GENERIC BIRKHOFF SPECTRA

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ABSTRACT. Suppose that $\Omega = \{0,1\}^{\mathbb{N}}$ and σ is the one-sided shift. The Birkhoff spectrum $S_f(\alpha) = \dim_H \left\{\omega \in \Omega : \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) = \alpha\right\}$, where $\dim_H$ is the Hausdorff dimension. It is well-known that the support of $S_f(\alpha)$ is a bounded and closed interval $L_f = [\alpha^*_{f,\min}, \alpha^*_{f,\max}]$ and $S_f(\alpha)$ on L_f is concave and upper semicontinuous. We are interested in possible shapes/properties of the spectrum, especially for generic/typical $f \in C(\Omega)$ in the sense of Baire category. For a dense set in $C(\Omega)$ the spectrum is not continuous on $\mathbb{R}$, though for the generic $f \in C(\Omega)$ the spectrum is continuous on $\mathbb{R}$, but has infinite one-sided derivatives at the endpoints of L_f. We give an example of a function which has continuous S_f on $\mathbb{R}$, but with finite one-sided derivatives at the endpoints of L_f. The spectrum of this function can be as close as possible to a "minimal spectrum". We use that if two functions f and g are close in $C(\Omega)$ then S_f and S_g are close on L_f apart from neighborhoods of the endpoints.

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1. Introduction

1.1. **Background.** Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving system. Birkhoff's Ergodic Theorem tells us that for μ-a.e. $x \in X$ and $f \in L^1(\mu)$, the limit $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^n x)$ exists, and is a T-invariant function. Furthermore, if T is ergodic with respect to μ, the limit equals the constant $\int f\, d\mu$. For the ergodic case, if we let $E_f(\alpha) := \{x \in X : \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^n x) = \alpha\}$ then $\mu(E_f(\alpha)) = 1$ if $\alpha = \int f\, d\mu$, and 0 otherwise.

Now consider (X, T) to be a topological dynamical system, and f be a continuous function on X. Instead measuring the level-set $E_f(\alpha)$ by the ergodic measure μ, one gets more interesting values by considering the Hausdorff dimension of the sets $E_f(\alpha)$ (including the irregular set $E'_f := \{x \in X : \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^n x)$ does not exist.$\}$). For a given measure μ the Birkhoff Ergodic Theorem selects just one α and gives zero μ measure to the other sets $E_f(\alpha')$ for $\alpha' \neq \alpha$. The function $S_f(\alpha) := \dim_H(E_f(\alpha))$ is called the *Birkhoff spectrum* for f, and it will be the primary object that we study in this paper.

Such kind of study is referred to as a multifractal analysis. Multifractal analysis on Birkhoff averages has been initiated by Y. Pesin and H. Weiss [14] for Hölder functions in the context of thermodynamic formalism. Birkhoff spectrum of continuous functions was studied by A.-H. Fan, D.-J. Feng, and J. Wu [6]. In their study (which we will recall precisely in Theorem 2.3.1), they have shown a variational formula between the dimension of the level set and the metric entropy. They have also shown that $S_f(\alpha)$ is concave and upper semicontinuous (hence continuous by the nature of concave functions; see [16, §10]) on the interior of the set $\{\alpha \in \mathbb{R}^d : E_f(\alpha) \neq \emptyset\}$, while remaining the question regarding the behavior of the spectrum at the boundary of its support open.

For other studies of the Birkhoff spectrum, we refer to, for instance, [1], [18], [3], [7], [11], [13], and [9]. For more information on multifractal analysis (especially with its relationship to thermodynamic formalism), we refer to [2], [15] and to a survey paper of V. Climenhaga [4].

The main objective of this paper is to better understand the Birkhoff spectrum for generic continuous functions. We recall that given a complete metric space (X, d), we say a set $A \subset X$ is *generic* (or *typical*) if A is a complement of a set of first category (i.e. a countable union of nowhere dense sets). The Baire category theorem asserts that a generic set A is dense in X. In our paper we will work with the full shift (Ω, σ) on the alphabet $\{0, 1\}$ and consider Birkhoff averages of real-valued continuous functions $f \in C(\Omega, \mathbb{R}) = C(\Omega)$. One of the main foci of this paper will be on the behavior of the spectrum of a generic continuous function at the boundary of the support of the spectrum. In case of one-dimensional range the support of the spectrum of $f \in C(\Omega)$ is always a (possibly degenerate) closed interval L_f and concave and upper semiconinuous functions are always continuous on such intervals. However, it may happen that S_f, as a function defined on $\mathbb{R}$ has a jump discontinuity at the endpoints of S_f. Such functions were called degenerate by J. Schmeling in [17]. We will show that for the generic $f \in C(\Omega)$ the spectrum is continuous, with infinite one-sided derivatives at the endpoints of L_f. Continuity of the spectrum for the generic Hölder function was proved by Schmeling in [17]. In fact, this combined with results in [12] and [6] imply the continuity of the spectrum for the generic continuous function in our setting. In this paper we give a direct proof of this fact.

1.2. **Summary of the main results, organization of the paper.** Let $\Omega = \{0,1\}^{\mathbb{N}}$, and σ be the shift map. We assume that (Ω, σ) is the full shift. The space of real-valued continuous functions on Ω (denoted $C(\Omega)$) is equipped with the usual supremum norm. We denote by $\alpha_{f,\max}$ (resp. $\alpha_{f,\min}$) the maximum (resp. minimum) value of $f \in C(\Omega)$. The level-sets of the Birkhoff averages are

$$E_f(\alpha) := \Big\{\omega \in \Omega : \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\sigma^n \omega) = \alpha\Big\}. \tag{1.1}$$

Let $\alpha^*_{f,\max} := \sup\{\alpha \in \mathbb{R} : E_f(\alpha) \neq \emptyset\}$, and $\alpha^*_{f,\min} := \inf\{\alpha \in \mathbb{R} : E_f(\alpha) \neq \emptyset\}$. We also put $L_f = [\alpha^*_{f,\min}, \alpha^*_{f,\max}]$. The Birkhoff spectrum is defined as $S_f(\alpha) := \dim_H E_f(\alpha)$, keeping in mind that the empty set has Hausdorff dimension zero S_f is defined on $\mathbb{R}$. Results on concavity of S_f and Birkhoff's Ergodic Theorem imply that L_f is the support of S_f. It is known, [6], that S_f is apart from being concave is also upper semicontinuous and hence it is continuous on the closed interval L_f. Often, for ease of terminology, we will mention the endpoints of the support of the spectrum as the endpoints of the spectrum.

In Section 2 after introducing some notation we give some simple examples and recall one of the main results of [6].

Next we discuss some tools used later. First, we show that given a continuous function f, any continuous function that is sufficiently close to f would have its Birkhoff spectrum also close to S_f on L_f except for a neighborhood of the endpoints of the spectrum. This will be proven in Theorem 3.1.1.

In Subsection 3.2 we prove some results about piecewise constant continuous (or simply PCC) functions, that is about functions which depend on finitely many coordinates. Among other results we show that for such functions f there is always a periodic ω in $E_f(\alpha^*_{f,\max})$.

The next two results will concern the continuity of a Birkhoff spectrum. Given $f \in C(\Omega)$, we say that the spectrum S_f is *continuous* if it is continuous on $\mathbb{R}$, and *discontinuous* otherwise. Equivalently, S_f is continuous when $S_f(\alpha^*_{f,\min}) = S_f(\alpha^*_{f,\max}) = 0$. We will first show that continuous, in fact PCC functions with discontinuous spectrum are dense in $C(\Omega)$ (Theorem 4.1.1). On the other hand, we give a direct proof of the fact that generic continuous functions have continuous spectrum (Theorem 4.2.1). In [6, §5, Item (2)] a question was raised about continuity of the spectrum at the boundary of its support. In the one-dimensional case, as we mentioned the answer is obvious if we consider the restriction of S_f onto L_f, however there might be discontinuity from the exterior side of L_f.

In Subsection 4.3 we show that for a dense open subset of $C(\Omega)$ the support of the spectrum is in the interior of $[\alpha_{f,\min}, \alpha_{f,\max}]$.

It is mentioned in the introduction of [6] that even for Hölder regular functions discussions of $S_f(\alpha)$ for boundary points of L_f are scarce, which is actually a subtle problem.

In the remainder of our paper, in Section 5 we will discuss one-sided derivatives of a Birkhoff spectrum at the endpoints/boundary points of the spectrum. Given $\varphi : \mathbb{R} \to \mathbb{R}$, we denote by $\partial^- \varphi(\alpha)$ the left-hand derivative of φ at α (if the value exists). Similarly, $\partial^+ \varphi(\alpha)$ denotes the right-hand derivative. We will show that the spectrum of a generic continuous function f has infinite one-sided derivatives at the endpoints of L_f, i.e. $\partial^+ f(\alpha^*_{f,\min}) = \infty$, and $\partial^- f(\alpha^*_{f,\max}) = -\infty$ (Theorem 5.1.1). We construct a continuous function with continuous spectrum for which the one-sided derivatives at the endpoints are finite (Theorem 5.2.1). This function will also have a very small spectrum. By concavity of the spectrum on its support there is always a triangle which should be under the graph of the spectrum. Our example will provide an example when the spectrum is very close to this lower estimate. In [18] Takens and Verbitsky calculated the spectrum of the Manneville-Pomeau map. It has a Birkhoff spectrum with a finite one-sided derivative at one of the endpoints.

It is not that obvious that functions with finite one-sided derivatives at the endpoints of the spectrum exist since for some well-known examples of functions with continuous spectrum, like the one discussed in Example 2.2.1 we have $\partial^+ f(\alpha^*_{f,\min}) = \infty$, and $\partial^- f(\alpha^*_{f,\max}) = -\infty$, however this function does not have a "generic spectrum" since $\alpha^*_{f,\min}$ equals $\alpha_{f,\min}$ and $\alpha^*_{f,\max}$ equals $\alpha_{f,\max}$. As we mentioned earlier for the generic continuous functions we always have $\alpha_{f,\min} < \alpha^*_{f,\min} < \alpha^*_{f,\max} < \alpha_{f,\max}$ see Theorem 4.3.1. In Theorem 5.2.4 we prove that for PCC functions f with continuous spectrum we always have $\partial^+ f(\alpha^*_{f,\min}) = \infty$, and $\partial^- f(\alpha^*_{f,\max}) = -\infty$. This illustrates that for the proof of Theorem 5.2.1 one needs to use a more involved construction than a PCC function.

2. Preliminaries

2.1. **Notation and terminology.** Let $\Omega = \{0,1\}^{\mathbb{N}}$, and σ be the shift map.

We introduce the usual metric d on Ω defined by

$$d(\omega, \omega') = \sum_{k=1}^{\infty} \frac{|\omega_k - \omega'_k|}{2^k},$$

where ω_k (resp. ω'_k) denotes the coordinates/entries of ω (resp. ω'). If $k \in \mathbb{N} \cup \{\infty\}$ and A is a finite string of 0s and 1s then A^k denotes the k-fold concatenation of

A and $[A]$ denotes the cylinder set $\{\omega : A\omega', \omega' \in \Omega\}$. Given $k, l \in \mathbb{N}$ and $\omega = (\omega_1\omega_2...) \in \Omega$ we put $\omega|k = \omega_1...\omega_k$ and $(\omega)_k^l := \omega_k\omega_{k+1}\ldots\omega_{l-1}\omega_l$, if $k \leq 0$ then $\omega|k$ is the empty string and analogously if $k > l$ then $(\omega)_k^l$ is also the empty string. The "conjugate" $\overline{\omega}$ is the string which we obtain from ω by swapping 0s and 1s, that is $\overline{\omega}_k = 1 - \omega_k$ for all k.

The s-dimensional Hausdorff measure of $A \subset \Omega$ is denoted by $\mathcal{H}^s(A)$ and recall that $\mathcal{H}^s(A) = \lim_{\delta\to 0+} \mathcal{H}^s_\delta(A)$ where $\mathcal{H}^s_\delta(A) = \inf\{\sum_i(\text{diam } U_i)^s : \text{where } A \subset \cup_i U_i$ and $\text{diam } U_i < \delta\}$. The Hausdorff dimension of $A \subset \Omega$ is $\dim_H A = \inf\{s : \mathcal{H}^s(A) = 0\}$. From this definition, it is a standard exercise to show that $\dim_H \Omega = 1$.

The complement of a set A is denoted by A^c.

Let $\text{PCC}^k(\Omega)$ be the set of those piecewise constant continuous functions in $C(\Omega)$, which depend only on cylinders of length/depth k. While the set of piecewise constant continuous functions in $C(\Omega)$, is denoted by $\text{PCC}(\Omega)$. Obviously $\text{PCC}(\Omega) = \cup_k \text{PCC}^k(\Omega)$.

The $(1/2, 1/2)$-Bernoulli measure, the "Lebesgue measure" on Ω is denoted by λ. In case we write $\int f$ for an $f : \Omega \to \mathbb{R}$ we always mean $\int_\Omega f d\lambda$.

We denote by $C_0(\Omega)$ the set of continuous functions for which $\int f = 0$, and $\text{PCC}^k_0(\Omega) = \text{PCC}^k(\Omega) \cap C_0(\Omega)$.

Given $f \in C(\Omega)$, we denote $\|f\| = \sup_{\omega\in\Omega} |f(\omega)|$, and for any $\delta > 0$, $B(f, \delta) = \{g \in C(\Omega) : \|f - g\| < \delta\}$.

Recall (1.1) and the subsequent definitions of $E_f(\alpha)$, $S_f(\alpha)$. We remark that our definition of $S_f(\alpha)$ is a bit different from the usual notation in multifractal analysis, since quite often $S_f(\alpha)$ is defined to be $-\infty$ when $E_f(\alpha)$ is empty.

As previously defined, we set $\alpha^*_{f,\max} = \sup\{\alpha \in \mathbb{R} : E_f(\alpha) \neq \emptyset\}$, where $\alpha^*_{f,\min} = \inf\{\alpha \in \mathbb{R} : E_f(\alpha) \neq \emptyset\}$. In general we have $\alpha_{f,\min} \leq \alpha^*_{f,\min} \leq \alpha^*_{f,\max} \leq \alpha^*_{f,\max}$, and it is possible for the strict inequalities to hold (including the first and the third inequality), as we will see in an example (cf. Example 2.2.3). In fact, as Theorem 4.3.1 shows this property is true for the generic continuous functions as well.

The σ-invariant Borel probability measures are denoted by $\mathcal{M}_\sigma$. By Birkhoff's Ergodic Theorem, we know that $\lambda(E_f(\int f)) = 1$. Furthermore, if $\{C_i\}_{i=1}^\infty$ are cylinders in Ω of length at least $k \in \mathbb{N}$ and $E_f(\int f) \subset \bigcup_{i=1}^\infty C_i$ then

$$1 = \lambda\left(E_f\left(\int f\right)\right) \leq \sum_{i=1}^{\infty} \lambda(C_i) = \sum_{i=1}^{\infty} \text{diam}(C_i),$$

which implies that $1 \leq \mathcal{H}_{2^{-k}}(E_f(\int f)) \leq \mathcal{H}_{2^{-k}}(\Omega)$ for any $k \in \mathbb{N}$, and thus $S_f(\int f d\lambda) = 1$. Given $f \in C(\Omega)$ and $\alpha \in \mathbb{R}$ we will also use the following subsets of $\mathcal{M}_\sigma$

$$\mathcal{F}_f(\alpha) := \left\{\mu \in \mathcal{M}_\sigma : \int f\, d\mu = \alpha\right\}. \tag{2.1}$$

2.2. **Examples.** We present a few examples of Birkhoff spectra of certain $PCC(\Omega)$ functions. We will first provide an example for a function with continuous spectrum.

Example 2.2.1. Let $f \in C(\Omega)$ be the function given by $f(\omega) = 1$ if $\omega_1 = 1$ and $f(\omega) = 0$ if $\omega_1 = 0$. Then for any $\alpha \in (0,1)$ we have

$$S_f(\alpha) = -\frac{\alpha \log(\alpha) + (1-\alpha)\log(1-\alpha)}{\log 2},$$

if $\alpha \notin (0,1)$ then $S_f(\alpha) = 0$. In particular, f has continuous spectrum, as $\alpha^*_{f,\min} = 0$, $\alpha^*_{f,\max} = 1$, and furthermore, $\partial^+ S_f(\alpha^*_{f,\min}) = +\infty$ and $\partial^- S_f(\alpha^*_{f,\max}) = -\infty$.

Verification of the properties of Example 2.2.1. We will prove two inequalities using suitably defined Hölder functions and the result of [5]. First, let us consider the function $h_1 : \Omega \to [0,1]$ defined by

$$h_1(\omega) = \sum_{i=1}^{\infty} \frac{\omega_i}{2^i}.$$

That is, h_1 takes a 0-1 sequence to the number with the corresponding binary expansion. We claim that h_1 is a Lipschitz function in fact. Indeed, if ω' differs from ω in its nth coordinate, but not before that point, then $d(\omega, \omega') \geq 2^{-n}$, while $|h_1(\omega) - h_1(\omega')| \leq 2^{-n+1}$, hence h_1 has Lipschitz constant 2. Moreover, $h_1(E_f(\alpha))$ equals the set of numbers in $[0,1]$ having a binary expansion in which the density of 1s equals α. Thus due to [5], the dimension of $h_1(E_f(\alpha))$ is given by the formula in the statement of the lemma, yielding

$$S_f(\alpha) \geq -\frac{\alpha \log(\alpha) + (1-\alpha)\log(1-\alpha)}{\log 2},$$

as h_1 is Lipschitz.

Concerning the other inequality, define $h_2 : C \to \Omega$ for the triadic Cantor set $C \subset [0,1]$: if the triadic expansion of $x \in C$ is

$$x = \sum_{i=1}^{\infty} \frac{x_i}{3^i},$$

then let $\omega = h_2(x)$ have coordinates $\frac{x_1}{2}, \frac{x_2}{2}, \ldots$. That is, h_2 is a one-to-one mapping between Ω and C. Now if x differs from x' in its nth coordinate, but not before that point, then $|x - x'| \geq 3^{-n}$. On the other hand, $d(h_2(\omega), h_2(\omega')) \leq 2^{-n+1}$. It quickly yields that h_2 is a Hölder function with exponent $\frac{\log 2}{\log 3}$. Moreover, $h_2^{-1}(E_f(\alpha))$ is the set of numbers in $[0,1]$ having a ternary expansion with no 1s, in which the density of 2s is α and the density of 0s is $1-\alpha$. Hence $h_2^{-1}(E_f(\alpha))$ is contained by the set of numbers in $[0,1]$ having a ternary expansion in which the density of 2s is α and the density of 0s is $1-\alpha$. Thus due to [5], the dimension of $h_2^{-1}(E_f(\alpha))$ is at most

$$-\frac{\alpha \log(\alpha) + (1-\alpha)\log(1-\alpha)}{\log 3}.$$

Hence as h_2 is $\frac{\log 2}{\log 3}$-Hölder, we obtain an upper estimate for $S_f(\alpha)$, that is the dimension of $E_f(\alpha)$, notably

$$S_f(\alpha) \leq -\frac{\alpha \log(\alpha) + (1-\alpha)\log(1-\alpha)}{\log 2}.$$

This shows that the desired equality holds, and the remaining claims clearly follow. □

Next, we will see examples of continuous functions with discontinuous spectra.

Example 2.2.2. If f is a constant function, i.e. $f \equiv C \in \mathbb{R}$, then $S_f(C) = 1$ and $S_f(\alpha) = 0$ otherwise. The same is true if f is cohomologous to a constant, i.e. there exists $g \in C(\Omega)$ for which $f = C + g - g \circ \sigma$ (we recall that if C is zero, f is called a coboundary).

Finally, we give an example where $\alpha_{f,\min} < \alpha^*_{f,\min} < \alpha^*_{f,\max} < \alpha_{f,\max}$ (that is, strict inequalities are satisfied), and the Birkhoff spectrum is discontinuous.

Example 2.2.3. There exists $f \in \mathrm{PCC}_0^3(\Omega)$ satisfying $\alpha_{f,\min} < \alpha^*_{f,\min} < \alpha^*_{f,\max} < \alpha_{f,\max}$ and $S_f(\alpha^*_{f,\min}), S_f(\alpha^*_{f,\max}) > 0$.

Proof. As $f \in \mathrm{PCC}_0^3(\Omega)$ we can define it by giving its values on 3-cylinders by abusing a bit the notation for f. We define f by $f([000]) = f([010]) = -2$, $f([001]) = -3$, $f([100]) = -1$, and $f(\overline{\omega}) = -f(\omega)$. Then we clearly have $\alpha_{f,\min} = -3$ while $\alpha_{f,\max} = 3$.

Now we claim $\alpha^*_{f,\min} = -2$, while $\alpha^*_{f,\max} = 2$, which would yield the inequalities $\alpha_{f,\min} < \alpha^*_{f,\min} < \alpha^*_{f,\max} < \alpha_{f,\max}$. Due to symmetry reasons, it suffices to verify $\alpha^*_{f,\min} = -2$. To this end, consider an arbitrary $\omega \in \Omega$. Now we are interested in the averages $\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega)$. In the sequence $f(\sigma^n\omega)$ each value is at least -2, except for the cases when the first three coordinates of $\sigma^n\omega$ are 001. However, in this case the first three coordinates of $\sigma^{n+2}\omega$ contain at least two 1s, or they are 100. In either case, $f(\sigma^{n+2}\omega) \geq -1$. This argument shows that in the sum $\sum_{n=1}^{N} f(\sigma^n\omega)$ the summands with value -3 can be paired with summands with value at least -1, except for possibly the last one, whose pair does not appear in the sum. Besides that, all the other summands have value at least -2. Consequently, the average $\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) \geq -2 - \frac{3}{N}$, hence the limit is at least -2, verifying $\alpha^*_{f,\min} \geq -2$. For the other inequality, we may simply consider the identically 0 sequence, hence $\alpha^*_{f,\min} = -2$. It proves the claim of this paragraph.

It remains to show that $S_f(\alpha^*_{f,\min}), S_f(\alpha^*_{f,\max}) > 0$. Due to symmetry reasons, these quantities are clearly equal, hence $S_f(\alpha^*_{f,\min}) > 0$ would be sufficient. Consider the following subset of Ω:

$$B = \{\omega \in \Omega : \omega_k = 0 \text{ for } k \equiv 1, 2 \mod 3\}.$$

Then for any $\omega \in B$ and n we have that at least two of the first three coordinates of $\sigma^n\omega$ equals 0. Consequently, $f(\sigma^n\omega) < 0$. Moreover, similarly to the previous argument we find that the in the sum $\sum_{n=1}^{N} f(\sigma^n\omega)$ the summands with value -3 can be paired with summands with value -1, except for possibly the last one. All the other summands have value -2. Hence we find

$$-2 - \frac{1}{N} \leq \frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) \leq -2.$$

It proves that $B \subset E_f(-2)$, hence $\dim_H B > 0$ would conclude the proof. However, this dimension can be calculated explicitly as B is a self-similar set, which equals the disjoint union of its 2 similar images, where the similarities have ratio $\frac{1}{8}$. Thus $\dim_H B = \frac{\log 2}{\log 8} = \frac{1}{3}$ by Hutchinson's Theorem [8]. $\square$

2.3. **Variational formula.** The following result was obtained by Fan, Feng, and Wu. We present this result in the context of the full-shift on an alphabet of two symbols (Ω, σ) (in [6], they proved the result for a topologically mixing subshift of finite type).

Theorem 2.3.1 ([6, Theorem A]). *Suppose that $f : \Omega \to \mathbb{R}^d$ is a continuous function. We denote $L_f := \{\alpha \in \mathbb{R}^d : \alpha = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\sigma^n \omega)$ for some $\omega \in \Omega\}$. There exists a concave and upper semi-continuous function Λ_f such that for any $\alpha \in L_f$*

$$S_f(\alpha) := \dim_H(E_f(\alpha)) = \Lambda_f(\alpha),$$

and

$$\Lambda_f(\alpha) = \max_{\mu \in \mathcal{F}_f(\alpha)} \frac{h_\mu}{\log 2}$$

where h_μ is the metric entropy of μ, and $\mathcal{F}_f(\alpha)$ can be defined analogously to (2.1).

The function $\Lambda_f(\alpha)$ is defined in the same paper [6, Proposition 5] using the cardinality of the cylinders of large length that contain at least one point ω for which the Birkhoff average of f of that length is close to α. It was later shown that the quantity $\Lambda_f(\alpha)$ indeed agrees with $S_f(\alpha)$ for all $\alpha \in L_f$ [6, Proposition 6].

3. Tools

3.1. **Norm Continuity Theorem.** We first prove that two Birkhoff spectra of two continuous functions are close (except near the endpoints) if those two functions are close in the supremum norm.

Theorem 3.1.1 (Norm continuity theorem). *Let $f \in C(\Omega)$ for which $\alpha^*_{f,\min} < \alpha^*_{f,\max}$, and $\varepsilon \in (0, (\alpha^*_{f,\max} - \alpha^*_{f,\min})/2)$ be given. Then there exists $\delta > 0$ such that for any $g \in B(f, \delta)$, we have $|S_f(\alpha) - S_g(\alpha)| < \varepsilon$ for all $\alpha \in (\alpha^*_{f,\min} + \varepsilon, \alpha^*_{f,\max} - \varepsilon)$.*

Remark 3.1.2. We will later learn that the generic continuous function satisfies the hypothesis of this theorem; see Theorem 4.3.1.

If one considers $f, g \in C(\Omega)$ with continuous spectrum then the above theorem can be used to show that for given $\varepsilon > 0$ one can find $\delta > 0$ such that $\|f - g\| < \delta$ implies that $\|S_f - S_g\| < \varepsilon$. On the other hand, if f has discontinuous spectrum, say $S_f(\alpha^*_{f,\max}) > 0$ then the density of functions with continuous spectrum (Theorem 4.2.1) and Remark 3.1.4 imply that arbitrary close to f one can find functions g such that $\|S_f - S_g\| > S_f(\alpha^*_{f,\max})/2$.

To proceed, we first prove the following lemma.

Lemma 3.1.3. *Let $\varepsilon > 0$ be given. Suppose that $f \in C(\Omega)$, and $\alpha \in [\alpha^*_{f,\min}, \alpha^*_{f,\max}]$. Then for any $g \in C(\Omega)$ such that $\|f - g\| < \varepsilon$, there exists $\alpha' \in (\alpha - \varepsilon, \alpha + \varepsilon)$ for which $S_g(\alpha') \geq S_f(\alpha)$. If $S_f(\alpha) = 0$, but $E_f(\alpha) \neq \emptyset$ then $E_g(\alpha') \neq \emptyset$.*

Remark 3.1.4. This implies that if $\|f-g\| < \varepsilon$ then $|\alpha^*_{f,\max} - \alpha^*_{g,\max}| < \varepsilon$ and $|\alpha^*_{f,\min} - \alpha^*_{g,\min}| < \varepsilon$.

Proof. Recall the definition of $\mathcal{F}_f(\alpha)$ from (2.1). By Theorem 2.3.1 there exists $\mu_0 \in \mathcal{F}_f(\alpha)$ for which

$$S_f(\alpha) = \frac{h_{\mu_0}}{\log 2} = \frac{\max_{\mu\in\mathcal{F}_f(\alpha)} h_\mu}{\log 2}.$$

Set $\alpha' = \int g\, d\mu_0$. Since $\|f-g\| < \varepsilon$, we have $\alpha' \in (\alpha-\varepsilon, \alpha+\varepsilon)$. For $\alpha \in [\alpha^*_{f,\min}, \alpha^*_{f,\max}]$ we have $E_f(\alpha) \neq \emptyset$. If $S_f(\alpha) = 0$, then $\alpha \in \{\alpha^*_{f,\min}, \alpha^*_{f,\max}\}$. Consider the map $f_* : \mathcal{M}_\sigma \to L_f$ for which that $f_*(\mu) = \int f\, d\mu$. Since the map f_* is affine and continuous, μ_0 must be one of the extremal points of the convex set $\mathcal{M}_\sigma$. This implies that μ_0 is ergodic, so we may apply Birkhoff's Ergodic Theorem to show that for μ_0 almost every ω we have $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N g(\sigma^n\omega) = \alpha'$ and hence $E_g(\alpha') \neq \emptyset$.

Hence, from now on we can suppose that $S_f(\alpha) > 0$. In that case since $\mu_0 \in \mathcal{F}_g(\alpha')$ by Theorem 2.3.1 we obtain that

$$S_g(\alpha') = \frac{\max_{\mu\in\mathcal{F}_g(\alpha')} h_\mu}{\log 2} \geq \frac{h_{\mu_0}}{\log 2} = S_f(\alpha).$$

□

Using this lemma, we will prove the theorem by using concavity of the spectrum.

Proof of Theorem 3.1.1. For some $L \in \mathbb{N}$, we consider a partition

$$\alpha^*_{f,\min} = \alpha_1 < \alpha_2 < \cdots < \alpha_L = \alpha^*_{f,\max}$$

for which for every $i = 1, 2, \ldots, L-1$, $|\alpha_{i+1} - \alpha_i| < \varepsilon/4$ is small enough such that for every $t \in [0,1]$, we have

$$(1-t)S(\alpha_i) + tS(\alpha_{i+1}) > S((1-t)\alpha_i + t\alpha_{i+1}) - \varepsilon/2.$$

For each α_i, we choose a positive number $\delta(\alpha_i) < \varepsilon/8$ as follows: For any $\alpha'_i \in (\alpha_i - \delta(\alpha_i), \alpha_i + \delta(\alpha_i))$, and $\beta'_i \geq S_f(\alpha_i)$, the line segments connecting the points (α'_i, β'_i) and $(\alpha'_{i+1}, \beta'_{i+1})$ are above the graph of $S_f(\alpha) - \varepsilon$ for $i = 2, ..., L-2$. We can also suppose that the intervals $(\alpha_i - \delta(\alpha_i), \alpha_i + \delta(\alpha_i))$ are disjoint. Then we set

$$\delta = \min\{\varepsilon/8, \delta(\alpha_1), \delta(\alpha_2), \ldots, \delta(\alpha_L)\}.$$

We apply Lemma 3.1.3 with $\varepsilon = \delta$ to show that there exists $\alpha'_i \in (\alpha_i - \delta, \alpha_i + \delta) \subset (\alpha_i - \delta(\alpha_i), \alpha_i + \delta(\alpha_i))$ such that $S_g(\alpha'_i) \geq S_f(\alpha_i)$ for $i = 1, ..., L-1$. Since $|\alpha'_1 - \alpha^*_{f,\min}| = |\alpha'_1 - \alpha_1| < \varepsilon/8$ and $|\alpha'_L - \alpha^*_{f,\max}| = |\alpha'_L - \alpha_L| < \varepsilon/8$ by using the concavity of S_g one can show that $S_g(\alpha) > S_f(\alpha) - \varepsilon$ for all $\alpha \in (\alpha^*_{f,\min} + (\varepsilon/2), \alpha^*_{f,\max} - (\varepsilon/2))$. By reversing the roles of f and g, by an analogous argument we can conclude that $S_f(\alpha) > S_g(\alpha) - \varepsilon$ for all $\alpha \in (\alpha^*_{g,\min} + (\varepsilon/2), \alpha^*_{g,\max} - (\varepsilon/2))$. Using Remark 3.1.4 we can conclude the proof. □

3.2. **Piecewise constant (PCC) functions.** We start with a lemma in which we show that $\alpha^*_{f,\max}$ is a uniform upper bound of the limit of the Birkhoff averages of any $f \in PCC^k$.

Lemma 3.2.1. *Assume $f \in \mathrm{PCC}^k(\Omega)$ and $\varepsilon > 0$. Then there exists N_0 such that for any $N \geq N_0$, for any $\omega \in \Omega$, we have*

$$\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) \leq \alpha^*_{f,\max} + \varepsilon, \tag{3.1}$$

which implies that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) \leq \alpha^*_{f,\max} \text{ uniformly for any } \omega \in \Omega. \tag{3.2}$$

Proof. Choose N_0 such that for any $N > N_0$

$$\frac{-k\,\|f\| + N(\alpha^*_{f,\max} + \varepsilon)}{N+k} > \alpha^*_{f,\max} + \frac{\varepsilon}{2}. \tag{3.3}$$

We claim that this N_0 satisfies the statement of the lemma. Proceeding towards a contradiction, assume the existence of a configuration ω and $N > N_0$ which refutes this claim, that is

$$\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) > \alpha^*_{f,\max} + \varepsilon. \tag{3.4}$$

Our goal is to construct $\omega' \in \Omega$, periodic by $N+k$ which will satisfy

$$\sum_{n=1}^{N} f(\sigma^n\omega') = \sum_{n=1}^{N} f(\sigma^n\omega) > N(\alpha^*_{f,\max} + \varepsilon), \tag{3.5}$$

and this will contradict the definition of $\alpha^*_{f,\max}$ as we will see in (3.7). In the ergodic sums we consider, the first coordinate has no importance, thus it is sufficient to construct $\sigma\omega'$. Let it be periodic with period $N+k$ (that is $\sigma^{N+k+1}\omega' = \sigma\omega'$), and define its first $N+k$ coordinates to be $\omega_2, \omega_3, ..., \omega_{N+k+1}$. Now if N' is arbitrary, express it modulo $N+k$ as $N' = p(N+k) + q$, where p is a nonnegative integer, while $0 \leq q < N+k$. Then the corresponding ergodic sum can be written as

(3.6)

$$\begin{aligned}\frac{1}{N'}\sum_{n=1}^{N'} f(\sigma^n\omega') &= \frac{1}{N'}\sum_{n=1}^{p(N+k)} f(\sigma^n\omega') + \frac{1}{N'}\sum_{n=1}^{q} f(\sigma^{p(N+k)+n}\omega') \\ &= \frac{p(N+k)}{N'}\left(\frac{1}{p(N+k)}\sum_{n=1}^{p(N+k)} f(\sigma^n\omega')\right) + \frac{1}{N'}\sum_{n=1}^{q} f(\sigma^{p(N+k)+n}\omega') = \circledast\end{aligned}$$

Using the periodicity of $\sigma\omega'$ in the first sum, and the boundedness of f in the second one we infer

$$\circledast = \frac{p(N+k)}{N'}\left(\frac{1}{N+k}\sum_{n=1}^{N+k} f(\sigma^n\omega')\right) + o(N').$$

Hence if $N' \to \infty$, the ergodic sum $\frac{1}{N'}\sum_{n=1}^{N'} f(\sigma^n\omega')$ converges to $\frac{1}{N+k}\sum_{n=1}^{N+k} f(\sigma^n\omega')$. Now by (3.4) and $f \in \mathrm{PCC}^k(\Omega)$, we have (3.5). Thus by (3.3), we deduce

$$\frac{1}{N+k}\sum_{n=1}^{N+k} f(\sigma^n\omega') > \frac{-k\|f\| + N(\alpha^*_{f,\max}+\varepsilon)}{N+k} > \alpha^*_{f,\max} + \frac{\varepsilon}{2}, \tag{3.7}$$

Hence $E_f(\alpha) \neq \emptyset$ for some $\alpha > \alpha^*_{f,\max} + \frac{\varepsilon}{2}$, which is obviously a contradiction. It concludes the proof. □

Next, we will show that if $f \in PCC(\Omega)$, then there exists a periodic point in Ω for which the limit of the Birkhoff averages of f equals $\alpha^*_{f,\max}$.

Lemma 3.2.2. *Let $f \in \mathrm{PCC}^k(\Omega)$. Then there exists a periodic configuration ω such that* $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) = \alpha^*_{f,\max}$.

Proof. We define a directed graph $G = (V, E)$ as follows: $V = \{0,1\}^k$, and there is an edge from $u \in V$ to $v \in V$ if roughly speaking v is one of the possible shifted images of u, that is $v_i = u_{i+1}$ for $i = 1, ..., k-1$. Now we can think of the values of f as weights on the vertices of G, while an arbitrary $\omega \in \Omega$ corresponds to an infinite walk Γ_ω in G. Moreover, the ergodic averages are simply the averages of weights along the vertices of finite subwalks of Γ_ω.

For technical reasons, it is advantageous to put the weights on the edges and work with those ones: one of the convenient ways to do so is putting weight $f(u)$ on all the edges *leaving* the vertex u. Denote the function $E \to \mathbb{R}$ obtained this way by f, too. Now the ergodic averages can be considered as the averages of weights along the edges of finite subwalks of ω.

Consider now $\omega \in \Omega$ such that $\frac{1}{N}\sum_{i=1}^{N} f(\sigma^i\omega) \to \alpha^*_{f,\max}$. Take the corresponding path Γ_ω. As V is finite, there exists a vertex which appears infinitely many times in Γ_ω. By erasing the first few entries of ω, or equivalently, erasing the first few edges of Γ_ω, we might assume by abuse of notation that the first vertex v of Γ_ω recurs infinitely many times. Now based on the recurrences of v, we can partition the infinite walk Γ_ω into closed, finite walks $\Gamma^{(1)}_\omega, \Gamma^{(2)}_\omega, ...$ such that each such walk starts and ends with v, and in the meantime it does not hit v. Now it is simple to verify that the edge set (counted with multiplicities from now on) of each $\Gamma^{(i)}_\omega$ is the union of graph cycles, or in other words, it is the union of closed walks containing each of their edges precisely once. (One cycle might also appear multiple times in this decomposition.) Indeed, we can find a subpath $e_1e_2...e_r$ such that $e_1 = e_r$, and there is no other repetition of edges in this subpath. Then $e_1e_2...e_{r-1}$ is a cycle, and its removal from $\Gamma^{(i)}_\omega$ results in a shorter closed walk starting and ending with v. Thus we can repeat the previous reasoning to find another cycle, if such exists and this procedure ends in finitely many steps.

Let us note now that there are only finitely many cycles in G as it is a finite graph. Denote their set by $\mathcal{C}$. By the previous paragraph, up to the last edge of any $\Gamma^{(i)}_\omega$, the edge set of Γ_ω can be written as the union of these cycles, such that $C \in \mathcal{C}$ is used $\rho_{C,i}$ times. Thus the ergodic average corresponding to the subpath of the

Γ_ω up to the last edge of $\Gamma_\omega^{(i)}$ is the following:

$$\frac{\sum_{C\in\mathcal{C}}\rho_{C,i}\sum_{e\in C}f(e)}{\sum_{C\in\mathcal{C}}\rho_{C,i}|C|}=\frac{\sum_{C\in\mathcal{C}}\rho_{C,i}|C|\sum_{e\in C}\frac{f(e)}{|C|}}{\sum_{C\in\mathcal{C}}\rho_{C,i}|C|}. \tag{3.8}$$

Notice that it is simply a convex combination of the cycle averages $\sum_{e\in C}\frac{f(e)}{|C|}$. Hence the ergodic average in (3.8) can be bounded from above by $\max_{C\in\mathcal{C}}\sum_{e\in C}\frac{f(e)}{|C|}$. Now by the choice of ω we also know that this ergodic average tends to $\alpha^*_{f,\max}$ as $i\to\infty$, hence

$$\alpha^*_{f,\max}\le\max_{C\in\mathcal{C}}\sum_{e\in C}\frac{f(e)}{|C|} \tag{3.9}$$

also holds.

Now consider the infinite walk which goes along a cycle C_0 over and over again, where C_0 is chosen so that the above maximum is attained. Then C_0 together with a starting point uniquely determines a periodic configuration $\omega^*\in\Omega$ for which $\sigma^i\omega^*$ always equals the respective vertex of C_0. Moreover, it is simple to check that the ergodic averages tend to $\sum_{e\in C_0}\frac{f(e)}{|C_0|}$. Hence this limit must be $\alpha^*_{f,\max}$ by (3.9), as it is an upper estimate for all ergodic limits. □

4. Continuity, discontinuity and support of the spectrum

By [6], we know that S_f is necessarily upper semi-continuous for any continuous function. Moreover, it is continuous on $[\alpha^*_{f,\min},\alpha^*_{f,\max}]$, while it vanishes outside of this interval. However it is not necessarily continuous at the endpoints of this interval.

4.1. **Denseness of** PCC **functions with discontinuous spectra.**

Recall an example of a $\mathrm{PCC}^3(\Omega)$ function with discontinuous spectrum from Example 2.2.3. In this section, we will show that functions in $\mathrm{PCC}(\Omega)$ with discontinuous spectrum form a dense subset of $C(\Omega)$.

Theorem 4.1.1. *Functions $h\in\mathrm{PCC}(\Omega)$ with $S_h(\alpha^*_{h,\max})>0$ are dense in $C(\Omega)$.*

Remark 4.1.2. Of course, a similar theorem is valid with $S_h(\alpha^*_{h,\min})>0$ in the conclusion and with a little extra technical effort one can show density in $C(\Omega)$ of those $f\in\mathrm{PCC}(\Omega)$ for which $S_h(\alpha^*_{h,\max})>0$ and $S_h(\alpha^*_{h,\min})>0$ hold simultaneously. As Theorem 4.2.1 shows functions satisfying the conclusion of Theorem 4.1.1, or any of its above mentioned variants form a first category set in $C(\Omega)$.

The main idea of the proof of Theorem 4.1.1 is to show that given any continuous function, we can approximate it by a PCC function, and we further "perturb" that PCC function in an appropriate way so that its spectrum will be discontinuous.

Proof of Theorem 4.1.1. Suppose $\varepsilon>0$ and $f_0\in C(\Omega)$ are arbitrary. We need to find an $h\in\mathrm{PCC}(\Omega)$ such that

$$\|f_0-h\|<\varepsilon \text{ and } S_f(\alpha^*_{h,\max})>0. \tag{4.1}$$

By using a suitably large $\mathbf{k}$ choose $f \in \mathrm{PCC}^{\mathbf{k}}(\Omega)$ such that $\|f - f_0\| < \varepsilon/2$. By Lemma 3.2.2 select a periodic ω' such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega') = \alpha^*_{f,\max}. \tag{4.2}$$

In this proof, as in (4.2) we prefer to take Birkhoff sums with indices between 0 and $N-1$, when taking limits it makes no difference. We can assume that there is a finite string of 0s and 1s, denoted by A such that $\omega' = A^\infty$, by not necessarily using the minimal period we can also suppose that $k_A = |A|$, the length of A is a multiple of $\mathbf{k}$.

Now we select a string B of length k_A. If $A \neq 0^{k_A}$ then we let $B = 0^{k_A}$, if $A = 0^{k_A}$ then we let $B = 1^{k_A}$. Without limiting generality in the sequel we assume that $B = 0^{k_A}$.

By using a suitably large number ℓ, to be fixed later, we consider strings $X = (A^{2\ell})AABAA$ and $Y = (A^{2\ell})ABAAA$.

Set $\mathbf{H} = \{X, Y\}^\infty$.

Observe that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega) =: \alpha_{XY} \leq \alpha^*_{f,\max} \text{ for any } \omega \in \mathbf{H}. \tag{4.3}$$

Put $m = \ell + 7$. We define the following finite union of cylinder sets in Ω

$$C_m = \{U_1 U_2 ... U_m \omega_0 \omega_1 ... : U_i \in \{X, Y\},\ i = 1, ..., m,\ \omega_j \in \{0, 1\},\ j = 0, 1, ...\}. \tag{4.4}$$

Put $\mathbf{P} = \cup_{i=0}^{\ell-1} \sigma^{i k_A} C_m$.

Next we define our perturbation function $g \in \mathrm{PCC}^{m k_A}(\Omega)$. If $\omega \in \mathbf{P}$ then we set $g(\omega) = \varepsilon/4$, otherwise put $g(\omega) = 0$.

It is easy to see that $\dim_H \mathbf{H} > 0$, since by Hutchinson's theorem $2 \cdot (2^{-(2\ell+5)k_A})^{\dim_H \mathbf{H}} = 1$, which gives $\dim_H \mathbf{H} = 1/((2\ell+5)k_A)$. Take and fix an arbitrary $\omega \in \mathbf{H}$. Recall that $|X| = |Y| = (2\ell+5)k_A$. By our definition of X and Y we have

$$\frac{1}{(2\ell+5)k_A} \sum_{j=0}^{(2\ell+5)k_A - 1} f(\sigma^{j + t(2\ell+5)k_A} \omega) = \alpha_{XY} \text{ for any } t \in \{0, 1, ...\}. \tag{4.5}$$

From the choice of ω and A it is also clear that

$$\frac{1}{2\ell k_A} \sum_{j=0}^{2\ell k_A - 1} f(\sigma^{j + t(2\ell+5)k_A} \omega) = \alpha^*_{f,\max} \text{ for any } t \in \{0, 1, ...\}. \tag{4.6}$$

Hence,

$$\alpha_{XY} \geq \frac{2\ell k_A \cdot \alpha^*_{f,\max} + 5 k_A \alpha_{f,\min}}{(2\ell+5)k_A} \to \alpha^*_{f,\max} \text{ as } \ell \to \infty. \tag{4.7}$$

Observe that if $U_i \in \{X, Y\}$ then there is a maximal substring of U_i which consists of consecutive zeros. This is the one which contains B, and of course might contain some zeros from the end/beginning of the As before/after B in U_i. This and the definition of $\mathbf{P}$ and g imply that for $\omega \in \mathbf{H}$

$$g(\sigma^j \omega) > 0 \text{ holds iff } j = ik_A + t(2\ell+5)k_A,\ i = 0, ..., \ell-1,\ t = 0, 1, \dots . \tag{4.8}$$

Therefore,

$$\frac{1}{(2\ell+5)k_A} \sum_{j=0}^{(2\ell+5)k_A-1} g(\sigma^{j+t(2\ell+5)k_A}\omega) = \frac{\ell\varepsilon}{4(2\ell+5)k_A} \text{ for any } t \in \{0, 1, ...\}. \tag{4.9}$$

Next we select ℓ. First we have to suppose that

$$\ell \cdot \frac{\varepsilon}{8} > 5k_A(\alpha^*_{f,\max} - \alpha_{f,\min}) \text{ and } \frac{\ell}{8(2\ell+5)} > \frac{1}{32}. \tag{4.10}$$

Then

$$\frac{2\ell k_A \alpha^*_{f,\max} + 5k_A \alpha_{f,\min} + \ell\frac{\varepsilon}{4}}{(2\ell+5)k_A} > \frac{2\ell k_A \alpha^*_{f,\max} + 5k_A \alpha^*_{f,\max} + \ell\frac{\varepsilon}{8}}{(2\ell+5)k_A} > \alpha^*_{f,\max} + \frac{\varepsilon}{32k_A}. \tag{4.11}$$

From (4.6), (4.7) and (4.11) it follows that if $h = f + g$ then for $\omega \in \mathbf{H}$

$$\frac{1}{(2\ell+5)k_A} \sum_{j=0}^{(2\ell+5)k_A-1} h(\sigma^{j+t(2\ell+5)k_A}\omega) = b^* > \alpha^*_{f,\max} + \frac{\varepsilon}{32k_A} \text{ for } t = 0, 1, \dots . \tag{4.12}$$

This obviously implies that $\mathbf{H} \subset E_h(b^*)$ and hence $S_h(b^*) = \dim_H E_h(b^*) > 0$.

If we can verify that $b^* = \alpha^*_{h,\max}$ then we are done. We need to show that if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} h(\sigma^n \omega) = \alpha \text{ then } \alpha \le b^*. \tag{4.13}$$

Suppose that we have a fixed $\omega \in \Omega$ for which the limit in (4.13) exists and equals α.

Now we subdivide ω into finitely or infinitely many substrings in the following way

$$\omega = Z_0 W_1 Z_1 W_2 Z_2 ...$$

where Z_0 might be the empty string, the other strings are non-empty. For any j the strings $W_j \in \{X, Y\}^{d_j}$, where $1 \le d_j \le +\infty$. The strings Z_j do not contain any substring of the form X or Y and they can be finite, or infinite. In case one of the Z_js is infinite then there exists N_1 such that for all $n \ge N_1$, $g(\sigma^n\omega) = 0$ and hence $\alpha \le \alpha^*_{f,\max} < b^*$.

Hence from now on we can suppose that the Z_js are finite.

If one of the W_js is infinite then one can find N_1 such that $\sigma^{N_1}\omega \in \mathbf{H}$ and hence $\alpha = b^*$ by (4.12).

Hence from now on we can suppose that all the W_js are finite.

Since for any $k \in \mathbf{N}$ we have $\omega \in E_h(\alpha)$ iff $\sigma^k\omega \in E_h(\alpha)$ we can suppose that $Z_0 = \emptyset$ and hence $\omega = W_1 Z_1 W_2 Z_2 ....$ Choose k_j, $j = 1, 2, ...$ such that the substring $W_j Z_j$ of ω starts at ω_{k_j}, that is $W_j Z_j = \omega_{k_j}\omega_{k_j+1}...\omega_{k_{j+1}-1}$. We denote by k'_j the place where Z_j starts, that is, $W_j = \omega_{k_j}\omega_{k_j+1}...\omega_{k'_j-1}$ and $Z_j = \omega_{k'_j}\omega_{k'_j+1}...\omega_{k_{j+1}-1}$.

Suppose that we have a j for which

$$\text{there exists } n \in \{k_j, ..., k_{j+1} - 1\} \text{ such that } g(\sigma^n \omega) > 0. \tag{4.14}$$

We denote the set of such js by $\mathbf{J}$.

Then $g(\sigma^n \omega) = \varepsilon/4$. We can assume that n_j is the maximal n satisfying the inequality in (4.14). Then $n_j < k'_j$. Moreover, by the definition of g and $\mathbf{P}$ we have

$$n_j = k'_j - m(2\ell + 5)k_A + (\ell - 1)k_A.$$

Put

$$k''_j = n_j - (\ell - 1)k_A + (2\ell + 5)k_A.$$

Then by the definition of g

$$\sigma^{k''_j} \omega | (2\ell + 5)k_A \in \{X, Y\} \text{ and } \sigma^{k_j} \omega | (k''_j - k_j) \in \{X, Y\}^{(k''_j - k_j)/(2\ell+5)k_A}, \tag{4.15}$$

where $(k''_j - k_j)/(2\ell + 5)k_A$ is an integer, that is $\sigma^{k_j} \omega | (k''_j - k_j)$ starts with a long string of Xs and Ys. Hence

$$\frac{1}{k''_j - k_j} \sum_{n=k_j}^{k''_j - 1} g(\sigma^n \omega) = \frac{\ell \varepsilon}{4(2\ell + 5)k_A}. \tag{4.16}$$

It is also clear that

$$\frac{1}{k_{j+1} - k''_j} \sum_{n=k''_j}^{k_{j+1} - 1} g(\sigma^n \omega) = 0. \tag{4.17}$$

Suppose that $\delta > 0$ is given. We want to find N_δ such that for $N \geq N_\delta$ we have

$$\frac{1}{N} \sum_{n=0}^{N-1} h(\sigma^n \omega) < b^* + \delta. \tag{4.18}$$

We can suppose that $\mathbf{J}$ is infinite since otherwise there exists N_1 such that $h(\sigma^n \omega) = f(\sigma^n \omega)$ for $n \geq N_1$ and $\alpha \leq \alpha^*_{f,\max} < b^*$ holds.

Now we split ω into two infinite substrings ω^g, the "good part" of ω can be obtained as the concatenation of the substrings $\sigma^{k_j} \omega | (k''_j - k_j)$, $j \in \mathbf{J}$.

While ω^b, the "bad part" of ω is the "rest" of ω, that is what is left of ω if we delete from it the good part. To be more specific if $j \notin \mathbf{J}$ then we take the string $\sigma^{k_j} \omega | (k_{j+1} - k_j)$, otherwise if $j \in \mathbf{J}$ then we take the string $\sigma^{k''_j} \omega | (k_{j+1} - k''_j)$ and concatenate these strings.

Using (4.12), (4.16), and the definition of the strings X and Y it is clear that if $j \in J$ then

$$\frac{1}{k''_j - k_j} \sum_{n=k_j}^{k''_j - 1} h(\sigma^n \omega) = b^*. \tag{4.19}$$

We also know that if $n \notin \cup_{j \in \mathbf{J}} [k_j, k''_j - k_j)$ then $g(\sigma^n \omega) = 0$ and hence $h(\sigma^n \omega) = f(\sigma^n \omega)$.

Moreover, if $(t+1)(2\ell+5)k_A \le k''_j - k_j$, for a $j \in \mathbf{J}$ then

$$\frac{1}{(2\ell+5)k_A} \sum_{n=k_j+t(2\ell+5)k_A}^{k_j+(t+1)(2\ell+5)k_A-1} h(\sigma^n\omega) = b^* \tag{4.20}$$

holds as well.

We introduce the notation $\mathbf{N}^g = \cup_{j\in\mathbf{J}}\{k_j, ..., k''_j - 1\}$ and $\mathbf{N}^b = \{0,1,...\} \setminus \mathbf{N}^g$. From (4.20) and the boundedness of h it follows that we can select N'_δ such that for $N > N'_\delta$

$$\frac{1}{\#\{n \in \mathbf{N}^g : n < N\}} \sum_{n\in\mathbf{N}^g,\ n<N} h(\sigma^n\omega) < b^* + \frac{\delta}{2}. \tag{4.21}$$

Denote $\#\{n \in \mathbf{N}^b : n < N\}$ by $\nu_b(N)$.

Next we need to estimate

$$\frac{1}{\nu_b(N)} \sum_{n\in\mathbf{N}^b,\ n<N} h(\sigma^n\omega) = \frac{1}{\nu_b(N)} \sum_{n\in\mathbf{N}^b,\ n<N} f(\sigma^n\omega). \tag{4.22}$$

A little later we will show that

$$\frac{1}{\nu_b(N)} \sum_{n\in\mathbf{N}^b,\ n<N} f(\sigma^n\omega) = \frac{1}{\nu_b(N)} \sum_{n=0}^{\nu_b(N)-1} f(\sigma^n\omega^b). \tag{4.23}$$

Next we show that if we verified this then we can complete our proof. Indeed by Lemma 3.2.1

$$\limsup_{N'\to\infty} \frac{1}{N'} \sum_{n=0}^{N'-1} f(\sigma^n\omega^b) \le \alpha^*_{f,\max}$$

and hence we can select $N_\delta \ge N'_\delta$ such that if $N \ge N_\delta$ then $\nu_b(N)$ is sufficiently large to have

$$\frac{1}{\nu_b(N)} \sum_{n=0}^{\nu_b(N)-1} f(\sigma^n\omega^b) \le \alpha^*_{f,\max} + \frac{\delta}{2}.$$

By (4.23) this yields that

$$\frac{1}{\nu_b(N)} \sum_{n\in\mathbf{N}^b,\ n<N} f(\sigma^n\omega) < \alpha^*_{f,\max} + \frac{\delta}{2} < b^* + \frac{\delta}{2}.$$

From this, (4.21), and (4.22), it follows that for $N > N_\delta$

$$\frac{1}{N} \sum_{n=0}^{N-1} h(\sigma^n\omega) < b^* + \frac{\delta}{2}.$$

Since a suitable N_δ can be chosen for any $\delta > 0$ we proved that $\alpha \le b^*$.

Hence, to complete the proof of the theorem we need to verify (4.23). But this is not difficult. Since $f \in \mathrm{PCC}^{\mathbf{k}}(\Omega)$ we know that $f(\sigma^n\omega)$ depends only on the string $\sigma^n\omega|\mathbf{k}$.

Observe that during the definition of ω^b we concatenate strings which start with a string A and A is of length $k_A > \mathbf{k}$. Indeed, if $j \notin \mathbf{J}$ then during the definition we concatenate the string $\sigma^{k_j}\omega|(k_{j+1} - k_j) = W_j Z_j$, and W_j starts with X or Y and they both start with A.

If $j \in \mathbf{J}$ then we take the string $\sigma^{k_j''}\omega|(k_{j+1} - k_j'')$ and by (4.15) this string starts with A.

We can define a function $\psi : \{0, 1, ...\} \to \mathbf{N}^b$ the following way. For $n \in \{0, 1, ...\}$ if we take ω_n^b then this entry corresponded to exactly one entry $\omega_{\psi(n)}$ of ω and belonged to a concatenated string making up ω^b. Suppose that $k_j \leq \psi(n) < k_{j+1}$. If $\psi(n) \leq k_{j+1} - \mathbf{k}$ then the strings $\sigma^n\omega^b|\mathbf{k}$ and $\sigma^{\psi(n)}\omega|\mathbf{k}$ are identical and hence $f(\sigma^n\omega^b) = f(\sigma^{\psi(n)}\omega)$. If $\psi(n) > k_{j+1} - \mathbf{k}$ then there is an $n' < n + \mathbf{k}$ such that $\psi(n') = k_{j+1}$. By our concatenation procedure it is clear that the strings $\sigma^n\omega^b|(n' - n)$ and $\sigma^{\psi(n)}\omega|(n' - n)$ are identical. It is also clear that $\psi(n') = k_{j+1}$ and $\sigma^{\psi(n')}\omega|k_A = A$, since we take the first k_A entries of a string which equals X or Y. Now recall our earlier observation that ω^b was obtained by the concatenation of strings which start with A. Hence $\sigma^{n'}\omega^b$ starts with the string A. This implies again that $f(\sigma^n\omega^b) = f(\sigma^{\psi(n)}\omega)$. □

4.2. **A generic continuous function has a continuous Birkhoff spectrum.** We have seen in the previous subsection that functions with dicontinuous spectrum form a dense set in $C(\Omega)$. Next we will show that the set of such functions is of first category.

Theorem 4.2.1. *For the generic continuous function $f \in C(\Omega)$, we have that S_f is continuous on $\mathbb{R}$.*

Remark 4.2.2. This theorem implies that the set of continuous functions with discontinuous Birkhoff spectrum is a set of first category. This set includes functions which are cohomologous to a constant, as we observed in Example 2.2.2, hence this is a possible way to see that these functions form a set of first category.

To prove Theorem 4.2.1, we need the following lemma, which shows that one can "perturb" a PCC function so that the new function would have a continuous spectrum.

Lemma 4.2.3. *Let $f \in \mathrm{PCC}^k(\Omega)$ and let $\varepsilon > 0$. Then there exists $p \in \mathbb{N}$ and $g \in C(\Omega)$ such that $\|g\| < \varepsilon$, S_{f+g} vanishes at $\alpha^*_{f+g,\max}$ and $\alpha^*_{f,\min} - \varepsilon \leq \alpha^*_{f+g,\min} \leq \alpha^*_{f+g,\max} \leq \alpha^*_{f,\max} + \varepsilon$.*

Proof. Let $f \in PCC^k(\Omega)$ and let $\varepsilon > 0$. Let ω^* be a periodic point with prime period p for which $\frac{1}{p}\sum_{n=1}^{p} f(\sigma^n\omega^*) = \alpha^*_{f,\max}$ (which exists by Lemma 3.2.2). Suppose $g_0(\omega) = \min_{i=1,\dots,p}\{d(\omega, \sigma^i\omega^*)\}$, and let $g = -\varepsilon g_0 + c$ for some $c \in (0, \varepsilon)$ chosen in a way that $\int g d\lambda = 0$. Since $\lambda(\Omega) = \mathrm{diam}(\Omega) = 1$, it is clear that $\|g\| < \varepsilon$.

Given $E \subset \mathbb{N}$, we denote by $\mathbf{d}(E)$ the density of the set E, that is $\lim_{N\to\infty} \frac{\#(E\cap[1,N])}{N}$ (if it exists). We let

$$H_{\omega^*} := \{\omega \in \Omega : \omega|_E = \omega^* \text{ where } \mathbf{d}(E^c) = 0\},$$

where $\omega|_E$ denotes the concatenation of ω_j, $j \in E$. We will show that $E_{f+g}(\alpha^*_{f+g,\max}) \subset H_{\omega^*}$, and then we observe that $\dim_H H_{\omega^*} = 0$.

By using (3.2) from Lemma 3.2.1 one can see that $\alpha^*_{f+g,\max} \le \alpha^*_{f,\max} + c$. Since $g_0(\sigma^n\omega^*) = 0$ for any n, we obtain $\alpha^*_{f+g,\max} \ge \alpha^*_{f,\max} + c$, and hence $\alpha^*_{f+g,\max} = \alpha^*_{f,\max} + c$. Let $\omega \in E_{f+g}(\alpha^*_{f+g,\max})$. Then we must have

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) - \frac{\varepsilon}{N}\sum_{n=1}^{N} g_0(\sigma^n\omega)\right) = \alpha^*_{f,\max},$$

and this is only possible if $\frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega) \to \alpha^*_{f,\max}$, and, in particular,

$$\frac{1}{N}\sum_{n=1}^{N} g_0(\sigma^n\omega) \to 0 \text{ as } N\to\infty.$$

This implies that the set

$$J_\omega := \{n\in\mathbb{N} : g_0(\sigma^n\omega) \ge 2^{-p}\}$$

has zero density. Observe that if $g_0(\sigma^n\omega) < 2^{-p}$ for $n = j', ..., j'+l$ then there exists $i \in \{0, ..., p-1\}$ such that

$$(\sigma^n\omega)_{j'}^{j'+l+p} = \sigma^i\omega^*|l+p+1. \tag{4.24}$$

The case when J_ω is finite is much easier and is left to the reader, we detail only the case when J_ω is infinite.

Suppose we enumerate $J_\omega = \{j_1, j_2, j_3, \ldots\}$ in the increasing order and we set $j_0 := 1$. Then for each $k \in \mathbb{N}\cup\{0\}$, there exists $i_k \in \{0, ..., p-1\}$ such that the (possibly empty) string $\gamma(j_k) := (\omega)_{j_k+1}^{j_{k+1}-1}$ equals $\sigma^{i_k}\omega^*|j_{k+1} - j_k - 1$. Hence, we have

$$\omega|_{J_\omega^c} = \gamma(j_0)\gamma(j_1)\gamma(j_2)\cdots.$$

Since ω^* is periodic we can choose $m_k \in \{0, ..., p-1\}$ such that if $\gamma^*(j_k) = \sigma^{m_k}\gamma(j_k)$, that is we throw away the first m_k entries of $\gamma(j_k)$, then

$$\gamma^*(j_0)\gamma^*(j_1)\gamma^*(j_2)\cdots = \omega^*.$$

Put $F = \cup_k\{j_k, ..., j_k + m_k\}$. Then $F \subset \bigcup_{i=0}^{p-1} J_\omega + i$ (where $A + b = \{a + b : a \in A\}$ for any $A \subset \mathbb{N}$ and $b \in \mathbb{N}$), which has a zero density. Setting $E = F^c$, we get $\omega|_E = \omega^*$. Hence, $\omega \in H_{\omega^*}$, which shows that $E_{f+g}(\alpha^*_{f+g,\max}) \subset H_{\omega^*}$.

We now show that $\dim_H H_{\omega^*} = 0$. Consider the set $H_{\mathbf{0}} := \{\omega\in\Omega : \mathbf{d}(\{i\in\mathbb{N} : \omega_i = 1\}) = 0\}$. Due to Example 2.2.1 we see that $\dim_H(H_{\mathbf{0}}) = 0$ as it equals $S_f(0)$ for f defined in that example. Given $\omega\in\Omega$ and $i\in\mathbb{N}$ we set $\nu(i,\omega) = \#\{j : \omega_j = 0,\ j \le i\}$. We define a map $h : \Omega\to\Omega$ as follows: $h(\omega) = h_1h_2h_3\ldots$, where

$$h_i := \begin{cases} \omega^*_{\nu(i,\omega)} & \text{if } \omega_i = 0 \\ 1 - \omega^*_{\nu(i,\omega)+1} & \text{if } \omega_i = 1. \end{cases}$$

It is easy to see that h is Lipschitz. One can also verify easily that $h(H_{\mathbf{0}}) \supset H_{\omega^*}$. Therefore, $0 \le \dim_H(H_{\omega^*}) \le \dim_H(h(H_{\mathbf{0}})) = 0$. □

What remains from the proof of Theorem 4.2.1 is rather standard:

Proof of Theorem 4.2.1. It suffices to prove that a generic continuous function h has continuous spectrum at the points $\alpha^*_{h,\min}$ and $\alpha^*_{h,\max}$, and due to symmetry reasons, it suffices to prove the continuity in $\alpha^*_{h,\max}$ (if it holds in a residual set, the other also does in another residual set, and the intersection of these sets is still residual). We will prove in fact that the set

$$Z = \{h \in C(\Omega) : S_h \text{ is not continuous at } \alpha^*_{h,\max}\}$$

is meager. Note that we know that S_h is concave and achieves its maximum at $\int h \, d\lambda$, hence

$$Z = \bigcup_{n=1}^{\infty} Z_{\frac{1}{n}},$$

where

$$Z_\theta = \left\{ h \in C(\Omega) : S_h(x) > \theta \text{ for all } x \in \left[\int h \, d\lambda, \alpha^*_{h,\max} \right] \right\}.$$

Now it suffices to prove that each Z_θ is nowhere dense, and clearly it is enough to consider small enough $\theta < 1$. To this end, take arbitrary $f \in \mathrm{PCC}^k(\Omega)$ for some k, and $\varepsilon > 0$. By Lemma 4.2.3, we can find $f + g$ in the ε-neighborhood of f such that it has continuous spectrum at $\alpha^*_{f+g,\max}$. Then $\alpha^*_{f+g,\max} > \int(f+g) \geq \alpha^*_{f+g,\min}$ necessarily holds, as $S_{f+g}(\int(f+g)d\lambda) = 1$. Now by continuity, we can take $x \in \left[\int h, \alpha^*_{f+g,\max}\right]$ such that $0 < S_{f+g}(x) < \frac{\theta}{2}$. By its concavity S_{f+g} is monotone decreasing on $[\int h, \alpha^*_{f+g,\max}]$ hence we can assume that

$$x - \alpha^*_{f,\min} \geq \alpha^*_{f,\max} - x.$$

Now apply Theorem 3.1.1 for $f + g$ with

$$\varepsilon' = \min\left\{\frac{\theta}{2}, \alpha^*_{f,\max} - x\right\}. \tag{4.25}$$

It guarantees that $0 < S_h(x) < \theta$ for any h with $\|h - (f+g)\| < \delta'$ for some $\delta' > 0$. Moreover, if h and $f + g$ are close enough to each other, we also have that their integral cannot differ by much, hence we also have that $x \in \left[\int h, \alpha^*_{h,\max}\right]$. Consequently, if h is in a sufficiently small neighborhood of $f + g$ satisfying both this integral condition and what is given by (4.25), then h is not in Z_θ. It yields that Z_θ is nowhere dense, as $PCC(\Omega)$ is dense, and in the neighborhood of an arbitrary f belonging to this set we constructed an open ball which is disjoint from Z_θ. It concludes the proof. □

4.3. **Supports of generic spectra are in** $(\alpha_{f,\min}, \alpha_{f,\max})$**.** In Example 2.2.1 we saw a very simple PCC function for which the range of the function $[\alpha_{f,\min}, \alpha_{f,\max}]$ coincides with the support of the spectrum $[\alpha^*_{f,\min}, \alpha^*_{f,\max}]$. In this subsection we verify that for the generic continuous function this is not true, we have (4.26), in fact we prove a little more, we show that the set of functions having this property is comeager.

Theorem 4.3.1. *For a dense open set $\mathcal{G} \subset C(\Omega)$ we have*

$$\alpha_{f,\min} < \alpha^*_{f,\min} < \alpha^*_{f,\max} < \alpha_{f,\max} \tag{4.26}$$

hence the generic $f \in C(\Omega)$ satisfies (4.26).

Proof. It suffices to prove that each inequality in (4.26) holds in a dense open subset of $C(\Omega)$, and due to symmetry, it is sufficient to prove that $\alpha^*_{f,\min} < \alpha^*_{f,\max}$ and $\alpha^*_{f,\max} < \alpha_{f,\max}$ hold in dense open subsets, respectively. Given Remark 3.1.4, it immediately follows that each of these inequalities holds in an open subset, thus we only have to keep an eye on denseness.

Consider first $\alpha^*_{f,\min} < \alpha^*_{f,\max}$. By Theorem 4.2.1 we know that S_f is continuous for $f \in G_1$ with a dense subset $G_1 \subset C(\Omega)$. However, for $\alpha_\lambda = \int f d\lambda$ we have $S_f(\alpha_\lambda) = 1$, and $S_f(\alpha^*_{f,\min}) = S_f(\alpha^*_{f,\max}) = 0$, hence

$$\alpha^*_{f,\min} < \alpha^*_{f,\max}. \tag{4.27}$$

It yields that for any $f \in G_1$ we have $\alpha^*_{f,\min} < \alpha^*_{f,\max}$, thus this inequality holds in a dense subset indeed.

Let us consider now $\alpha^*_{f,\max} < \alpha_{f,\max}$. We know that functions $f \in \mathrm{PCC}(\Omega)$ are dense in $C(\Omega)$. Consider such a function f, we have $f \in \mathrm{PCC}^k(\Omega)$ for some $k > 0$. By Lemma 3.2.2 we know that there exists a periodic configuration ω_f with $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega_f) = \alpha^*_{f,\max}$. If $\alpha^*_{f,\max} < \alpha_{f,\max}$ then we are done. Hence we can suppose that $\alpha^*_{f,\max} = \alpha_{f,\max}$. Assume first that ω_f can be chosen such that ω_f is neither identically 1^∞ nor 0^∞. Then we can choose a substring A of length k such that f is maximal on $[A]$ and A is neither $[11\cdots1]$ nor $[00\cdots0]$ (i.e. cylinders of k many 1s or 0s, respectively). Now for given $\varepsilon > 0$ define $g \in \mathrm{PCC}^k(\Omega)$ such that $f = g$ except on the cylinder $[A]$ where $g = f + \varepsilon$. Set ω_g to be a periodic configuration for which $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g(\sigma^n\omega_g) = \alpha^*_{g,\max}$, which is again guaranteed to exist by Lemma 3.2.2. The relative frequency of the substring A in ω_g is strictly smaller than 1, as A contains both 0s and 1s, hence at least $1/k$ of the substrings start with a binary digit different from the first entry in A. Thus we can conclude

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g(\sigma^n\omega_g) - \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\sigma^n\omega_f) < \|g - f\| = \varepsilon,$$

hence

$$\alpha^*_{g,\max} - \alpha^*_{f,\max} < \varepsilon.$$

However,

$$\alpha_{g,\max} - \alpha_{f,\max} = \varepsilon$$

by definition. Hence we can find g arbitrarily close to f with $\alpha^*_{g,\max} < \alpha_{g,\max}$ in this case.

Assume now that the only possible choices for ω_f are amongst 1^∞ and 0^∞. If A can be chosen as in the first case, differing from the identically 1 and identically 0 strings of length k, then the previous argument might be repeated, thus it suffices to observe the cases when ω_f and A can only be identically 1 or identically 0. Clearly without loss of generality we can assume that the former one holds. In this case we perturb f as follows: let $g \in \mathrm{PCC}^{k+1}(\Omega)$ such that it equals f everywhere, except on the $(k+1)$-cylinder which starts with k many 1s and ends with a 0. On this cylinder let $g = f + \varepsilon$ such that g is very close to f. Then $\alpha_{g,\max} - \alpha_{f,\max} = \varepsilon$ as previously. Moreover, if $\varepsilon = \|g - f\|$ is sufficiently small, by the conditions of this case the only maximizing periodic configuration for g is 1^∞, too. Hence $\alpha^*_{g,\max} = \alpha^*_{f,\max}$, which immediately yields $\alpha^*_{g,\max} < \alpha_{g,\max}$ again.

Thus in both cases we showed that any $f \in \mathrm{PCC}^k(\Omega)$ can be approximated by functions satisfying $\alpha^*_{g,\max} < \alpha_{g,\max}$. It yields that such functions also form a dense set, which concludes the proof. □

Remark 4.3.2. In ergodic optimization, a function $f \in C(\Omega)$ for which $\alpha^*_{f,\max} = \alpha_{f,\max}$ is called *revealed* (cf. [10, §5]). Theorem 4.3.1 tells us that the set of revealed functions in $C(\Omega)$ forms a nowhere dense set.

5. One-sided derivatives of the Birkhoff spectra at endpoints

In this section for functions with continuous spectrum we are interested in the one-sided derivatives of the spectrum at the endpoints of its support in the direction of the interior of the support.

5.1. **One-sided derivatives at the endpoints of spectra for generic functions.** For the generic continuous function we have already seen in Theorem 4.2.1 that the spectrum is continuous at these endpoints, and as in the direction of the exterior of L_f the spectrum is constant zero, the one-sided derivative is also zero. On the other hand, towards the interior of the support it is of infinite absolute value as we see in the next theorem.

Theorem 5.1.1. *For the generic continuous function $f \in C(\Omega)$, we have $\partial^- S_f(\alpha^*_{f,\max}) = -\infty$, while $\partial^+ S_f(\alpha^*_{f,\min}) = \infty$.*

We start with a lemma which will be the building block for the proof of the above theorem.

Lemma 5.1.2. *Let $f_0 \in C(\Omega)$, $\varepsilon > 0$, and $\nu \in \mathbb{N}$ be given. Then there exists $f_2 \in C(\Omega)$ and $\delta > 0$ such that $\|f_0 - f_2\| < \varepsilon/2$, $\delta < \varepsilon/2$, and for any $f \in B(f_2, \delta) \subset B(f_0, \varepsilon)$ there exists $\alpha' < \alpha^*_{f,\max}$ such that*

$$\frac{S_f(\alpha') - S_f(\alpha^*_{f,\max})}{\alpha' - \alpha^*_{f,\max}} < -\nu. \tag{5.1}$$

Remark 5.1.3. As S_f is concave on the interval L_f, the inequality (5.1) in the lemma implies $\partial^- S_f(\alpha^*_{f,\max}) < -\nu$.

Proof. Using Theorem 4.1.1 choose $f_1 \in PCC(\Omega)$ with $\|f_0 - f_1\| < \varepsilon/4$ such that $\varepsilon_1 = S_{f_1}(\alpha^*_{f_1,\max}) > 0$.

Set $\varepsilon_2 = \min\{\varepsilon_1, \frac{\varepsilon}{2}, 1/2\}$.

Using Theorem 4.2.1 choose $f_2 \in C(\Omega)$ such that

$$\|f_1 - f_2\| < \frac{\varepsilon_2}{10\nu} \text{ and } S_{f_2}(\alpha^*_{f_2,\max}) = 0.$$

By Lemma 3.1.3 and Remark 3.1.4 applied to f_1 and f_2 we obtain that $\alpha^*_{f_2,\max} < \alpha^*_{f_1,\max} + \frac{\varepsilon_2}{10\nu}$ and there exists $\alpha' > \alpha^*_{f_1,\max} - \frac{\varepsilon_2}{10\nu}$ such that

$$S_{f_2}(\alpha') \geq S_{f_1}(\alpha^*_{f_1,\max}) = \varepsilon_1 \geq \varepsilon_2. \tag{5.2}$$

Then

$$\alpha^*_{f_2,\max} - \alpha' < 2 \cdot \frac{\varepsilon_2}{10\nu}. \tag{5.3}$$

Keep in mind that $S_{f_2}(\alpha^*_{f_2,\max}) = 0$ and choose $\delta_1 > 0$ such that

$$S_{f_2}(\alpha) < \frac{\varepsilon_2}{20} \text{ holds for } \alpha \in (\alpha^*_{f_2,\max} - \delta_1, \alpha^*_{f_2,\max}]. \tag{5.4}$$

Observe that from (5.2) it also follows that $\alpha^*_{f_2,\min} \leq \alpha' < \alpha^*_{f_2,\max} - \delta_1$. Now choose $\delta_2 > 0$ such that

$$\delta_2 < \min\left\{\frac{\alpha^*_{f_2,\max} - \alpha'}{10}, \frac{\delta_1}{5}, \frac{\varepsilon_2}{20\nu}\right\}. \tag{5.5}$$

Using this δ_2 as ε in Theorem 3.1.1 select $\delta \in (0, \delta_2)$ such that for $f \in B(f_2, \delta)$ we have

$$|S_f(\alpha) - S_{f_2}(\alpha)| < \delta_2 \text{ for } \alpha \in (\alpha^*_{f_2,\min} + \delta_2, \alpha^*_{f_2,\max} - \delta_2). \tag{5.6}$$

Suppose $f \in B(f_2, \delta)$. Then by Lemma 3.1.3, Remark 3.1.4, (5.3) and (5.5) we obtain

$$|\alpha^*_{f,\max} - \alpha^*_{f_2,\max}| < \delta_2 \text{ and hence } |\alpha' - \alpha^*_{f,\max}| < 1.1(\alpha^*_{f_2,\max} - \alpha') < 1.1 \cdot \frac{\varepsilon_2}{5\nu}.$$

By (5.4), $S_{f_2}(\alpha^*_{f_2,\max} - \delta_1/2) < \varepsilon_2/20$ and then by (5.6), $S_f(\alpha^*_{f_2,\max} - \delta_1/2) < \varepsilon_2/10 < 1$. By concavity of S_f and $S_f(\int f) = 1$ it is clear that S_f is monotone decreasing on $[\alpha^*_{f,\max} - \delta_1/2, \alpha^*_{f,\max}]$ and hence

$$S_f(\alpha^*_{f,\max}) < \frac{\varepsilon_2}{10}. \tag{5.7}$$

Using (5.2), (5.5) and (5.6) we infer

$$S_f(\alpha') > S_{f_2}(\alpha') - \delta_2 \geq 0.9\varepsilon_2.$$

By this, (5.7) and (5.3)

$$\frac{S_f(\alpha') - S_f(\alpha^*_{f,\max})}{\alpha' - \alpha^*_{f,\max}} < -\frac{0.8\varepsilon_2}{1.1 \cdot \frac{\varepsilon_2}{5\nu}} < -\nu.$$

□

Remark 5.1.4. We remark that due to symmetry reasons a version of Lemma 5.1.2 also holds at the other endpoint, $\alpha^*_{f,\min}$ of the spectrum yielding that for any $f \in B(f_2, \delta) \subset B(f_0, \varepsilon)$ there exists $\alpha' > \alpha^*_{f,\min}$ such that

$$\frac{S_f(\alpha') - S_f(\alpha^*_{f,\min})}{\alpha' - \alpha^*_{f,\min}} > \nu. \tag{5.8}$$

As we observed earlier in the one-dimensional case S_f is continuous on $[\alpha^*_{f,\min}, \alpha^*_{f,\max}]$ hence even in case of discontinuous spectra one can consider $\partial^- S_f(\alpha^*_{f,\max})$ and $\partial^+ S_f(\alpha^*_{f,\max})$, one might have a one-sided discontinuity only in the direction pointing towards the exterior of the support of the spectrum.

Lemma 5.1.2 easily implies Theorem 5.1.1:

Proof of Theorem 5.1.1. Consider an arbitrary $f_0 \in C(\Omega)$ and $\varepsilon > 0$. Fix $\nu \in \mathbb{N}$. We may apply Lemma 5.1.2 and Remark 5.1.3 to see that $B(f_0, \varepsilon)$ contains a smaller open set $B(f_2, \delta)$ of $C(\Omega)$ such that for any $f \in B(f_2, \delta)$ we have $\partial^- S_f(\alpha^*_{f,\max}) < -\nu$. It implies that the complement of

$$A_\nu = \{f \in C(\Omega) : \partial^- S_f(\alpha^*_{f,\max}) < -\nu\}$$

is nowhere dense for any ν. Hence $A = \bigcup_{\nu=1}^{\infty} A_\nu$ is a residual set of $C(\Omega)$, yielding that for the generic continuous function $f \in C(\Omega)$, we have $\partial^- S_f(\alpha^*_{f,\max}) = -\infty$.

However, by Remark 5.1.4 we may conclude the same way that for the generic continuous function $f \in C(\Omega)$, we have $\partial^+ S_f(\alpha^*_{f,\min}) = \infty$. Thus for the generic continuous function, we have both of these prescribed equalities, which concludes the proof. $\square$

5.2. **Finite one-sided derivatives at the endpoints of the spectrum.** Now our goal is to construct a continuous function f with the property that the spectrum S_f is continuous, but it is not generic in the above sense, that is the one-sided derivatives in the endpoints $\alpha^*_{f,\min}$ and $\alpha^*_{f,\max}$ are finite.

Theorem 5.2.1. *There exists $f \in C_0(\Omega)$ such that S_f is continuous, $\alpha^*_{f,\min} = -1$ and $\alpha^*_{f,\max} = 1$, and $\partial^- S_f(\alpha^*_{f,\max}) > -\infty$, while $\partial^+ S_f(\alpha^*_{f,\min}) < \infty$. Moreover, these derivatives can be arbitrarily close to -1 and 1, respectively.*

The first step in this direction is the following lemma, in which we give upper bounds on a value of the spectrum for a suitably defined function. Since $S_f(\int f d\lambda) = 1$ if we have a function with continuous spectrum then by concavity of the spectrum $\partial^- S_f(\alpha^*_{f,\max}) \leq -1/(\alpha^*_{f,\max} - \int f d\lambda)$ and $\partial^+ S_f(\alpha^*_{f,\min}) \geq 1/(\int f d\lambda - \alpha^*_{f,\min})$.

In the next Lemma we define a PCC function with "very small" spectrum. This type of functions serve as building blocks in the proof of Theorem 5.1.1.

Lemma 5.2.2. *Let $b > a$, and let $f : \Omega \to \mathbb{R}$ be such that $f(\omega) = b$ if the first L coordinates of ω is 1, otherwise $f(\omega) = a$. Moreover, fix $\varepsilon > 0$ and $0 < \beta < 1$. Then if L is sufficiently large, then*

$$S_f(t) \leq \beta + \varepsilon \tag{5.9}$$

for $t = \beta a + (1-\beta)b$.

Remark 5.2.3. Observe that in the above lemma if L is large then $\int f d\lambda = b \cdot 2^{-L} + a(1 - 2^{-L})$ and hence $S_f(b \cdot 2^{-L} + a \cdot (1 - 2^{-L})) = 1$. The point $b \cdot 2^{-L} + a \cdot (1 - 2^{-L})$ is very close to $a = \alpha_{f,\min}$. It is also clear that $E_f(b) \neq \emptyset$, since 1^∞ belongs to it. By also considering 0^∞ we see that $[a, b] = [\alpha^*_{f,\min}, \alpha^*_{f,\max}]$. Hence the line segment connecting $(b \cdot 2^{-L} + a \cdot (1-2^{-L}), 1)$ to $(b, 0)$ should be under the graph of S_f on $[b \cdot 2^{-L} + a \cdot (1-2^{-L})]$. If β is small then t is very close to b and by concavity of the spectrum on $[b \cdot 2^{-L} + a \cdot (1-2^{-L}), t]$ the graph of S_f should be under the line segment connecting $(t, \beta+\varepsilon) = (\beta a + (1-\beta)b, \beta+\varepsilon)$ to $(b, 0)$. This implies that for small β and large L apart from a very short interval near the endpoint a the spectrum S_f is very close to the line segment connecting $(a, 1)$ to $(b, 0)$ and on $[a, b]$ approximates the upper part of the boundary of the right angled triangle with vertices $(a, 0)$, $(a, 1)$ and $(b, 0)$.

Proof. Let $t = \beta a + (1-\beta)b$. Clearly it suffices to prove the statement of the lemma for small enough ε, thus we might assume that $\beta^* = \beta + \frac{\varepsilon}{2} < 1$. We would like to estimate the dimension of

$$E_f(t) = \left\{\omega : \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\sigma^n \omega) = t\right\}.$$

This set contains ω if and only if it contains $\sigma(\omega)$, thus we can shift the sum by one for technical convenience. Moreover, if we replace the lim by a lim inf, we can deduce that this set is contained by

$$\left\{\omega : \liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t\right\}.$$

If ω is in this set, then for large enough N the corresponding ergodic average exceeds $t^* = \beta^* a + (1-\beta^*)b < t$, that is

$$E_f(t) \subset \bigcup_{m=1}^{\infty} \bigcap_{N=m}^{\infty} \left\{\omega : \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t^*\right\}. \tag{5.10}$$

In the sequel for ease of notation we will use $\left\{\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t^*\right\}$ instead of $\left\{\omega : \frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t^*\right\}$. The union in (5.10) is the union of a growing sequence of sets, thus the dimension is simply the limit of $\dim_H A_m$, where

$$A_m = \bigcap_{N=m}^{\infty} \left\{\frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t^*\right\}.$$

Now we focus on estimating the dimension of this set. To this end, we would like to count the cylinder sets of length $N+L-1$ which intersect $\left\{\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t^*\right\}$ for large N, as they give a cover of A_m for any $N \geq m$. (We are concerned with cylinders of length $N+L-1$ instead of the ones with length N as the first $N+L-1$ coordinates affect $\sum_{n=0}^{N-1} f(\sigma^n \omega)$.) For our purposes it suffices to choose N such that $L | N+L-1$, as we can diverge to infinity with N even under this restriction and we need an upper estimate of the dimension.

The number of blocks consisting of at least L consecutive 1s is at most $\frac{N+L-1}{L}$. If $L \geq 2$, and there are i such blocks, the number of ways to place them among the $N+L-1$ coordinates is at most $\binom{N+L-1}{2i}$, since the placement of each block can be uniquely specified by the coordinates for which the first and the last coordinates of the block occupy. (We note that it is indeed an upper estimate: this expression does not deal with the length of the blocks, neither with the fact that blocks are separated from each other with at least one intermediate coordinate.) Moreover, if $L \geq 5$, then for the largest possible value of i, that is for $i = \frac{N+L-1}{L}$ we still have

$$2i = 2 \cdot \frac{N+L-1}{L} < \frac{N+L-1}{2}.$$

Thus the number of ways we can arrange the blocks of at least L consecutive 1s is at most

$$\sum_{i=0}^{\frac{N+L-1}{L}} \binom{N+L-1}{2i} \leq \left(\frac{N+L-1}{L} + 1\right) \cdot \binom{N+L-1}{2 \cdot \frac{N+L-1}{L}} \tag{5.11}$$

$$\leq (N+L-1) \cdot \binom{N+L-1}{2 \cdot \frac{N+L-1}{L}},$$

as the binomial coefficients are increasing until the middle ones.

We should also give a bound on the number of ways we can choose the other coordinates. Since $\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t^*$, we know that most of the coordinates belong to one of the above blocks. More specifically, in the first N coordinates there are at most $\beta^* N$ not covered by them, as otherwise the number of terms in $\sum_{n=0}^{N-1} f(\sigma^n \omega)$ with $f(\sigma^n \omega) = a$ exceeds $\beta^* N$, which yields that

$$\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n \omega) < \beta^* a + (1-\beta^*) b = t^*.$$

Thus a raw upper estimate for the number of the ways we can choose the remaining coordinates in order to have an $N+L-1$-cylinder intersecting

$$\left\{ \frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t^* \right\}$$

is $2^{\beta^* N} \cdot 2^{L-1}$, where the last factor is simply the number of ways we can choose the last $L-1$ coordinates.

Combining the results of the preceding two paragraphs yields that

$$\left\{ \frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t^* \right\}$$

is covered by at most

$$(N+L-1) \cdot \binom{N+L-1}{2 \cdot \frac{N+L-1}{L}} \cdot 2^{\beta^* N + L - 1}$$

many cylinders of diameter $2^{-(N+L-1)}$. By using the standard $\binom{a}{b} \leq \left(\frac{ae}{b}\right)^b$ bound on the binomial coefficients, we can relax this upper bound to

$$(N+L-1) \cdot \left(\frac{eL}{2}\right)^{2 \cdot \frac{N+L-1}{L}} \cdot 2^{\beta^* N + L - 1} = k \cdot \left(\frac{eL}{2}\right)^{\frac{2k}{L}} \cdot 2^{\beta^* k} \cdot 2^{(1-\beta^*)(L-1)}, \tag{5.12}$$

where $k = N + L - 1$. Notice that for large enough L (and consequently, large enough k) we have

$$2^{\frac{\varepsilon}{2}} > \sqrt[k]{k} \left(\frac{eL}{2}\right)^{\frac{2}{L}},$$

as both factors on the right tend to 1. Fix L to be sufficiently large in order to guarantee this. Consequently, (5.12) can be estimated from above by

$$2^{(\beta^*+\frac{\varepsilon}{2})k}\cdot 2^{(1-\beta^*)(L-1)}. \tag{5.13}$$

Hence

$$\left\{\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega)\geq t^*\right\}$$

can be covered by at most $2^{(\beta^*+\frac{\varepsilon}{2})k}\cdot 2^{(1-\beta^*)(L-1)}$ many cylinders of diameter 2^{-k} for any k with $L|k$. It immediately yields

$$\mathcal{H}^{\beta^*+\frac{\varepsilon}{2}}_{2^{-k}}\left(\left\{\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega)\geq t^*\right\}\right)\leq 2^{(1-\beta^*)(L-1)}$$

where $N=k-L+1$ as before. However, this set contains A_m for large enough k, N, thus

$$\mathcal{H}^{\beta^*+\frac{\varepsilon}{2}}_{2^{-k}}(A_m)\leq 2^{(1-\beta^*)(L-1)}.$$

As k, N can be arbitrarily large, it shows that in fact

$$\mathcal{H}^{\beta^*+\frac{\varepsilon}{2}}(A_m)\leq 2^{(1-\beta^*)(L-1)}$$

and consequently,

$$\dim_H(A_m)\leq \beta^*+\frac{\varepsilon}{2}=\beta+\varepsilon.$$

Consequently, by our initial observations

$$S_f(t)\leq\beta+\varepsilon,$$

as stated. □

We do not know whether there is a PCC function with finite one-sided derivatives at the endpoints of the spectrum. The following theorem might make one believe that the answer to this question is negative:

Theorem 5.2.4. *Assume that $f\in\mathrm{PCC}(\Omega)$ and S_f is continuous. Then $\partial^- S_f(\alpha^*_{f,\max})=-\infty$, while $\partial^+ S_f(\alpha^*_{f,\min})=\infty$.*

Proof. Choose k such that $f\in\mathrm{PCC}^k(\Omega)$. By symmetry, it clearly suffices to prove $\partial^- S_f(\alpha^*_{f,\max})=-\infty$. Consider the directed graph $G=(V,E)$ defined in the proof of Lemma 3.2.2, and the set $\mathcal{C}$ of its cycles. By that reasoning it is clear that there exist cycles with distinct weight averages as otherwise for any infinite path Γ we would get the same weight average in limit, which means that the ergodic averages have the same limit for all configurations, hence S_f cannot be continuous. Moreover, as G is connected as a directed graph, the graph of cycles $G_{\mathcal{C}}$ is also connected, in which the vertices are the elements of $\mathcal{C}$, and two of them are connected if they have a common vertex. This, together with our previous observation implies that we can choose cycles C and C' such that they have a common vertex v, the cycle C has maximal weight average amongst the elements of $\mathcal{C}$, while C' does not. Now consider the set of infinite paths in G denoted by H_β which consists of the paths which start from v, and can be partitioned into finite pieces $\Gamma_1,\Gamma_2,\ldots$ such that each

Γ_i equals either C or C', and the density $\mathbf{d}(\{i : \Gamma_i = C\}) = \beta$. Then it is obvious to see that the weight average along any $\Gamma \in H_\beta$ tends to

$$\beta \cdot \frac{1}{|C|}\sum_{e\in C} f(e) + (1-\beta)\cdot \frac{1}{|C'|}\sum_{e\in C'} f(e) = \beta\alpha^*_{f,\max} + (1-\beta)\alpha',$$

where $\alpha' < \alpha^*_{f,\max}$ by the choice of C'. Thus if we take the corresponding configuration $\omega(\Gamma)$, and in the ergodic averages we shift the indexing again by one, we see that

$$\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega(\Gamma)) \to \beta\alpha^*_{f,\max} + (1-\beta)\alpha'.$$

That is, if Ω_β denotes the set of $\omega(\Gamma)$s for which $\Gamma \in H_\beta$, we have

$$\Omega_\beta \subseteq E_f(\beta\alpha^*_{f,\max} + (1-\beta)\alpha'). \tag{5.14}$$

However, the dimension of Ω_β is easy to estimate from below using the following mapping: for $\omega(\Gamma) \in \Omega_\beta$ define $h(\omega(\Gamma)) = h_1h_2...$ by

$$h_i := \begin{cases} 1 & \text{if } \Gamma_i = C \\ 0 & \text{if } \Gamma_i = C'. \end{cases}$$

Now h is a Hölder-mapping. Note that the starting point of Γ determines the first k coordinates of $\omega(\Gamma)$, and then going along C (resp. C') determines the next $|C|$ (resp. $|C'|$) coordinates. By reversing this argument, if $K = \max\{|C|, |C'|\}$, the first $k + mK$ coordinates of $\omega(\Gamma)$ uniquely determine the cycles $\Gamma_1, ..., \Gamma_m$ in the decomposition of Γ. In other words, the first m coordinates of $h(\omega(\Gamma))$ are uniquely determined by the first $k + mK$ coordinates of $\omega(\Gamma)$. From this, one easily obtains that h is a Hölder-$1/K$ mapping.

Moreover, by the definition of H_β and Ω_β, it is clear that $h(\Omega_\beta)$ equals the set of configurations in which the density of 1s equals β. Thus by Example 2.2.1, we can deduce that

$$\dim_H(h(\Omega_\beta)) = -\frac{\beta\log(\beta) + (1-\beta)\log(1-\beta)}{\log 2}.$$

Hence as h was Hölder-$1/K$:

$$\dim_H(\Omega_\beta) \geq -\frac{\beta\log(\beta) + (1-\beta)\log(1-\beta)}{K\log 2}.$$

Thus by (5.14):

$$S_f(\beta\alpha^*_{f,\max} + (1-\beta)\alpha') \geq -\frac{\beta\log(\beta) + (1-\beta)\log(1-\beta)}{K\log 2}.$$

Consequently, also using that by continuity of S_f we have $S_f(\alpha^*_{f,\max}) = 0$ we infer

$$\frac{S_f(\alpha^*_{f,\max}) - S_f(\beta\alpha^*_{f,\max} + (1-\beta)\alpha')}{\alpha^*_{f,\max} - (\beta\alpha^*_{f,\max} + (1-\beta)\alpha')} \leq \frac{\beta\log(\beta) + (1-\beta)\log(1-\beta)}{(1-\beta)(\alpha^*_{f,\max} - \alpha')K\log 2}$$

However, the right hand side can be estimated from above by omitting the negative first term, and after simplifying by $1-\beta$ we see that it tends to $-\infty$ as $\beta \to 1$. Hence the same holds for the left hand side, showing that $\partial^- S_f(\alpha^*_{f,\max}) = -\infty$. $\square$

Given Theorem 5.2.4, it seems to be reasonable to look for a function verifying the statement of Theorem 5.2.1, which is not in PCC(Ω). Hence we need to "iterate" the idea used in Lemma 5.2.2.

Proof of Theorem 5.2.1. We define f to be a more elaborate variant of the function appearing in Lemma 5.2.2. More specifically, we will define a strictly increasing sequence (t_j) with terms in $(0,1)$ such that $t_j \to 1$, and we will also define a strictly increasing sequence (L_j) of positive integers, to be fixed later and chosen recursively. We can suppose that $L_1 > 5$.

Now we let $f(\omega) = t_j$ if ω starts with a block of 1s of length at least L_j, but less than L_{j+1}. Moreover, $f(\omega) = -t_j$ if ω starts with a block of 0s of length at least L_j, but less than L_{j+1}. Finally, let $f(1^\infty) = 1$ and $f(0^\infty) = -1$ for the constant sequences, and let $f(\omega) = 0$ for any remaining ω. Due to symmetry, it is clear that $\int f = 0$, and it is straightforward to check continuity. It remains to prove that the relevant derivatives are finite. By symmetry again, it suffices to verify $\partial^- S_f(\alpha^*_{f,\max}) > -\infty$. To this end, we will use an argument similar to the one seen in the proof of Lemma 5.2.2. The importance of the actual choice of the sequence (t_j) is limited to technicalities, in the following we will choose $t_j = 1 - 2^{-j}$.

As in (5.10), we can deduce

$$E_f(t_{j+1}) \subset \bigcup_{m=1}^{\infty} \bigcap_{N=m}^{\infty} \left\{ \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t_j \right\}.$$

This union is the union of a growing sequence of sets, thus the dimension is simply the limit of $\dim_H A_m$, where

$$A_m = \bigcap_{N=m}^{\infty} \left\{ \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n \omega) \geq t_j \right\}.$$

In order to estimate this dimension, we first introduce an auxiliary function, which is easier to examine. Explicitly, we let $f_j = 0$, if $f \leq 0$, and we let $f_j = 1$ if $f \geq t_j$. In any other case we let $f_j = f$. Then $f_j \geq f$, consequently

$$A_{m,j} = \bigcap_{N=m}^{\infty} \left\{ \frac{1}{N} \sum_{n=0}^{N-1} f_j(\sigma^n \omega) \geq t_j \right\}$$

contains A_m. Thus it suffices to estimate the dimension of $A_{m,j}$. The argument is similar to the one in the proof of Lemma 5.2.2. We would like to count the cylinder sets of length $N + L_j - 1$ which intersect $\left\{ \frac{1}{N} \sum_{n=0}^{N-1} f_j(\sigma^n \omega) \geq t_j \right\}$ for large N, as they give a cover of $A_{m,j}$ for any $N \geq m$. In order to avoid the inconvenience caused by integer parts, we will only consider Ns with certain divisibility properties, as before.

First of all, the number of blocks consisting of at least L_j consecutive 1s is at most $\frac{N+L_j-1}{L_j}$, which is an integer for infinitely many N. Thus the number of ways we can arrange the blocks of at least L_j consecutive 1s is at most

$$(5.15)\qquad \sum_{i=0}^{\frac{N+L_j-1}{L_j}} \binom{N+L_j-1}{2i} \le \left(\frac{N+L_j-1}{L_j}+1\right)\cdot\binom{N+L_j-1}{2\cdot\frac{N+L_j-1}{L_j}} \le (N+L_j-1)\cdot\binom{N+L_j-1}{2\cdot\frac{N+L_j-1}{L_j}},$$

using $L_j \ge L_1 > 5$, as in (5.11). We call these blocks j-blocks.

The novelty of cylinder counting in this proof compared to the previous one is that we have to take into account the blocks responsible for the values of f_j between 0 and t_{j-1}. As $\frac{1}{N}\sum_{n=0}^{N-1} f_j(\sigma^n\omega) \ge t_j$, in the first N coordinates there are at most $\frac{1-t_j}{1-t_{j-1}}N = \frac{N}{2}$ not covered by the j-blocks, as otherwise the number of terms in $\sum_{n=0}^{N-1} f(\sigma^n\omega)$ with $f(\sigma^n\omega) \le t_{j-1}$ is too large and we have $\frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n\omega) < t_j$. Thus beside the already placed j-blocks, there are at most $\frac{1-t_j}{1-t_{j-1}}N + L_j - 1 = \frac{N}{2} + L_j - 1$ coordinates remaining, which might contain some $(j-1)$-blocks of at least L_{j-1} consecutive 1s. By a similar estimate to (5.15) we find that the number of possible arrangements of these $(j-1)$-blocks is at most

$$(5.16)\qquad \sum_{i=0}^{\frac{\frac{N}{2}+L_j-1}{L_{j-1}}} \binom{\frac{N}{2}+L_j-1}{2i} \le \left(\frac{\frac{N}{2}+L_j-1}{L_{j-1}}+1\right)\cdot\binom{\frac{N}{2}+L_j-1}{2\cdot\frac{\frac{N}{2}+L_j-1}{L_{j-1}}} \le \left(\frac{N}{2}+L_j-1\right)\cdot\binom{\frac{N}{2}+L_j-1}{2\cdot\frac{\frac{N}{2}+L_j-1}{L_{j-1}}},$$

using $L_{j-1} \ge L_1 > 5$.

Suppose that $j_0 \in \{0, ..., j-1\}$. Proceeding recursively, by the same argument we can conclude that the union of the $(j-i)$-blocks taken for $i = 0, 1, ..., j_0 - 1$ cover all but at most $\frac{1-t_j}{1-t_{j_0}}N = \frac{N}{2^{j_0}}$ of the first N coordinates. Thus beside these blocks there are at most $\frac{N}{2^{j_0}} + L_j - 1$ coordinates remaining, which yields similarly to (5.16) that the number of possible arrangements of the $(j-j_0)$-blocks is at most

$$(5.17)\qquad \left(\frac{N}{2^{j_0}}+L_j-1\right)\cdot\binom{\frac{N}{2^{j_0}}+L_j-1}{2\cdot\frac{\frac{N}{2^{j_0}}+L_j-1}{L_{j-j_0}}} < (N+L_j-1)\cdot\binom{\frac{N}{2^{j_0}}+L_j-1}{2\cdot\frac{\frac{N}{2^{j_0}}+L_j-1}{L_{j-j_0}}}.$$

We can use this bound for $j_0 = 0, 1, ..., j-1$. (We note that for infinitely many values of N each number appearing in the above binomial coefficients is an integer.) Finally, there can be coordinates which are not contained by any such block. At most $(1-t_j)N$ of them in the first N coordinates, and arbitrarily many of them in the last $L_j - 1$ coordinates. Thus they can be chosen at most $2^{(1-t_j)N+L_j-1}$ different ways. Hence the number of cylinders which intersect $\left\{\frac{1}{N}\sum_{n=0}^{N-1} f_j(\sigma^n\omega) \ge t_j\right\}$ can be bounded by taking the product of the estimates in (5.17), and multiplying it by $2^{(1-t_j)N+L_j-1}$. Hence $\left\{\frac{1}{N}\sum_{n=0}^{N-1} f_j(\sigma^n\omega) \geq t_j\right\}$ can be covered by at most

$$(N+L_j-1)^j \cdot 2^{(1-t_j)N+L_j-1} \cdot \prod_{j_0=0}^{j-1} \binom{\frac{N}{2^{j_0}}+L_j-1}{2\cdot\frac{\frac{N}{2^{j_0}}+L_j-1}{L_{j-j_0}}} \tag{5.18}$$

many cylinders of diameter $2^{-(N+L_j-1)}$. Observe that the $j_0 = 0$ case in (5.18) includes the estimate (5.15). By the standard estimate of binomial coefficients we can estimate it further from above by

$$(N+L_j-1)^j \cdot 2^{(1-t_j)N+L_j-1} \prod_{j_0=0}^{j-1} \left(\frac{eL_{j-j_0}}{2}\right)^{2\cdot\frac{\frac{N}{2^{j_0}}+L_j-1}{L_{j-j_0}}}. \tag{5.19}$$

Introduce the notation $k = N + L_j - 1$ again. By factoring out constants depending on $L_1, ..., L_j$ into a constant denoted by $C(L_1, ..., L_j)$, and rearranging (5.19) one can obtain that it equals

$$C(L_1, ..., L_j) \cdot k^j \cdot 2^{(1-t_j)k} \prod_{j_0=0}^{j-1} \left(\frac{eL_{j-j_0}}{2}\right)^{\frac{2k}{2^{j_0}L_{j-j_0}}}. \tag{5.20}$$

This formulation leads us to a suitable choice of L_n: for an arbitrary fixed $\tau > 0$, define L_n large enough to guarantee that

$$\left(\frac{eL_n}{2}\right)^{\frac{2}{L_n}} < 2^{\tau/2^{2n}}. \tag{5.21}$$

With this choice, (5.20) can be estimated by

$$\begin{aligned} C(L_1, ..., L_j) \cdot k^j \cdot 2^{(1-t_j)k} \prod_{j_0=0}^{j-1} 2^{\tau k/2^{2j-j_0}} &\leq C(L_1, ..., L_j) \cdot k^j \cdot 2^{(1-t_j+\frac{\tau}{2^j})k} \\ &\leq C(L_1, ..., L_j) \cdot 2^{(1-t_j+\frac{2\tau}{2^j})k}, \end{aligned} \tag{5.22}$$

where the last inequality holds for large enough N, k. It immediately yields

$$\mathcal{H}_{2^{-k}}^{1-t_j+\frac{2\tau}{2^j}} \left(\left\{\frac{1}{N}\sum_{n=0}^{N-1} f_j(\sigma^n\omega) \geq t_j\right\}\right) \leq C(L_1, ..., L_j)$$

where $N = k - L_j + 1$ as before. However, this set contains $A_{m,j}$ for large enough k, N, thus

$$\mathcal{H}_{2^{-k}}^{1-t_j+\frac{2\tau}{2^j}} (A_m) \leq C(L_1, ..., L_j).$$

As k, N can be arbitrarily large, it shows that in fact

$$\mathcal{H}^{1-t_j+\frac{2\tau}{2^j}} (A_m) \leq C(L_1, ..., L_j)$$

and consequently,

$$\dim_H(A_{m,j}) \leq 1 - t_j + \frac{2\tau}{2^j}.$$

Consequently, by our initial observations

$$S_f(t_j) \leq 1 - t_j + \frac{2\tau}{2^j},$$

that is, using $t_j = 1 - 2^{-j}$ we have

$$S_f(1 - 2^{-j}) \leq \frac{1 + 2\tau}{2^j}.$$

Thus if we calculate the left derivative of S_f at 1 by going along the sequence t_j, we find that it is at most $-(1 + 2\tau) > -\infty$, which concludes the proof. □

Remark 5.2.5. We note that as the spectrum is concave, for any function $f \in C_0(\Omega)$ such that $\alpha^*_{f,\min} = -1$ and $\alpha^*_{f,\max} = 1$ we have that the graph of S_f is above the triangle graph with vertices $(-1, 0), (0, 1), (1, 0)$. On the other hand, it must be below the constant 1 function in the interval $[-1, 1]$. It is natural to ask whether these extremes can be attained/approximated. We do not give the complete answer for these questions, but make a few observations.

First of all, Theorem 5.2.1 easily yields that S_f can be arbitrarily close to the triangle graph: notably for the function f constructed in the previous proof, S_f is contained by the triangle with vertices $(-1, 0), (0, 1 + 2\tau), (1, 0)$ due to concavity. Thus the theoretic minimum can be approximated.

On the other hand, if we would like to construct some f such that S_f is considerably large, we can consider a function similar to the one in Example 2.2.3. More explicitly, let $f \in \mathrm{PCC}^{2k+1}(\Omega)$ be such that it takes the value -1 on cylinders which contain more 0s than 1s in their first $2k + 1$ coordinates, and $f(\omega) = 1$ otherwise. As in the proof of Example 2.2.3, we can show by Hutchinson's theorem that $S_f(-1) = S_f(1)$ is at least $\frac{k}{2k+1}$. Thus the piecewise linear graph determined by the vertices (-1,1/2), (0,1), (1,1/2) can be arbitrarily close to a lower estimate of the spectrum, which means that S_f is considerably large, even though it is far from what we strived for.

We also provide another example, which displays that $S_f(\alpha^*_{f,\max})$ can be arbitrarily close to 1 even for nonconstant functions, if we drop the condition that $\alpha^*_{f,\max} = 1$. Notably, let $f \in \mathrm{PCC}^k(\Omega)$ such that it takes the value -1 if the first k coordinates equal 0, while it takes the value $\frac{1}{2^k-1}$ if these coordinates contain at least one 1. Then similarly to the previous argument we have that $S_f\left(\frac{1}{2^k-1}\right) \geq \frac{k-1}{k}$. It would be interesting to see how large $S_f(\alpha^*_{f,\max})$ can be if $f \in C_0(\Omega)$ such that $\alpha^*_{f,\min} = -1$ and $\alpha^*_{f,\max} = 1$.

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