2$ and then the countable union for $t = 1, 2, ...$, we will still have a set in Σ_α and as we have already seen it concludes the proof of the first direction.

For the other direction, let us assume $gr(f) = \bigcap_{j=1}^{\infty} G_j$ where the set G_j is in Σ_α for each j, their sequence is nested, and they satisfy the three conditions of the theorem. Let us define F_j as it follows:

$$F_j = \bigcup_{x \in X} \overline{G_j(x)},$$

thus F_j is the closure by coordinates. If we regard it as a multivalued function defined on X with range 2^Y, whose values are naturally the vertical sections of the set, we can easily verify that it satisfies the conditions of Proposition 2.2. Indeed, it has clearly nonempty, closed values, and as the projection of $(X \times U) \cap G_n$ to X is in Σ_α for each open subset U of Y, the inverse image $F_j^{-1}(U)$ is in Σ_α for the open subsets of Y. Hence F_j has a Borel-α selection f_j. As α is a successor ordinal, $\alpha - 1$ makes sense and Proposition 2.3 can be applied, yielding f_j is Baire-$(\alpha - 1)$. The conclusion is the same as it was in the proof of Theorem 1.3: as the intersection of the sets $F_j(x)$ is only $\{f(x)\}$ and their diameter tends to 0, $f_j(x) \to f(x)$ must hold, and as a consequence, f is the pointwise limit of Baire-$(\alpha - 1)$ functions, meaning f is Baire-α. $\square$

5. ACCUMULATION POINTS OF GRAPHS

Before we start the proof of Theorem 1.9, we would like to remark that the conditions concerning T are clearly necessary, even if we do not require f to be Baire-2:

Proposition 5.1. *In the setting of the first case of Theorem 1.9, if a subset T of $X \times Y$ equals L_f for a function $f : X \to Y$, then T is closed and $T \cap (\{x\} \times Y) \neq \emptyset$ for each $x \in X$.*

Furthermore, in the setting of the second case of Theorem 1.9, if a subset T of $X \times Y$ equals L_f for a function $f : X \to Y$, then T is closed and there is a countable set $D \subseteq X$ such that $T \cap (\{x\} \times Y) \neq \emptyset$ for each $x \in X \setminus D$.

Proof. As L_f is the set of the accumulation points of $gr(f)$, it must be closed in both cases. On the other hand, in the first case, if we consider any $x \in X$, by our conditions there is a sequence $(x_n)_{n=1}^{\infty}$ with elements from X distinct from x and with limit x. Thus by the compactness of Y, for any f the sequence $(f(x_n))_{n=1}^{\infty}$ has a limit point, implying the sequence $(x_n, f(x_n))_{n=1}^{\infty}$ has a limit point in $\{x\} \times Y$. Hence if $T = L_f$, the set T has to intersect any vertical line in the first case.

In the second case, proceeding towards a contradiction, let us assume the set D of points in X satisfying $T \cap (\{x\} \times Y) = \emptyset$ is uncountable and there exists a function $f : X \to Y$ for which $T = L_f$ holds. As both X and Y is σ-compact, it implies the existence of compact sets $C_X \subseteq X$ and $C_Y \subseteq Y$ such that $C_X \cap D$ is uncountable and the cardinality of $D^* = \{x : x \in C_X \cap D, f(x) \in C_Y\}$ is also uncountable. Thus by the separability of X, the set D^* contains one of its accumulation points, d. Therefore there exists a sequence (d_i) in D^*, $(d_i \neq d)$ with limit d. Since all the elements of the sequence $(f(d_i))$ are in the compact set C_Y, it has a convergent subsequence, therefore $L_f(d)$ cannot be empty, while $T(d)$ is, a contradiction. $\qquad\square$

However, Theorem 1.9 states for such a set T we have a Baire-2 function satisfying $L_f = T$, yielding the following:

Corollary 5.2. *In the setting of any case of Theorem 1.9, if a subset T of $X \times Y$ equals L_f for a function $f : X \to Y$, then there exists a Baire-2 function such that $L_f = T$.*

Proof of Theorem 1.9. Let us regard the first case. Consider a metric on X. By Lemma 2.5, there exists a countable set $A \subseteq X$ and a function $f_0 : A \to Y$ satisfying $L_{f_0} = T$. We wish to extend this function to $f : X \to Y$ such that f is Baire-2 without making L_f larger. In order to do this, define a multifunction $F : X \to 2^Y$ the following way:

$$F(x) = \begin{cases} \{f_0(x)\} & \text{if } x \in A \\ T \cap (\{x\} \times Y) & \text{if } x \in X \setminus A. \end{cases}$$

As T is closed and its vertical sections are nonempty, F has nonempty closed values. Furthermore, $F^{-1}(G)$ is in Σ_3 for each open subset G of Y. Indeed, $T \cap (X \times G)$ is a set in Σ_2. Next we show that $\pi(T \cap (X \times G))$ is also in Σ_2. Let us recall that as $X \times Y$ is σ-compact, any closed set is the union of countably many compact sets, implying any set in Σ_2 is also the union of countably many compact sets. However, the projection of a compact set is obviously compact, thus closed. Hence $\pi(T \cap (X \times G))$ is in Σ_2 as we stated. Furthermore, one can easily verify that $F^{-1}(G)$ and $\pi(T \cap (X \times G))$ can differ only in the points of A, because if we regard T as a multifunction whose values are its vertical sections, T and F differ only in A. Thus $F^{-1}(G)$ differs only in a countable set from a set in Σ_2, yielding it is in Σ_3: indeed, a countable set is always in Σ_2, thus if we add a countable set to $\pi(T \cap (X \times G))$ we obtain another set in Σ_2, while removing a countable set is equivalent to intersecting with its complement, which is in Π_2. Hence the set we are interested in is the intersection of a set in Σ_2 and a set in Π_2, which are both in Σ_3 as X is metrizable. As a consequence, the intersection is also in Σ_3, as

we stated. Hence F satisfies all the conditions of Proposition 2.2, yielding it admits a Borel-3 selection f. This function f is also Baire-2 since the conditions of Theorem 1.9 satisfy the conditions of Proposition 2.3. What remains to show that is $L_f = T$. We have already seen $T \subseteq L_f$ as $T = L_{f_0}$ by the construction of f_0 and $L_{f_0} \subseteq L_f$ clearly holds. For the other inclusion, we only have to verify that there is no sequence in $gr(f)$ with limit outside of T. However, in that case there would be such a sequence in $gr(f_0)$ as every point of $gr(f)$ is in the closed set T, except for the ones in $gr(f_0)$. Nevertheless that would imply L_{f_0} is already larger than T, which is a contradiction.

In the second case, we can proceed almost the same way. Let us define f_0 on a countable set A provided by Lemma 2.5. As Y is not compact, there exists a sequence $(y_1, y_2, ...)$ in Y without any accumulation point. Furthermore, as $D \setminus A$ is a countable set, we can enumerate its elements, possibly finitely: $(d_1, d_2, ...)$. Let us define the multifunction $F : X \to 2^Y$ as it follows:

$$F(x) = \begin{cases} \{f_0(x)\} & \text{if } x \in A \\ \{y_i\} & \text{if } x = d_i \\ T \cap (\{x\} \times Y) & \text{if } x \in X \setminus (A \cup D). \end{cases}$$

The steps of the previous case can be repeated to show that we can apply Proposition 2.2 to F without any difficulty, yielding the existence of a Borel-3 selection f, which is also Baire-2 by Proposition 2.3. What is a difference from the previous case, that in the proof of $L_f = T$ we have to take into account those sequences of points of $gr(f)$ which contain infinitely many points above $D \setminus A$. However, as the sequence $(y_1, y_2, ...)$ has no accumulation point in Y, such a sequence cannot have an accumulation point in $X \times Y$, thus $L_f = T$, indeed. □

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