A topological variant of first passage percolation

MSc Thesis

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Chapter 1

Introduction

First passage percolation was introduced by Hammersley and Welsh in 1965 as a model to describe fluid flows through porous medium. It quickly became a popular area of probability theory, as one can easily ask very difficult questions. Many of these have still remained unsolved despite the growing interest from mathematicians, physicists and biologists. The main setup is the following: we have a given graph, usually we like to consider the lattice $\mathbb{Z}^d$. We denote the set of nearest neighbor edges by E. We place independent, identically distributed, non-negative random variables with a distribution law μ on each edge $e \in E$, which is called the passage time of e, and denoted by $\tau(e)$. We think about it as the time needed to traverse e. Based on this, we can define the passage time of any finite path Γ of consecutive edges as the sum of the passage times of contained edges:

$$\tau(\Gamma) = \sum_{e \in \Gamma} \tau(e).$$

Using this definition, we might define the passage time between any two points, or in other words the T-distance of any two points $x, y \in \mathbb{R}^d$

$$T(x, y) = \inf_{\Gamma} \tau(\Gamma),$$

where the infimum is taken over all the paths connecting x' to y', where x' and y' are the unique lattice points such that $x \in x' + [0,1)^d$, $y \in y' + [0,1)^d$. The term "distance" is appropriate here: one can easily show that $T : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ is a pseudometric, that is an "almost metric" in which the distance of distinct points might be 0.

In brief, this is the probabilistic setup. In the sequel when we recall results related to this theory, for the sake of brevity we will often omit the precise technical conditions, such as conditions about the finiteness of certain moments or the value of the distribution function in the infimum of its support. Instead of it we will simply refer to "some mild conditions" about the distribution function and cite the source of the result. For the reader interested in the details the recently published book [2] is also warmly recommended.

The topological setup was introduced by the author in papers [10] and [11], which works form the content of this thesis. Roughly, instead of non-negative random variables on each edge, we consider some $A \subseteq \mathbb{R}_{\geq 0}$. To exclude trivialities, let A have at least two elements. The passage time of any edge will be an element of A, and passage times of paths and between points are defined as in the probabilistic setup. Formally, the space of configurations is $\Omega = \times_{e \in E} A$. To define topology, we equip A by its usual subspace topology inherited from $\mathbb{R}$, and equip $\Omega = \times_{e \in E} A$ with the product topology. If there might be ambiguity, we will write T_ω and τ_ω for the passage times in the $\omega \in \Omega$ configuration. We call this model ordinary topological first passage percolation, as in Chapter 3 we will define Hilbert first passage percolation, which is a possible generalization of the concept. By simply saying topological first passage percolation, we always mean the ordinary one.

Primarily we are interested in the classical questions of the probabilistic setup which might make sense in the topological setup as well. Clearly estimates on variances of certain passage times have no direct analogues. On the other hand, quite a few results of probability theory are 0-1 laws, whose meaning is roughly that the model *typically* has certain properties. That is, with probability 1 certain events occur. Such questions can also be examined in the topological setup, once we have defined another notion of "smallness" which can be defined in terms of topology without having any measure on the space. This is how we arrive to the basic concepts of Baire category:

Definition 1.0.1. *Let X be a topological space. We say that $A \subseteq X$ is nowhere dense if for any nonempty open set $U \subseteq X$ there exists a nonempty open set $V \subseteq U$ such that $A \cap V = \emptyset$.*

Definition 1.0.2. *Let X be a topological space. We say that $A \subseteq X$ is meager (or of first category) if there exists a sequence $(A_n)_{n=1}^{\infty}$ of nowhere dense sets such that $A = \bigcup_{n=1}^{\infty} A_n$. We say that A is of second category if it is not of first category. Finally, A is residual if its complement is of first category.*

For example, if the underlying topological space is $X = \mathbb{R}$, then $\mathbb{Z}$ or the triadic Cantor set is nowhere dense. Meanwhile, $\mathbb{Q}$ is dense and meager, as a countable set, and $\mathbb{R} \setminus \mathbb{Q}$ is residual. We also use the terminology that a property is generic (or typical) if it holds in a residual subset of the topological space.

By definition, it is plain to see that the empty set and subsets of a meager set are also meager. Moreover, if A_n is meager for $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is meager. Due to these properties, we say the meager sets form a σ-ideal. The most famous and important example of such is the σ-ideal of sets of measure zero in a measure space. However, the σ-ideal of meager sets also prove to be useful in numerous occasions and hence it is widely examined in many situations.

It is worth mentioning at this point that in the literature it is rather usual to introduce restrictions on the space in which we examine residuality. Notably, we would like to think about meager sets as small, negligible sets, but this way of thinking might be quite far from truth: an extreme example to display this phenomenon is that if $X = \mathbb{Q}$, the whole space is meager. To this end, one can introduce the notion of Baire spaces: we say that X is a Baire space if the interior of any meager set is empty. Due to Baire's category theorem, we know that a complete metric space is always a Baire space, and in many cases Baire category is considered in complete metric spaces exclusively. If the space in question is not a Baire space, residuality might be meaningless. However, our arguments are mostly correct even without this assumption, thus we will not restrict our observations to Baire spaces unless it is truly needed.

Now we can formulate what kind of questions we will investigate: we will consider 0-1 laws of the probability model and examine what is the generic behavior of the topological model in these situations. We will see that certain classical almost sure events, as the existence of finite geodesics have residual counterparts, while the notion of the limit shape or time constants gets as chaotic as possible.

Chapter 2

Ordinary topological first passage percolation

2.1 A note on negative passage times

Before going into further depths of topological first passage percolation, it is useful to say a few words about what happens if we allow negative passage times. In this case, it is plain to see that apart from a nowhere dense set of Ω, the passage time between any two points would be $-\infty$. To verify this, we declare at this point how we will think about the topology on Ω. The most convenient way for us is to consider cylinder sets as the basis of the topology, that is the basis sets are of the form

$$U = \times_{e \in E} U_e,$$

where each U_e is open in A and with at most finitely many exceptions $U_e = A$. We say that U_e is the projection of U to the edge e.

Using this, we can easily verify our previous claim. We need that if U is a nontrivial open set of Ω, then there exists a nontrivial open set $V \subseteq U$ such that on V, the passage time between any two points is $-\infty$. It clearly suffices to show this for a cylinder set U, which is rather straightforward: as there are only finitely many edges for which U has nontrivial projection, we can choose an edge e with trivial projection. Then we define V to have the same projections everywhere as U, except for e, where the projection contains only negative values. Then in any configuration in V, the passage time between two lattice points x, y is $-\infty$: indeed, we can take paths of arbitratily low passage time by going to one of the vertices of e from x on a fixed route, then go along it back and forth as many times as we wish, and then finally go to y on a fixed route. The first and the last part of this path has a fixed passage time in a given configuration, while the middle term can be arbitrarily low. Thus apart from a nowhere dense set of Ω, the passage time between any two lattice points is $-\infty$ indeed, and it quickly yields the same for any two points.

One may wonder what happens if we allow negative passage times, but we only permit self-avoiding paths, except for that the starting and the ending point of a path may coincide. This restriction clearly rules out our previous argument, however, we might expect that passage times are still $-\infty$ in a considerably large set if $d \geq 2$. (If $d = 1$, we have only one possible path between any two vertices, thus in a reasonably small open set there are vertices whose T-distance is quite well determined. As a consequence, it is something we are not interested in.) The following theorem shows that the above expectation is true.

Theorem 2.1.1. *Suppose that A contains a negative value, and we define $T(x, y)$ by considering the infimum only for the paths which might contain only their starting point and endpoint twice. Then generically we have $T(x, y) = -\infty$ for any two points x, y.*

Proof. As the passage time between any two non-lattice points equals the passage time between certain lattice points, it suffices to prove that in a residual subset of Ω we have $T(x, y) = -\infty$ for any two lattice points x, y. Denote the subset where this holds by S. Furthermore, let us denote the set of configurations satisfying $T(x, y) < -n$ for some $n \in \mathbb{N}$ by $S(x, y, n)$. Using this notation, we have

$$S = \bigcap_{x,y\in\mathbb{Z}^d,\ n\in\mathbb{N}} S(x, y, n),$$

which is a countable intersection. As a consequence, it suffices to prove that each $S(x, y, n)$ is residual. By definition, this is equivalent to $Q(x, y, n) = \Omega \setminus S(x, y, n)$ is meager. In fact, we will prove that $Q(x, y, n)$ is nowhere dense. Fix U to be a cylinder set. Let us denote the set of edges belonging to nontrivial projections of U by $E_U = \{e_1, e_2, ..., e_k\}$. By shrinking the projections $U_{e_1}, ..., U_{e_k}$, we can achieve that all of them are bounded in $\mathbb{R}$. Denote these new projections by U'_{e_i}, $i = 1, ..., k$, and the cylinder set defined by them by U'. Then for any configuration in U', the sum of passage times over the edges $e_1, ..., e_k$ is bounded by a constant C.

Let $a \in A$ be negative. Note that we might construct a self-avoiding path from x to y of arbitrarily large ℓ^1 length, or in other words, of arbitrarily large number of edges. Indeed, we can go arbitrarily far along the direction of one axis, and then if we forget about these edges, the remaining graph is still connected as $d \geq 2$. Thus we might consider a path Γ from x to y with length m for large enough m. We determine m later. Now we define $V \subseteq U'$ to have the same projections as U', except for the edges in $\Gamma \setminus E_U$: here we define the projections to be a subset of

$(-\infty, \frac{a}{2})$. In V, we can bound the passage time of Γ as it follows:

$$\sum_{e\in\Gamma} t(e) = \sum_{e\in\Gamma\setminus E_U} t(e) + \sum_{e\in\Gamma\cap E_U} t(e) \leq (m-k)\frac{a}{2} + C < -n,$$

if m is large enough, as $\frac{a}{2} < 0$ and k, C are fixed. Thus the configurations in V cannot be in $Q(x,y,n)$, yielding $Q(x,y,n)$ is nowhere dense, which is what we wanted to prove. □

2.2 Finite geodesics

As it was proved in [12] for $d = 2$ and any distribution, and in [8] for arbitrary d under mild conditions on the distribution, with probability 1 there exists an optimal path between any two lattice points, which is called a geodesic. Furthermore, if the probability distribution function is continuous, geodesics are unique with probability 1. As the next theorem displays, these properties have their respective topological analogues. The underlying idea is also similar: geodesics do not tend to use too many edges.

Theorem 2.2.1. *Generically there exists a geodesic between any two lattice points. Furthermore, if A has no isolated points then generically these geodesics are unique.*

Proof. Consider the first statement. We will prove that for given $x, y \in \mathbb{Z}^d$, apart from a nowhere dense set of Ω, there exists a geodesic between x and y. As there are countably many such pairs, it would be sufficient. The idea of the proof is that usually the paths with reasonably low passage times lie in a bounded set containing x and y, thus if we are interested in $T(x,y)$, we have to consider only finitely many paths, hence the infimum is the minimum.

To verify our claim, fix lattice points x, y and a cylinder set U. Let us denote the set of edges belonging to nontrivial projections of U by $E_U = \{e_1, e_2, ..., e_k\}$. As in the proof of Theorem 2.1.1, we can construct a smaller cylinder set by shrinking the projections $U_{e_1}, ..., U_{e_k}$, such that all of these projections are bounded in $\mathbb{R}$. We denote again these new projections by U'_{e_i}, $i = 1, ..., k$, and the cylinder set defined by them by U'. Then for any configuration in U', the sum of passage times over the edges $e_1, ..., e_k$ is bounded by a constant C. Choose an $a \in A$ such that $a > 0$. We will fix an $n \in \mathbb{N}$ later. Consider all the edges with ℓ^1 distance at most n from x. Denote their set by E^*. If n is large enough, there is an optimal ℓ^1 path from x to y using only edges in E^*. We will define $V \subseteq U'$ as a cylinder set which has nontrivial projections to the edges in $E_U \cup E^*$. Concerning the edges in E_U, we define V to have the same projections as U', meanwhile for the edges in $E^* \setminus E_U$, we require that the projections of V equal $(a-\varepsilon, a+\varepsilon) \cap A$, where $0 < \varepsilon < a$. Consider now any configuration in V, and take a path Γ from x to y with ℓ_1-length $|x-y|$, using only edges in E^*. Then its passage time is at most $Ck + |x-y|(a+\varepsilon) = C_1$, a constant independent from the actual configuration in V. Meanwhile if we consider any path from x to y which uses an edge which is not in E^* its passage time is at least $(n-k)(a-\varepsilon)$ for any configuration in V, as it has to use at least n edges to leave E^*, and apart from the at most k edges in E_U they have passage time at least $a-\varepsilon$. However, for large enough n, this passage time eventually surpasses C_1. As a consequence, if we define E^* and then V using this n, we will know that for any configuration in V, the paths leaving E^* have passage times higher than the passage time of Γ. Hence in the definition of $T(x,y)$, we have to consider only the paths connecting x, y which use only edges in E^*. There are finitely many of them, thus in fact the infimum is the minimum, yielding we have a geodesic between x, y in V. As a consequence, as we claimed, there exists a geodesic between x and y apart from a nowhere dense set of Ω.

What remains to prove is the uniqueness part. It suffices to prove that for given lattice points x, y, apart from a nowhere dense set there is a unique geodesic between x and y. Fix a cylinder set U. By the previous argument, we know that there exists of a cylinder set $V \subseteq U$ such that in V, there is a geodesic between x and y. We will shrink this cylinder set further to arrive at a cylinder set W in which there is always a unique geodesic between x and y. In order to do so, define for each path Γ connecting x and y the number $\tau(\Gamma, V)$ as the infimum of passage times of Γ for configurations in V. Let $\tau(V) = \inf_\Gamma \tau(\Gamma, V)$. By the definition of V, this is determined by finitely many paths from x to y in fact, as for any configuration in V, the too long paths have too large passage times. Thus $\tau(V)$ equals a minimum, and in the sequel, we might focus only on these paths. Let Γ_0 be one of the paths for which $\tau(\Gamma_0, V) = \tau(V)$. It would be nice to have a unique path with this property: from this point, the construction of W would be more or less straightforward. We claim that for an appropriate $V' \subseteq V$ we can have $\tau(\Gamma_0, V') = \tau(V') = \tau(V)$ while for any $\Gamma \neq \Gamma_0$ we have $\tau(\Gamma, V') > \tau(V')$. Indeed, if we define V' to have the same projections as V to the edges contained by Γ_0, we immediately have our first requirement. Furhermore, if $\Gamma \neq \Gamma_0$ with $\tau(\Gamma, V) = \tau(V)$, there is at least one edge $e \in \Gamma \setminus \Gamma_0$. We will shrink the projection to this edge: as A has no isolated points, we can choose some nonempty $V'(e) \subseteq V(e) \cap A$ with higher infimum than $\inf V(e)$, which results in $\tau(\Gamma, V') > \tau(\Gamma, V) \geq \tau(V')$. Repeating the same step for each $\Gamma \neq \Gamma_0$ with $\tau(\Gamma, V) = \tau(V)$ (which means only finitely many steps) we obtain some V' with the above property.

In the final step we will only shrink the projections of V' to the edges in Γ_0. As $\tau(\Gamma, V') > \tau(\Gamma_0, V')$ for any $\Gamma_0 \neq \Gamma$, and there are only finitely many paths we are interested in by now, for some $\varepsilon > 0$ we have $\tau(\Gamma, V') > \tau(\Gamma_0, V') + \varepsilon$. We will shrink the projections of V' to the edges in Γ_0 based on this bound. Namely, if Γ_0 contains the edges $e'_1, ..., e'_m$, and the infimum of $V'(e'_i)$ is a_i, we will define $W(e_i)$ as $\left(a_i, a_i + \frac{\varepsilon}{m}\right) \cap A$. Then as

$$\tau(\Gamma_0, V') \geq \sum_{i=1}^{m} a_i,$$

we have that for any configuration $\omega \in W$ the passage time of Γ_0 is at most

$$\tau_\omega(\Gamma_0) \leq \sum_{i=1}^{m} \left(a_i + \frac{\varepsilon}{m}\right) \leq \tau(\Gamma_0, V') + \epsilon < \tau(\Gamma, V') \leq \tau_\omega(\Gamma)$$

for any $\Gamma \neq \Gamma_0$, as a configuration in W is also in V', hence $\tau(\Gamma, V') \leq \tau_\omega(\Gamma)$. Thus for any configuration in W, the unique geodesic from x to y is Γ_0. This concludes the proof. □

Remark 2.2.2. In the proof we clearly used that A has no isolated points to be able to nontrivially shrink open sets in A. By a similar argument, one can quickly check that if A has an isolated point a, then for any two lattice points x, y such that the line segment $[x, y]$ is not parallel to any of the coordinate axis (i.e. there are multiple optimal ℓ_1 paths from x to y), there exists a cylinder set U such that for any configuration in U there are multiple geodesics from x to y. Indeed, we can define U to have projections containing only a to the set of edges within a given large ℓ_1 distance to $[x, y]$, similarly to the definition of V in the previous proof. Then it is easy to see that the geodesics between x and y are precisely the optimal ℓ_1 paths, of which there are more than one.

2.3 Infinite geodesics

We turn our attention to infinite geodesics, which are self-avoiding paths of infinitely many edges such that each of their finite subpaths are finite geodesics. We distinguish two types of infinite geodesics: the ones indexed by $\mathbb{N}$, informally which are infinite in only one direction, and the ones indexed by $\mathbb{Z}$, informally which are infinite in both directions. We call the former ones geodesic rays while the latter ones are the geodesic lines.

In the probabilistic setup, one might easily check by Kőnig's lemma the almost sure existence of a geodesic ray, using that with probability 1 there is a geodesic between any two points. In the topological setup, we can use the same argument to prove the same in a residual subset of Ω. Namely, denote the first coordinate vector by ξ_1 in $\mathbb{R}^d$ and observe a finite geodesic from 0 to $n\xi_1$ for $n = 1, 2, ...$ As there are finitely many edges having the origin as one of its endpoints, there are infinitely many of these paths which start with the same edge, then there are infinitely many of them which continue with the same edge, etc. This way one might verify the existence of a geodesic ray. Now it is a natural question whether there are more distinct geodesic rays, where by distinct we mean that they share only finitely many edges. In the probabilistic setup it is conjectured that for continuous distributions there are infinitely many of them with probability 1. For d=2 and a certain class of distribution functions this claim was verified in [1]. However, in the topological setup we encounter a completely different phenomenon, which was displayed by the following theorems proved in [10]:

Theorem 2.3.1. *If $d \geq 2$ and $\sup A > 5 \inf A$ then generically there is no more than one geodesic ray in $\mathbb{Z}^d$.*

Theorem 2.3.2. *For arbitrary A, generically there exists only a bounded number of distinct geodesic rays in $\mathbb{Z}^d$, more precisely, there are no more than $4d^2$ distinct geodesic rays.*

The proofs were based on quite similar geometric ideas. However, we omit the proofs here, as in [11] we could give the neat and complete answer to the problem in the form of a common sharpened version of both Theorem 2.3.1 and Theorem 2.3.2:

Theorem 2.3.3. *For arbitrary A, generically there exists exactly one geodesic ray in $\mathbb{Z}^d$ if $d \geq 2$*

Corollary 2.3.4. *If $d \geq 2$, there can be no geodesic lines, as a geodesic line can be dissected into two distinct geodesic rays.*

We note that $d \geq 2$ is obviously necessary in Theorem 2.3.3 and Corollary 2.3.4: if $d = 1$, the lattice consists of a single line, which must be a geodesic line.

Proof of Theorem 2.3.3. The outline of the proof is similar to the ones of its weaker counterparts, however, it relies on a bit more elaborate geometric construction.

As earlier, first we will prove that if x is a fixed lattice point then apart from a meager subset of Ω there is no more than one geodesic ray starting from x. Clearly it suffices to prove this claim for $x = 0$. Let $F(0)$ denote the set of configurations in which there are at least two distinct geodesic rays starting from the origin. Then $F(0) = \bigcup_{m=1}^{\infty} F_m(0)$ where $F_m(0)$ stands for the set of configurations in which there are at least two distinct geodesics starting from the origin such that they have at most m edges in common. We claim that for any m we have that $F_m(0)$ is a nowhere dense set in Ω, which would verify our preliminary statement about the meagerness of $F(0)$.

Fix U to be a cylinder set, and denote the set of edges belonging to nontrivial projections of U by $E_U = \{e_1, e_2, ..., e_N\}$. We can simply construct a smaller cylinder set U' by shrinking the projections $U_{e_1}, ..., U_{e_N}$, such that all of these projections are bounded in $\mathbb{R}$. Then for any configuration in U', the sum of passage times over the edges $e_1, ..., e_N$ is bounded by a constant C. The novelty appears at this point: instead of considering concentric hypercubes centered at the origin, we take a skew construction. More precisely, let $K_1 = [-p, p]^d$ and $K_2 = [-q', q] \times [-r, r]^{d-1}$ where $p, q, q', r \in \mathbb{N}$ and p is chosen such that the edges in E_U are in the interior of K_1. The values $q < r < q'$ are to be fixed later. Let us denote the set of edges in K_2 which are not in the interior of K_1 by E^*. We will define $V \subseteq U'$ as a cylinder set which has nontrivial projections to the edges in $E_U \cup E^*$. The underlying concept is borrowed from the proof given for the case $\sup A < 5 \inf A$: for the configurations in V we would like to have essentially one (and the same) geodesic from the boundary ∂K_1 to the boundary ∂K_2, notably the line segment connecting $p\xi_1$ and $q\xi_1$ (in general ξ_i denotes the ith coordinate vector). By this we mean that for any lattice points $x_1 \in \partial K_1$ and $x_2 \in \partial K_2$, a geodesic Γ from x_1 to x_2 eventually arrives in $p\xi_1$, and then it goes along the line segment $[p\xi_1, q\xi_1]$. It would be sufficient: any geodesic ray starting from the origin eventually leaves K_1 and K_2, and a geodesic ray is a geodesic between any two of its points, the previous properties would guarantee that any geodesic ray starting from the origin would go along the line segment $[p\xi_1, q\xi_1]$. However, that would mean that our configuration cannot be in $F_m(0)$ for $q - p > m$ as there would not exist at least two distinct geodesics starting from the origin such that they have at most m edges in common.

Let us make the above argument rigorous. Fix $a < b$ in A. Moreover, fix $\varepsilon > 0$ and $\lambda > 1$ such that $(a + \varepsilon)\lambda < b - \varepsilon$ holds. Finally, for later usage define a small value ε_e for each edge $e \in E$ such that $\sum_{e \in E} \varepsilon_e < \varepsilon$. We will have small passage times on the edges of ∂K_1, ∂K_2, and along the line segment $[p\xi_1, q\xi_1]$ to guarantee a path with considerably low passage time between any two points of ∂K_1 and ∂K_2. We call these edges cheap. Meanwhile on other edges between the two boundaries (e.g. the expensive edges) we would like to have considerably larger passage times. Thus for every cheap edge e we define the relatively open set

$$V_e = (a - \varepsilon_e, a + \varepsilon_e) \cap A,$$

and for any expensive edge we define

$$V_e = (b - \varepsilon_e, b + \varepsilon_e) \cap A.$$

By this, we have defined V. Now consider any configuration in V. For technical convenience we will prove the following claim, which is formally stronger than what we stated before: if $x_1 \in \partial K_1 \cup [p\xi_1, (q-1)\xi_1]$, while $x_2 \in [(p+1)\xi_1, q\xi_1] \cup \partial K_2$, then there is no geodesic from x_1 to x_2 which uses expensive edges. Proceeding towards a contradiction, assume the existence of x_1, x_2, and a geodesic Γ which refutes this claim. As any subpath of a geodesic is also a geodesic, by passing to a suitable subpath we can assume that Γ uses expensive edges only. Hence the passage time of Γ can be estimated from below by

$$\tau(\Gamma) \geq |x_2 - x_1|b - \varepsilon \geq |x_2 - x_1|(b - \varepsilon).$$

We will arrive at a contradiction by constructing a cheaper Γ' from x_1 to x_2 which does not use expensive edges. In the following we will separate cases based on the position of x_2. The figure below displays how we will construct Γ' with the desired properties in one of the cases and it also helps understanding the other constructions. If $x_2 \in [p\xi_1, q\xi_1]$, we have a simple task. Indeed, in this case Γ can be replaced by a

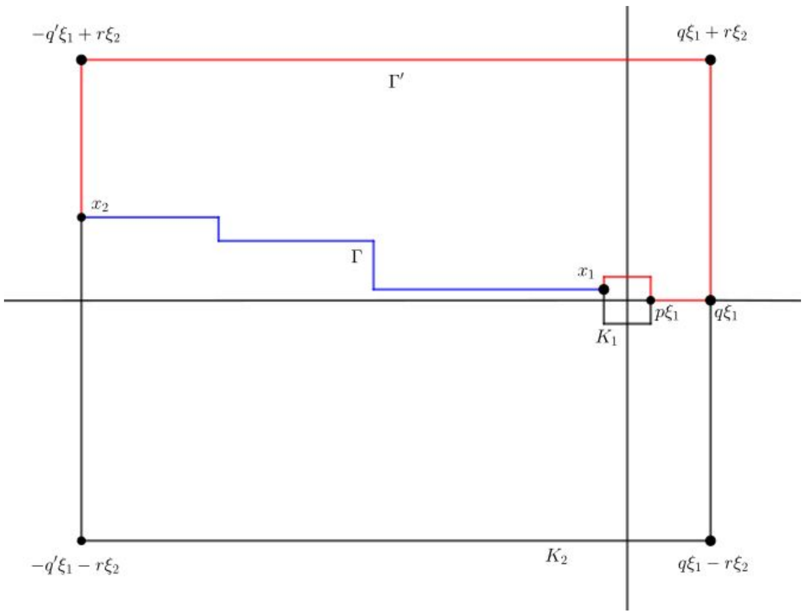

Figure 2.1: The case when $x_2 \in \partial K_2$ is on the facet containing $-q'\xi_2$ and $d = 2$.

path Γ' which is not longer in ℓ_1, and instead of using expensive edges only, it uses cheap edges exlusively. Thus $\tau(\Gamma') < \tau(\Gamma)$ clearly holds.

Now assume that x_2 is on the same facet of ∂K_2 as $q\xi_1$. We separate two subcases:

- $x_1 \in [p\xi_1, (q-1)\xi_1]$. In this case there exists an ℓ_1-optimal path from x_1 to x_2 which consists of cheap edges exclusively. Choose such a path to be Γ', it contains $|x_2 - x_1|$ edges. The passage time of Γ' can be estimated from above by

$$\tau(\Gamma') \leq |x_2 - x_1|a + \varepsilon \leq |x_2 - x_1|(a + \varepsilon).$$

Comparing the bounds gives

$$|x_2 - x_1|(b - \varepsilon) \leq |x_2 - x_1|(a + \varepsilon).$$

However, as $a+\varepsilon < b-\varepsilon$, it is clearly impossible, thus we have handled this case.

- $x_1 \in \partial K_1$. In this case we define Γ' by joining together two shorter paths Γ_1 and Γ_2. The path Γ_1 will run on ∂K_1 from x_1 to $p\xi_1$ such that it uses as few edges as it is possible. Consequently, $|\Gamma_1| \leq 2dp$. Meanwhile the path Γ_2 will go from $p\xi_1$ to x_2 such that it is ℓ_1-optimal and uses cheap edges exclusively. (By the choice of the facet containing x_2 it is clearly possible.) Now by the triangle inequality we have

$$|\Gamma_2| \leq |x_2 - x_1| + 2dp.$$

Using the estimate for the number of edges in Γ_1 and Γ_2 we can obtain an upper bound for the passage time of Γ':

$$\tau(\Gamma') \leq (|x_2 - x_1| + 4dp)a + \varepsilon \leq (|x_2 - x_1| + 4dp)(a+\varepsilon).$$

Comparing the bounds gives

$$|x_2 - x_1|(b-\varepsilon) \leq (|x_2 - x_1| + 4dp)(a+\varepsilon).$$

As $(a+\varepsilon)\lambda < b-\varepsilon$ now we can obtain after division

$$\lambda|x_2 - x_1| \leq |x_2 - x_1| + 4dp,$$

or equivalently,

$$|x_2 - x_1| \leq \frac{4dp}{\lambda - 1}.$$

However, $|x_2 - x_1| \geq q - p$ necessarily holds. Hence if q is chosen to be sufficiently large compared to p we get a contradiction, which concludes this case.

Assume now that x_2 lies on a facet of K_2 neighboring the one containing $q\xi_1$. In this case we construct Γ' by joining at most three paths $\Gamma_1, \Gamma_2, \Gamma_3$: if $x_1 \in \partial K_1$ we define Γ_1 in order to reach $p\xi_1$ as in the second subcase of the previous case. Next we use Γ_2 to reach $q\xi_1$ using the edges of $[p\xi_1, q\xi_1]$. Finally, we define Γ_3 to reach x_2 so that it is optimal in ℓ_1 and uses only the edges of ∂K_2. The first two parts use at most $2dp + q - p$ edges, while we can get a simple upper estimate for the length of Γ_3 using the triangle inequality, notably

$$|\Gamma_3| \leq |x_2 - x_1| + 2dp + q - p.$$

Consequently, we have

$$|\Gamma'| \leq |x_2 - x_1| + 4dp + 2q - 2p.$$

As all these edges are cheap, we deduce the following bound:

$$\tau(\Gamma') \leq (|x_2 - x_1| + 4dp + 2q - 2p)a + \varepsilon \leq (|x_2 - x_1| + 4dp + 2q - 2p)(a + \varepsilon).$$

Comparing to the lower bound given for $\tau(\Gamma)$ we gain

$$|x_2 - x_1|(b - \varepsilon) \leq (|x_2 - x_1| + 4dp + 2q - 2p)(a + \varepsilon).$$

Given the ratio bound on $a + \varepsilon$ and $b - \varepsilon$ it yields

$$\lambda|x_2 - x_1| \leq |x_2 - x_1| + 4dp + 2q - 2p.$$

Thus, simple rearrangement yields

$$|x_2 - x_1| \leq \frac{4dp + 2q - 2p}{\lambda - 1}.$$

The right hand side expression is already fixed, while for the left hand side we have $|x_2 - x_1| \geq r - q$. Thus if r is chosen so that it is sufficiently large compared to the already fixed q, then we get a contradiction, which concludes this case.

The final case to consider is when x_2 is on the same facet of K_2 as $-q'\xi_1$. In this case we define Γ' as the union of at most four shorter paths $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$. Concerning Γ_1 and Γ_2 we resort to the previous case in order to get to $q\xi_1$ from x_1, using at most $2dp + q - p$ cheap edges. Then we define Γ_3 to be the line segment $[q\xi_1, q\xi_1 + r\xi_2]$, thus we reach a facet neighboring to the one containing x_2 using $r + 2dp + q - p$ cheap edges. Finally we define Γ_4 to reach x_2 so that it is optimal in ℓ_1 and uses only the edges of ∂K_2. By the triangle inequality we have

$$|\Gamma_4| \leq |x_2 - x_1| + r + 2dp + q - p,$$

and hence

$$|\Gamma'| \leq |x_2 - x_1| + 2r + 4dp + 2q - 2p.$$

As all these edges are cheap, we deduce the following bound:

$$\tau(\Gamma') \leq (|x_2 - x_1| + 2r + 4dp + 2q - 2p)a + \varepsilon \leq (|x_2 - x_1| + 2r + 4dp + 2q - 2p)(a + \varepsilon).$$

Comparing to the lower bound given for $\tau(\Gamma)$ we gain

$$|x_2 - x_1|(b - \varepsilon) \leq (|x_2 - x_1| + 2r + 4dp + 2q - 2p)(a + \varepsilon).$$

By the ratio bound on $a+\varepsilon$ and $b-\varepsilon$ it yields

$$\lambda|x_2 - x_1| \leq |x_2 - x_1| + 2r + 4dp + 2q - 2p.$$

Simple rearrangement yields

$$|x_2 - x_1| \leq \frac{2r + 4dp + 2q - 2p}{\lambda - 1}.$$

The right hand side expression is already fixed, while for the left hand side we have $|x_2 - x_1| \geq q' - p$. Thus if q' is chosen so that it is sufficiently large compared to r, then we get a contradiction, which concludes this case, and also the proof of the fact that $F(0)$ is meager.

The final step of the proof does not differ at all from the final step of the proof given for the case $5 \inf A < \sup A$. Namely, let $F \subseteq \Omega$ be the set of configurations in which there are at least two geodesic rays. Moreover, let $F(x)$ be the set of configurations in which there are at least two distinct geodesic rays with starting point x, and F_m be the set of configurations in which there exist two disjoint geodesic rays with starting point in the cube $[-m, m]^d$. Then

$$F = \left(\bigcup_{x \in \mathbb{Z}^d} F(x) \right) \cup \left(\bigcup_{m=1}^{\infty} F_m \right)$$

holds: if there exist at least two geodesic rays they are either disjoint or have a common point x, and in the latter case we have two geodesic rays starting from x if we forget about the initial parts of these geodesics. Furthermore, we know that each of the sets $F(x)$ are meager by our argument up to this step. Thus if we could obtain that each F_m is nowhere dense, that would conclude the proof. However, having seen the proof of the first part we do not have a difficult task as we can basically repeat that argument. Indeed, in that proof we showed that for a given cylinder set U one can construct boxes K_1, K_2 and another cylinder set $V \subseteq U$ such that for configurations in V any geodesic from ∂K_1 to ∂K_2 goes along the line segment $[p\xi_1, q\xi_1]$. Thus if we choose $p > m$ during the construction we will obtain that none of the configurations in F_m can appear in V as in V there cannot be two disjoint geodesic rays starting from $[-m, m]^d$, given they all meet in $p\xi_1$. Thus F_m is nowhere dense indeed, which concludes the proof of the theorem. □

Theorem 2.3.3 implies that geodesic rays starting from any two distinct points meet after a finite number of edges, which is a rather interesting, extraordinary behavior. It is useful to think a bit about how we should imagine the unique geodesic ray then, what it looks like. It is tempting to imagine a picture in which it has some asymptotic direction. However, using similar techniques to the one seen in the proof, one can easily verify that generically the unique geodesic ray intersects any path of infinite length infinitely many times. Consequently, it is more appropriate to think about it as an infinite path which looks somewhat spiralic in the long run.

Remark 2.3.5. We note that in each case of the proof, in the estimates we only needed that the Γ' we construct is optimal in ℓ_1 between its endpoints amongst paths contained by $\partial K_1 \cup [p\xi_1, q\xi_1] \cup \partial K_2$, while Γ does not use any edge contained by this set. We will refer back to this remark in the proof of Theorem 3.4.1.

2.4 The behavior of $\frac{T(0,\mu x)}{\mu|x|}$

In this section we revisit an old basic result of first passage percolation, that is the existence of the time constants. More explicitly, if we consider any vector x, then under mild conditions on the distribution, the function $\frac{T(0,tx)}{t}$ has an almost sure limit in ∞ which is usually denoted by $\mu(x)$. One may wonder if it also holds in a large subset of Ω in the topological setup. The following theorem shows the converse. For a vector $x \in \mathbb{R}^d$ we denote by $|x|$ the ℓ_1 norm of x, that is the sum of the absolute values of the coordinates of x.

Theorem 2.4.1. *Fix any nonzero vector x. Then generically for any λ with*

$$\inf A \leq \lambda \leq \sup A$$

there exists a sequence $(\mu_k)_{k=1}^{\infty}$ with $\mu_k \to \infty$ such that

$$\lim_{k\to\infty} \frac{T(0,\mu_k x)}{\mu_k|x|} = \lambda.$$

Before proving Theorem 2.4.1, it is worth mentioning that a sequence of the form $\frac{T(0,\mu_k x)}{\mu_k|x|}$ cannot have a limit smaller than $\inf A$ or larger than $\sup A$, regardless of which configuration we observe. Indeed, choose μ large and let us denote by $p(\mu,x)$ the lattice point with the property $\mu x \in p(\mu,x) + [0,1)^d$, that is the lattice point which was used to define the passage time $T(0,\mu x)$. Then we have $|p(\mu,x) - \mu x| < d$. Thus we have

$$(\mu|x| - d)\inf A \leq T(0,\mu x) \leq (\mu|x| + d)\sup A,$$

where we obtain the first inequality by considering any path from 0 to $p(\mu,x)$ and the second one by considering a path between these points with minimal ℓ_1-length. A simple rearrangement verifies our claim. It shows that in Theorem 2.4.1 the λs we consider are the only ones which we have to observe.

Proof of Theorem 2.4.1. By a simple rescaling it is easy to see that it suffices to prove the statement for $x \in \mathbb{R}^d$ with $|x| = 1$. Indeed, if for a given λ the sequence of coefficients μ_k yields the given limit for the point $\frac{x}{|x|}$ then the sequence of coefficients $\mu_k|x|$ will be fine for the point x. In the spirit of this remark let us fix $x \in \mathbb{R}^d$ with $|x| = 1$. We say that $\frac{T(0,\mu x)}{\mu}$ is a normalized passage time in the direction of x.

Let us denote by S the set of configurations that are "bad" for us, namely the subset of Ω in which there exists some (finite or infinite) λ with $\inf A \le \lambda \le \sup A$ such that there is no sequence $(\mu_k)_{k=1}^{\infty}$ with $\mu_k \to \infty$ such that $\lim_{k\to\infty} \frac{T(0,\mu_k x)}{\mu_k} = \lambda$. In this case, there is surely such a finite λ, thus in our further arguments we think about S this way. Our aim is to express S as a countable union of sets which are easier to handle and prove that these sets are nowhere dense. Having this purpose in mind, we will denote by $S(\lambda, \delta, M)$ the set of configurations in which for any $\mu > M$ we have that the distance of $\frac{T(0,\mu x)}{\mu}$ and λ is larger than δ. The following equation clearly holds:

$$S = \bigcup_{\inf A<\lambda<\sup A} \bigcup_{\delta>0} \bigcup_{M>0} S\left(\lambda, \delta, M\right).$$

Indeed, by the definition of convergence if there is no appropriate sequence of coefficients for a given $\lambda \in (\inf A, \sup A)$ then there exists a neighborhood of it such that $\frac{T(0,\mu x)}{\mu}$ is not in this neighborhood for large enough μ. However, by basic separability arguments on the real line we have that it further equals

$$S = \bigcup_{\lambda\in\mathbb{Q}, \inf A<\lambda<\sup A} \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} S\left(\lambda, \frac{1}{n}, m\right),$$

which is a decomposition we pursued.

Having this knowledge it suffices to prove that all the sets $S\left(\lambda, \frac{1}{n}, m\right)$ are nowhere dense. In order to prove this, fix λ, n, m, and fix real numbers a, b with

$$\inf A \le a < \lambda < b \le \sup A.$$

(This step has importance only if A is unbounded, and its sole technical role is that we cannot calculate with $\sup A$ in this case, thus it needs to be replaced by a finite quantity.)

Clearly it suffices to prove our claim for large enough n, as for fixed λ and m the sequence $S\left(\lambda, \frac{1}{n}, m\right)$ is growing as n tends to infinity. Thus without loss of generality it suffices to consider the case when $a + \frac{1}{n} < \lambda < b - \frac{1}{n}$.

As usual, fix U to be a cylinder set, and denote the set of edges belonging to nontrivial projections of U by $E_U = \{e_1, e_2, ..., e_k\}$. As in the proof of Theorem 2.1.1, we can construct a smaller cylinder set by shrinking the projections $U_{e_1}, ..., U_{e_k}$, such that all of these projections are bounded in $\mathbb{R}$. Again, we denote these new projections by U'_{e_i}, $i = 1, ..., k$, and the cylinder set defined by them by U'. Then for any configuration in U', the sum of passage times over the edges $e_1, ..., e_k$ is bounded by a constant C. Our goal is to find a cylinder set $V \subseteq U'$ and some $\mu > m$ such that the distance of $\frac{T(0,\mu x)}{\mu}$ and λ is at most $\frac{1}{n}$ for any configuration in V. We state that for suitably large μ it is possible to find such V. Consider a large $\mu > 1$, its exact value is to be determined later.

Now fix a path Γ_0 with minimal ℓ_1-length from the origin to $p(\mu, x)$. Roughly we would like to define V such that it has nontrivial projections to the edges in E_U and to the edges in a large box K containing 0 and $p(\mu, x)$. (The size of K is also to be fixed later.) Concerning the edges in $\Gamma_0 \setminus E_U$, we would like to define the projections so that the passage time of Γ_0 is close to $\lambda\mu$, by having projections close to a or b with a suitable frequency. For the other edges in K we would like to have projections close to b in order to guarantee that the passage time between 0 and $p(\mu, x)$ is not reduced too much by another path.

Rigorously speaking, choose μ sufficiently large so that $|p(\mu, x)| = N_1 + N_2$ for some positive integers satisfying

$$\frac{aN_1 + bN_2}{|p(\mu, x)|} \in \left(\lambda - \frac{1}{4n}, \lambda + \frac{1}{4n}\right).$$

As $|p(\mu, x)|$ can be arbitrarily large and the length of the interval we aim at is fixed, it is simple to see that we can choose μ, N_1, N_2 to satisfy this relation. Moreover, as the distance of $|p(\mu, x)|$ and μ is bounded by d, for suitably large μ this yields

$$\frac{aN_1 + bN_2}{\mu} \in \left(\lambda - \frac{1}{2n}, \lambda + \frac{1}{2n}\right). \tag{2.1}$$

Now we choose N_1 edges of Γ_0, and for the ones not in E_U, we require V to have projection $(a - \varepsilon_e, a + \varepsilon_e) \cap A$ to any such edge e, such that the sum of these ε_es is at most $\frac{1}{4n}$. These are the cheap edges. We proceed similarly for all the other edges in K: for the ones not in E_U, we require V to have projection $(b - \varepsilon_e, b + \varepsilon_e) \cap A$ to any such edge e, such that the sum of these ε_es is at most $\frac{1}{4n}$. These are the expensive edges, and by the choice of n, they are bounded away from the cheap ones. As the number of edges in E_U is fixed and N_1, N_2 can grow arbitrarily large for large $|p(\mu, x)|$, the projection to the majority of the edges in Γ_0 will be either cheap or expensive. We fix K now: define it such that any path leaving K contains at least $|p(\mu, x)|$ expensive edges.

Now our only remaining task is to estimate the passage time between 0 and $p(\mu, x)$ for configurations in V. Our aim is to verify that we have

$$T(0, p(\mu, x)) \in \left[\mu\left(\lambda - \frac{1}{n}\right), \mu\left(\lambda + \frac{1}{n}\right)\right], \tag{2.2}$$

which would follow from

$$\tau(\Gamma) > \mu\left(\lambda - \frac{1}{n}\right) \tag{2.3}$$

for any path Γ from 0 to $p(\mu, x)$ and

$$\tau(\Gamma_0) < \mu\left(\lambda + \frac{1}{n}\right). \tag{2.4}$$

In order to check (2.3), consider now any path Γ from 0 to $p(\mu, x)$. If Γ leaves K, it contains at least $N_1 + N_2$ expensive edges, which results in

$$\frac{\tau(\Gamma)}{\mu} \geq \frac{b(N_1 + N_2) - \frac{1}{4n}}{\mu} > \frac{aN_1 + bN_2 - \frac{1}{4n}}{\mu} > \lambda - \frac{1}{n},$$

by (2.1), $\mu > 1$ and the condition on the expensive edges. Thus we have (2.3) for these paths. Assume now that Γ stays in K. Then $|\Gamma| \geq |p(\mu, x)|$, and at most k edges of Γ is in E_U. Thus Γ has at least $N_1 + N_2 - k$ edges which are either cheap or expensive. As amongst these at most N_1 are cheap, we have the following lower bound on the passage time of Γ if we forget about the edges in $E_U \cap \Gamma$ and consider the trivial lower estimates for the number and passage times of cheap and expensive edges:

$$\frac{\tau(\Gamma)}{\mu} \geq \frac{aN_1 + b(N_2 - k) - \frac{1}{4n}}{\mu} > \lambda - \frac{1}{n},$$

by (2.1) for large enough μ as $\frac{bk}{\mu}$ tends to 0. It verifies (2.3) for any path from 0 to $p(\mu, x)$, hence it remains to show (2.4). However, it can be done similarly. We know that Γ_0 contains at most N_1 cheap edges, N_2 expensive edges, and the sum of passage times on the edges in $E_U \cap \Gamma$ is bounded by C for any configuration in V. Thus we have

$$\frac{\tau(\Gamma)}{\mu} \leq \frac{aN_1 + bN_2 + \frac{1}{4n} + C}{\mu} < \lambda + \frac{1}{n},$$

by (2.1) for large μ, which verifies (2.4), and concludes the proof. □

2.5 The behavior of $\frac{B(t)}{t}$

In probability theory, a fundamental result related to the existence of time constants is the Cox–Durrett shape theorem. Let us denote by $B(t)$ the ball of radius t centered at the origin in the pseudometric T, that is the subset of $\mathbb{R}^d$ we might reach from the origin in time t. A truly interesting result of the theory (see [5]) is that there exists a so-called limit shape B_μ, which has the property that as t tends to infinity, with probability one $\frac{B(t)}{t}$ tends to B_μ in some sense. Moreover, B_μ depends only on the distribution μ. Various works can be found in the literature based on this theorem about the speed of this convergence for example. We might ask if a similar statement holds generically in the topological setup. The results of this section point out it is quite far from the truth: along suitable sequences (t_n) we might obtain different sets as the Hausdorff limit of $\frac{B(t_n)}{t_n}$ simultaneously. To make the discussion simpler, we introduce the following definition:

Definition 2.5.1. *We say that K is a limit set of the percolation in some configuration, if there exists a sequence (t_n) with $t_n \to +\infty$ such that $\frac{B(t_n)}{t_n} \to K$ in the Hausdorff metric.*

Let us denote by D_r the ℓ_1 closed ball of radius r centered at 0, and let $\mathcal{K}_A^d$ be the set of connected compact sets in $\mathbb{R}^d$ satisfying

$$D_{\frac{1}{\sup A}} \subseteq K \subseteq D_{\frac{1}{\inf A}},$$

where the leftmost set is replaced by $\{0\}$ if $\sup A = \infty$, and the rightmost set is replaced by $\mathbb{R}^d$ if $\inf A = 0$. Furthermore, we say that $K \in \mathcal{P}_A^d$ if $K \in \mathcal{K}_A^d$, if for each $x \in K$ there is a "topological path" in K of ℓ_1-length at most $\frac{1}{\inf A}$ from 0 to x. (From now on, we use the terms path and topological path in order to clearly distinguish paths in graph theoretical sense and paths in topological sense.) Its closure in $\mathcal{K}_A^d$ with respect to the Hausdorff metric is denoted by $\overline{\mathcal{P}_A^d}$. Moreover, it is worth mentioning that $\overline{\mathcal{P}_A^d}$ also equals the closure of $\mathcal{P}_A^{d,-}$, which is the set of Ks satisfying that there exists $\alpha_K > 0$ such that for each $x \in K$ there is a topological path in K of ℓ_1-length at most $\frac{1}{\inf A} - \alpha_K$ from 0 to x. This remark proves to be technically helpful.

First let us note that if K is a limit set then $K \in \mathcal{K}_A^d$. This claim can be verified similarly as the necessity of the conditions of Theorem 2.4.1. For example even if every passage time would be $\inf A$, which yields that $B(t)$ is as large as can be for each t, the limit of $\frac{B(t)}{t}$ would be $D_{\frac{1}{\inf A}}$, and not larger. The connectedness of K is also obvious as this property is conserved by taking Hausdorff limit. What is a bit more difficult and intriguing that the limit sets are also in the smaller family $\overline{\mathcal{P}_A^d}$. If $\inf A = 0$, we have $\mathcal{P}_A^d = \mathcal{K}_A^d$, thus it does not require further explanation. Consider the general case, and proceeding towards a contradiction assume the converse. Denote by $\tilde{B}(t)$ the subgraph of $\mathbb{Z}^d$ which is accessible from the origin in time t. Then as the Hausdorff distance of $B(t)$ and $\tilde{B}(t)$ is uniformly bounded by a constant dependent only on the dimension, we have that $\frac{\tilde{B}(t_n)}{t_n}$ also converges to K in Hausdorff distance. However, we know that $\tilde{B}(t_n)$ is a connected subgraph of $\mathbb{Z}^d$, and each of its points is accessible from the origin using a path with ℓ_1-length $\frac{t_n}{\inf A}$. Thus any point of $\frac{\tilde{B}(t_n)}{t_n}$ is accessible from the origin using a topological path, which stays in the set, and has ℓ_1-length at most $\frac{1}{\inf A}$. It immediately yields that

$$\frac{\tilde{B}(t_n)}{t_n} \in \mathcal{P}_A^d.$$

As a consequence, $K \in P_A^d$ also holds and K is the Hausdorff limit of the sets $\frac{\tilde{B}(t_n)}{t_n}$, a contradiction. Hence $K \in \overline{\mathcal{P}_A^d}$ holds indeed if K is a limit set.

We also note that if $d = 1$, then $\mathcal{K}_A^d = \mathcal{P}_A^d$. To conclude the list of our initial observations about $\mathcal{P}_A^d$, we point out that it contains certain natural classes of sets, even if $\inf A \neq 0$. First of all, it is quite obvious that it contains all the convex sets of $\mathcal{K}_A^d$. Moreover, it contains the star domains of $\mathcal{K}_A^d$ with respect to the origin. We also mention a less natural class: we introduce the notion of star domains in ℓ_1 sense. The set $S \subseteq \mathbb{R}^d$ is a star domain with respect to $x_0 \in S$ in ℓ_1 sense (or generalized star domain with respect to x_0), if for any $x \in S$ there is a topological path from x_0 to x in S with ℓ_1-length $|x - x_0|$. In other words, each of the coordinate functions of the topological path are monotone. We denote the subset of $\mathcal{K}_A^d$ containing the generalized star domains with respect to 0 by $\mathcal{K}_A^{d,*}$. Then $\mathcal{K}_A^{d,*} \subseteq \mathcal{P}_A^d$ also holds.

Now our aim is to prove the following theorem which completely characterizes the generic behavior of limit sets if $\inf A = 0$, or $\sup A = \infty$:

Theorem 2.5.2. *Assume that* $\inf A = 0$, *or* $\sup A = \infty$. *Then generically we have that K is a limit set if and only if* $K \in \overline{\mathcal{P}_A^d}$.

In the following we will prove this theorem through a few steps. Denote by $\mathcal{P}_{A,0}^{d,-}$ the set that contains those sets of $\mathcal{P}_A^{d,-}$ which can be expressed as the closure of a connected open set. It is easy to see that $\mathcal{P}_{A,0}^{d,-}$ is dense in $\mathcal{P}_A^{d,-}$. Indeed, if $K \in \mathcal{P}_A^{d,-}$, denote by $K(r)$ the set of points which are in $D_{\frac{1}{\inf A}}$ and at most r apart from K in ℓ_1. Then for sufficiently small r the relation $K(r) \in \mathcal{P}_A^{d,-}$ holds: for any point $x \in K(r)$ we might choose $x' \in K$ within distance r. Then there is a topological path from the origin to x' in K of ℓ_1-length at most $\frac{1}{\inf A} - \alpha_K$, which can be continued by a line segment of ℓ_1-length r to x. (Here we use the fact that $D_{\frac{1}{\inf A}}$ is convex.) Moreover, $K(r)$ is the closure of a connected open set: by the compactness of K, it is simply the closure of $K_0(r)$, the set of points which are in $D_{\frac{1}{\inf A}}$ and less than r apart from K in ℓ_1. Consequently, $\overline{\mathcal{P}_{A,0}^{d,-}} = \overline{\mathcal{P}_A^d}$ also holds. Thus if we could verify the modified statement of Theorem 2.5.2 which we obtain by replacing $\mathcal{P}_A^d$ by $\mathcal{P}_{A,0}^{d,-}$, that would be sufficient. We also recall that instead of the desired convergence of $\frac{B(t_n)}{t_n}$, it suffices to prove the same for $\frac{\tilde{B}(t_n)}{t_n}$.

Now by a standard argument about the separability of $\mathcal{P}_{A,0}^{d,-}$ it suffices to prove that for a given set $K \in \mathcal{P}_{A,0}^{d,-}$ we can find a suitable sequence of times in a residual subset of Ω. (We know that $\mathcal{P}_{A,0}^{d,-}$ is separable as it is a subspace of the separable $\mathcal{K}_A^d$, which is a metric space.) Our proof will rely on constructing cylinder sets in which we have a large control on $\tilde{B}(t)$. In other words, we desire to construct subgraphs of $\mathbb{Z}^d$ which are close to tK. To formalize this idea, we will need the following lemma:

Lemma 2.5.3. *Let $K \in \mathcal{P}^{d,-}_{A,0}$ and $\varepsilon > 0$ fixed. Denote by G_n the embedded graph whose vertices are the vertices of $\frac{\mathbb{Z}^d}{n}$ in K, and edges are those nearest neighbor edges which lie entirely in K. Then for infinitely many n the graph G_n has a connected subgraph H_n such that it contains all the vertices and edges in $D_{\frac{1}{\sup A}}$, satisfies $d_H(K, H_n) < \varepsilon$, and to all of its vertices there is a path from the origin of ℓ_1-length smaller than $\frac{1}{\inf A}$.*

If $\inf A = 0$, we have $\mathcal{P}^{d,-}_{A,0} = \mathcal{K}^d_{A,0}$, and the last condition on H_n is tautological. In this neater form, we find this lemma interesting in its own right as a nice exercise of a course in analysis. (It is likely to be known in some form, but we could not find a reference for it.)

Proof of Lemma 2.5.3. As $\operatorname{int} K$ is a connected open set, the points with rational coordinates in $\operatorname{int} K$ form a dense subset of K. Let us consider now a open ball of radius ε centered at each point with rational coordinates in $\operatorname{int} K$. These balls give an open cover of the compact set K, thus we might choose a finite cover. Denote the centers of these balls by $v_1, ..., v_m$. The coordinates of these points might have only finitely many distinct denominators. Thus if n is chosen as a common multiple of them, H_n can contain all these points, which guarantees $d_H(K, H_n) < \varepsilon$. What remains to show that is for large enough such n, it is possible to choose a connected H_n satisfying the condition about the lengths of paths such that it contains the points $v_1, ..., v_m$, and the vertices and edges in $D_{\frac{1}{\sup A}}$.

As $K \in \mathcal{P}^{d,-}_{A}$, there are topological paths $\gamma_1, ...\gamma_m : [0, 1] \to \operatorname{int} K$ from 0 to $v_1, ...v_m$ with ℓ_1-length less than $\frac{1}{\inf A}$. As each of the sets $\gamma_i([0, 1])$ are compact and contained by $\operatorname{int} K$, it is possible to choose $r > 0$ such that their neighborhoods of radius r are also contained by $\operatorname{int} K$. Moreover, by the definition of ℓ_1-length we might choose points on $\gamma_i([0, 1])$ such that they can be connected by a broken line L_i with pieces parallel to the coordinate vectors, and its length is also less than $\frac{1}{\inf A}$. Furthermore, by the existence of r, if we choose a suitably fine partition of $\gamma_i([0, 1])$, we might have $L_i \in \operatorname{int} K$. For the sake of simplicity, denote the vertices of L_i by $0 = p_1, ..., p_k = v_i$. Now for $\beta > 0$ fixed, we can choose n so large that $\frac{\mathbb{Z}^d}{n}$ has vertices closer than β in ℓ_1 to any vertex of L_i. Denote such vertices of $\frac{\mathbb{Z}^d}{n}$ by $0 = q_1, ..., q_k = v_i$. Also if n is large enough, if we consider the smallest lattice hypercubes of $\frac{\mathbb{Z}^d}{n}$ crossed by L_i, they are still in $\operatorname{int} K$, and $q_1, ..., q_k$ might be chosen to be the vertices of these cubes. Thus using the edges of these cubes we can find a path $\Gamma_{i,n}$ in G_n from 0 to v_i, which stays in $\operatorname{int} K$, and optimal in ℓ_1 between any vertices q_j and q_{j+1}. Hence we can deduce by triangle inequality that

$$|\Gamma_{i,n}| = \sum_{j=1}^{k-1} |q_j q_{j+1}| \leq \sum_{j=1}^{k-1} |p_j p_{j+1}| + 2\sum_{j=1}^{k} |p_j q_j| \leq |L_i| + 2k\beta.$$

As k is fixed and β can be arbitrarily small, it guarantees that for large enough n the length $|\Gamma_{i,n}|$ is less than $\inf A$. We can define H_n for infinitely many n appropriately based on this argument: we require it to contain all the vertices and edges of $\frac{\mathbb{Z}^d}{n}$ in $D_{\frac{1}{\sup A}}$, it is clearly connected and all the vertices are accessible by a path of ℓ_1-length less than $\frac{1}{\inf A}$. Furthermore, we require it to contain the vertices $v_1, ..., v_m$ and the paths $\Gamma_{i,n}$, which does not mess up the condition about the distance of vertices from the origin, and guarantees the bound on the Hausdorff distance. $\square$

Proof of Theorem 2.5.2. By our previous remarks, it suffices to prove that if $K \in \mathcal{P}^{d,-}_{A,0}$, then in a residual subset of Ω there exists a suitable sequence $t_n \to \infty$ with $\frac{\tilde{B}(t_n)}{t_n} \to K$. Denote the set of configurations not having this property by $F(K)$. Then by the definition of convergence, $F(K)$ can be expressed as a countable union as follows:

$$F(K) = \bigcup_{i=1}^{\infty} \bigcup_{m=1}^{\infty} F\left(K, \frac{1}{i}, m\right),$$

where $F(K, \varepsilon, \mu_0)$ stands for the set of configurations in which for any $\mu > \mu_0$ we have

$$d_H\left(K, \frac{\tilde{B}(\mu)}{\mu}\right) > \varepsilon.$$

Verifying that $F\left(K, \frac{1}{i}, m\right)$ is nowhere dense for each i, m would conclude the proof. Clearly it suffices to do so for large enough i, m.

As usual, fix U to be a cylinder set, and denote the set of edges belonging to nontrivial projections of U by $E_U = \{e_1, e_2, ..., e_k\}$. As in the proof of Theorem 2.1.1, we can construct a smaller cylinder set by shrinking the projections $U_{e_1}, ..., U_{e_k}$, such that all of these projections are bounded in $\mathbb{R}$. Again, we denote these new projections by U'_{e_i}, $i = 1, ..., k$, and the cylinder set defined by them by U'. Then for any configuration in U', the sum of passage times over the edges $e_1, ..., e_k$ is bounded by a constant C. Our goal is to find a cylinder set $V \subseteq U'$ and some $\mu > m$ such that the Hausdorff distance of $\frac{\tilde{B}(\mu)}{\mu}$ and K is at most $\frac{1}{i}$ for any configuration in V. We distinguish the cases based on the value of $\inf A$ and $\sup A$. The idea will be the same in the three cases, but the realization will vary.

(i) Assume first that $\inf A = 0$ and $\sup A = \infty$, as technically it is the easiest. We pursue μ as a large enough $n \in \mathbb{N}$ for which $n > C$ and which satisfies Lemma

2.5.3 with $\varepsilon = \frac{1}{2i}$. Now we try to choose V such that for any configuration in V, the set $\tilde{B}(n)$ is close to nH_n, which is a subgraph of $\mathbb{Z}^d$. For this aim, denote the edge set of nH_n by $E(nH_n)$. For any edge $e \in E(nH_n) \setminus E_U$ we define V_e to be $[0, \varepsilon_e) \cap A$, where the ε_es are small enough to have a smaller sum than $n - C$. Furthermore, for any further edge e leaving the graph nH_n or neighboring to one of the edges in E_U, we define V_e to have strictly larger elements than n. By the first part of the definition $nH_n \subseteq \tilde{B}(n)$ obviously holds. Furthermore, $\tilde{B}(n)$ may differ from nH_n in only the edges of E_U, which yields that their Hausdorff distance is at most k. As a consequence, since the Hausdorff distance of nH_n and K is at most $\frac{n}{2i}$, by triangle inequality we have that

$$d_H\left(K, \frac{\tilde{B}(n)}{n}\right) \leq \frac{1}{2i} + \frac{k}{n} \leq \frac{1}{i},$$

if n is large enough. It concludes the proof in this case.

(ii) If $\inf A = 0$ and A is bounded, the proof relies on the same concept, but our task is a bit more difficult. We choose n as in (i), and look for V with a similar property. If n is large enough, we have $E_U \subseteq nH_n$. Choose $N \in \mathbb{N}$ with $N \sup A > C$. For an edge e in $D_{\frac{n}{\sup A} - N}$, which is not contained by E_U, we define V_e to be $(\sup A - \varepsilon_e, \sup A) \cap A$, where the ε_es are small enough, they are to be fixed later. Thus these are expensive edges. For any other edge e of $E(nH_n) \setminus E_U$ we stick to the definition in (i): $V_e = [0, \varepsilon_e) \cap A$, here ε_e is small again, these are cheap edges. Finally, for any further edge e with distance at most $2N$ from the graph nH_n, we define V_e to be $(\sup A - \varepsilon_e, \sup A) \cap A$, hence these are expensive edges again. Now if we consider any point $x \in nH_n$, there is a path Γ to it from the origin which might use the edges of E_U, and uses at most $\left\lceil \frac{n}{\sup A} - N \right\rceil$ expensive edges. All the other edges in Γ are cheap. Thus by the definition of N, if we choose the ε_es to have small enough sum, the passage time of Γ is bounded by n for any configuration in V, which results in $x \in \tilde{B}(n)$. Furthermore, if a point x is further from nH_n than $2N$, any path from 0 to x uses more than $\frac{n}{\sup A}$ expensive edges, which yields that if the ε_es have small enough sum, $x \notin \tilde{B}(n)$. As a consequence, the Hausdorff distance of $\tilde{B}(n)$ and nH_n is at most $2N$, which is fixed. The final step is the same triangle inequality as in (i).

(iii) Assume $\inf A > 0$ and $\sup A = \infty$. We might attempt to copy the argument of (i). The only difficulty is that we have to replace the projections $[0, \varepsilon_e) \cap A$ by $(\inf A, \inf A + \varepsilon_e) \cap A$. Now by the last condition on H_n in Lemma 2.5.3, for any of vertex x of nH_n there exists a path Γ from 0 to x with ℓ_1-length less than $\frac{n}{\inf A}$. Thus if $C < N \inf A$, we might obtain that for any such x with

$|x| < \frac{n}{\inf A} - N$, the passage time of Γ is at most n for a good choice of ε_e. This means that $\tilde{B}(n)$ contains all the points of nH_n for any configuration in V, except for possibly those ones which are closer to $\partial D_{\frac{n}{\inf A}}$ than N. On the other hand, $\tilde{B}(n)$ cannot contain a point which is further from nH_n than k. Thus the Hausdorff distance of nH_n and $\tilde{B}(n)$ can be bounded by $N + k$, and the proof might be finished using the triangle inequality.

□

One may wonder what happens in the case when A is bounded away both from 0 and $+\infty$. If $d = 1$, it does not require much effort to show that generically any $K \in \mathcal{K}_A^d = \mathcal{P}_A^d$ is a limit set. However, if $d \geq 2$, one faces difficulties as $\overline{\mathcal{P}_A^d}$ is not the family of limit sets anymore:

Proposition 2.5.4. *For suitable A, there exists $K \in \mathcal{P}_A^d$ which is not a limit set.*

Proof. Let $A = \{1, 2\}$, and let $K = D_{\frac{1}{2}} \cup [0, \xi_1]$. Then $K \in \mathcal{P}_A^d$ clearly holds, as K is a star domain with respect to 0. We state there is no configuration in Ω and a sequence $(t_n)_{n=1}^{\infty}$ tending to infinity with $\frac{B(t_n)}{t_n} \to K$. Assume the converse: there exists such a configuration and such a sequence of times. Then by the condition $\frac{B(t_n)}{t_n} \to K$, there exists a sequence of points $x_n = t_n\xi_1 + o(t_n)v_n$, where $|v_n| = 1$, and a path Γ_n from 0 to x_n with passage time $\tau(\Gamma_n) = t_n + o(t_n)$. (Here $o(t_n)$ denotes a sequence of quantities which satisfies $\frac{o(t_n)}{t_n} \to 0$ as $n \to \infty$.) This guarantees that Γ_n contains at most $o(t_n)$ edges with passage time 2. Moreover, for large enough n, these paths cross the boundary of $D_{\frac{1}{2}t_n}$ at some point y_n. Denote the piece of Γ_n from 0 to y_n by Γ'_n. As Γ'_n also contains at most $o(t_n)$ edges with passage time 2, it is simple to check that it guarantees

$$T(0, y_n) \leq \tau(\Gamma'_n) = \frac{t_n}{2} + o(t_n).$$

Without loss of generality, we can assume that each y_n lies in the upper half-plane. Based on the previous inequality, for arbitrary fixed $\alpha > 0$ we have

$$\begin{aligned} T(0, y_n + [\alpha t_n](\xi_1 + \xi_2)) &\leq T(0, y_n) + T(y_n, y_n + [\alpha t_n](\xi_1 + \xi_2)) \\ &\leq \left(\frac{1}{2} + 4\alpha\right) t_n + o(t_n), \end{aligned} \tag{2.5}$$

if we estimate the second passage time of the middle expression by $4\alpha t_n$, which is a valid upper bound by the choice of A and the ℓ_1 distance of the two points observed. For large n and small enough α, it is strictly smaller than t_n, thus we have that $z_n = y_n + [\alpha t_n](\xi_1 + \xi_2)$ is in $B(t_n)$. Moreover, for large enough n, the Euclidean distance of z_n from both the first coordinate axis and $\partial D_{\frac{1}{2}t_n}$ is at least $\frac{\alpha t_n}{2}$. Thus the distance of $\frac{z_n}{t_n}$ from K is at least $\frac{\alpha}{2}$ for large n. On the other hand, the sequence $\left(\frac{z_n}{t_n}\right)_{n=1}^{\infty}$ is in D_1, thus it has a convergent subsequence with limit $z \in D_1$ with distance at least $\frac{\alpha}{2}$ from K. However, z is contained by the Hausdorff limit of $\frac{B_{t_n}}{t_n}$ by $z_n \in B_{t_n}$, which is K, a contradiction. □

This proposition shows that in the cases not handled by Theorem 2.5.2, we need to modify the statement itself. Requiring convexity might be an attractive idea, as one might feel that in the example above the failure is somewhat caused by the lack of it, however, it is not complicated to construct configurations in which $\frac{B_t}{t}$ tends to a concave shape. Thus it is not the proper way to completely overcome this difficulty. However, in [10] we did not even manage to prove that the convex sets of $\mathcal{K}_A^d$ are limit sets. The proof of this conjecture was one of the main results of [11]:

Theorem 2.5.5. *Let A be arbitrary. Then generically any convex $K \in \mathcal{K}_A^d$ is a limit set.*

From now on our goal is to prove Theorem 2.5.5. In the following we will assume that A is bounded away both from 0 and $+\infty$ since the other cases are covered by Theorem 2.5.2 as the convex sets of $\mathcal{K}_A^d$ are in $\overline{\mathcal{P}_A^d}$. What we gain by this assumption is that we circumvent certain technical difficulties, however, we note that some of the definitions and results could be generalized to these extremes.

We start our investigations by introducing two families of metrics in $\mathbb{R}^d$:

Definition 2.5.6. *Let f be a nonnegative, measurable function. The pseudometric d_{f,ℓ_1} induced by f is defined by*

$$d_{f,\ell_1}(x,y) = \inf_{\Gamma: x \to y} \int_\Gamma f(t) \mathrm{d}s,$$

where the arc length is considered in ℓ_1, and we consider piecewise linear topological paths with finitely many pieces upon taking infimum.

We note that we mean piecewise linearity in the usual general sense, that is we do not require the pieces to be parallel to the coordinate axes.

Definition 2.5.7. *We call a pseudometric ρ on $\mathbb{R}^d$ a percolation metric with support A if there exists a measurable function $f : \mathbb{R}^d \to \overline{A}$ so that $\rho = d_{f,\ell_1}$.*

In the sequel we omit ℓ_1 from the subscript as we are only concerned with ℓ_1 based metrics, and unless it may cause ambiguity we will not write out the suffix "with support A" either. The family of sets arising as closed unit balls of percolation metrics, centered at the origin, will be denoted by $\mathcal{W}_A^d$. As A is bounded away from both 0 and $+\infty$, the elements of $\mathcal{W}_A^d$ are compact sets, each of them is the closure of its interior. Moreover, each percolation metric with support A is a proper metric indeed. The closure of $\mathcal{W}_A^d$ in $\mathcal{K}_A^d$ is denoted by $\overline{\mathcal{W}_A^d}$.

The following theorem is simple to prove and displays how percolation metrics are related to the limit of sequences of the type $\frac{B(t_n)}{t_n}$:

Theorem 2.5.8. *Assume that $\frac{B(t_n)}{t_n} \to K$ in the Hausdorff metric in some configuration for a sequence t_n diverging to $+\infty$. Then $K \in \overline{\mathcal{W}_A^d}$.*

Remark 2.5.9. Due to our earlier remarks, this theorem yields $\overline{\mathcal{W}_A^d} \subseteq \overline{\mathcal{P}_A^d}$.

Proof of Theorem 2.5.8. Consider the subgraph $\tilde{B}(t_n)$ of $\mathbb{Z}^d$ accessible in time t_n from the origin. We obviously have $\frac{\tilde{B}(t_n)}{t_n} \to K$ by assumption. Now we define f_n as follows: in the relative interior of an edge of the graph $\frac{\mathbb{Z}^d}{t_n}$ we define f_n to have the same value as the passage time of the corresponding edge in $\mathbb{Z}^d$. (In the endpoints this definition would not give a unique value, but the value on a discrete set will not have any importance anyway.) For any remaining point $x \in \mathbb{R}^d$ we set $f_n(x) = \sup A$. We state that the Hausdorff distance of the closed unit ball B_n of d_{f_n} and $\frac{\tilde{B}(t_n)}{t_n}$ converges to 0: as $B_n \in \mathcal{W}_A^d$ that would conclude the proof. As the containment $\frac{\tilde{B}(t_n)}{t_n} \subseteq B_n$ is obvious, we only have to examine how far a point of B_n can lie from the graph $\frac{\tilde{B}(t_n)}{t_n}$. We also note that the value of f_n only matters in $D_{\frac{1}{\inf A}}$, as $B_n \subseteq D_{\frac{1}{\inf A}}$ necessarily holds.

First let us notice that if x is a vertex of $\frac{\mathbb{Z}^d}{t_n}$ then in the definition of $d_{f_n}(0, x)$ it suffices to consider topological paths which are also paths in the graph $\frac{\mathbb{Z}^d}{t_n}$. Indeed, by definition for any $\varepsilon > 0$ there exists a topological path Γ from the origin to x so that we have

$$\int_\Gamma f_n(t)\mathrm{d}s < d_{f_n}(0, x) + \varepsilon.$$

Our aim is to show that there exists a Γ' which is a path in the graph $\frac{\mathbb{Z}^d}{t_n}$ and the integral of f_n on Γ' does not exceed the integral of f_n on Γ. If Γ itself is such a path then we are done. Moving towards a contradiction, assume that there is a point x and a path Γ connecting the origin to x which is not such a path, and there is no such Γ'. Let $N(\Gamma)$ be the number of pieces of Γ in the graph $\frac{\mathbb{Z}^d}{t_n}$, where by such pieces we mean largest connected components in one of the edges of $\frac{\mathbb{Z}^d}{t_n}$. Meanwhile let $M(\Gamma)$ be the number of complementary components of Γ. Now choose a contradictory x, Γ so that $M(\Gamma)$ is minimal, and amongst these one for which $N(\Gamma)$ is minimal. As Γ is not a path in the graph $\frac{\mathbb{Z}^d}{t_n}$, we can choose line segments $[y, y']$ and $[z, z']$ so that they are consecutive pieces in the above sense, that is they are respectively contained by some edges $\frac{e_y}{t_n}, \frac{e_z}{t_n}$, and the subpath Γ_1 of Γ between y' and z might hit the graph $\frac{\mathbb{Z}^d}{t_n}$ in a discrete set only. Now choose Γ' to be a modification of Γ:

from the origin to y and from z to x we do not alter Γ, but we replace some of the remaining parts based on the relation between $\tau(e_y)$ and $\tau(e_z)$ and the relative position of these edges. Notably, unless e_y is an orthogonal translated image of e_z, it is simple to check that there is an ℓ_1-optimal topological path Γ_1' from y' to z which is in fact a path in the graph $\frac{\mathbb{Z}^d}{t_n}$. Moreover, as Γ_1 does not hit the graph $\frac{\mathbb{Z}^d}{t_n}$, we know $f_n = \sup A$ in Γ_1 almost everywhere, while it is at most $\sup A$ in Γ_1'. Hence, if we define Γ' as the topological path gained from Γ by replacing Γ_1 with Γ_1', we reduce the d_{f_n}-length, thus Γ' also has to be a contradictory topological path from the origin to x. (Or it is already a path in the graph, which would also be a contradiction.) However, we reduced $M(\Gamma)$ by one, which contradicts the choice of x, Γ. Thus we have a contradiction in the case when e_y cannot be obtained from e_z by an orthogonal translation.

Let us consider the other case. Let us also assume $\tau(e_y) \leq \tau(e_z)$, in the other case we can use the same argument by symmetry. In this case we proceed the following way: we project orthogonally z' to $\frac{e_y}{t_n}$ to gain z^*, and we gain Γ' by replacing the subpath of Γ from y to z' by $[y, z^*] \cup [z^*, z']$. Now

$$|y - z^*| \leq |y' - y| + |z' - z|,$$

hence by $\tau(e_y) \leq \tau(e_z)$ we have that the integral of f_n on $[y, z^*] \subseteq \Gamma'$ cannot exceed the integral of f_n on $[y, y'] \cup [z, z'] \subseteq \Gamma$. On the other hand, the part of Γ' connecting the edges $\frac{e_y}{t_n}$ and $\frac{e_z}{t_n}$ is ℓ_1-optimal, hence the integral of f_n here cannot exceed the integral of f_n on the corresponding part of Γ, either. Consequently, $\int_{\Gamma'} f_n \leq \int_\Gamma f_n$, thus Γ' also has to be a contradictory topological path from the origin to x. (Or it is already a path in the graph, which would also be a contradiction.) However, we eliminated the piece $[z, z']$ of Γ, hence we reduced $N(\Gamma)$ by one while not increasing $M(\Gamma)$. (Except for the case when z^*, z' are both vertices of the graph $\frac{\mathbb{Z}^d}{t_n}$, that is $[z^*, z']$ is the union of a few consecutive edges, but in this case, we reduce $M(\Gamma)$ by the previous step.) It contradicts the choice of x, Γ. Thus the claim of the previous paragraph holds.

Now let $x \in B_n$ arbitrary. Consider the smallest lattice hypercube of $\frac{\mathbb{Z}^d}{t_n}$ containing x. Denote an arbitrary vertex of it by x'. Now $|x - x'| \leq dt_n^{-1}$, which simply yields that $d_{f_n}(x, x') \leq dt_n^{-1} \sup A$. Consequently,

$$d_{f_n}(0, x') \leq 1 + dt_n^{-1} \sup A$$

by the triangle inequality. Thus by the claim of the previous paragraph for any $\varepsilon > 0$ there exists a path Γ from the origin to x' so that it is in the graph $\frac{\mathbb{Z}^d}{t_n}$ and

$$\int_\Gamma f(t) \mathrm{d}s < 1 + dt_n^{-1} \sup A + \varepsilon.$$

Choose for example $\varepsilon = dt_n^{-1} \sup A$. Now if we go back on Γ from x' by $k \leq |\Gamma|$ edges, where k is to be precised later, we get back to a vertex y of $\frac{\mathbb{Z}^d}{t_n}$, and the subpath of Γ from 0 to y guarantees that

$$d_{f_n}(0, y) < 1 + 2dt_n^{-1} \sup A - kt_n^{-1} \inf A = 1 + (2d \sup A - k \inf A)t_n^{-1}.$$

Thus we can choose k such that the y we obtain satisfies $d_{f_n}(0, y) < 1$, and k is bounded by

$$k \leq \left\lceil \frac{2d \sup A}{\inf A} \right\rceil .$$

Hence by the definition of f_n in the graph $\frac{\mathbb{Z}^d}{t_n}$ we have that $y \in \frac{\tilde{B}(t_n)}{t_n}$. However, by the choice of k we have that $|x' - y| \leq \left\lceil \frac{2d \sup A}{\inf A} \right\rceil t_n^{-1}$. Adding it to the upper bound on $|x - x'|$ we obtain

$$|y - x| \leq dt_n^{-1} + \left\lceil \frac{2d \sup A}{\inf A} \right\rceil t_n^{-1}.$$

This quantity is a uniform bound: for any n and $x \in B_n$ we have such a $y \in \frac{\tilde{B}(t_n)}{t_n}$. Thus as we have that $\frac{\tilde{B}(t_n)}{t_n} \subseteq B_n$, and $t_n \to +\infty$, we obtain that the elementwise Hausdorff distance of these sequences tends to 0. Consequently, $B_n \to K$ as well. As $B_n \in \mathcal{W}_A^d$, it concludes the proof. □

In some sense the above theorem gives a necessary condition on a set $K \in \mathcal{K}_A^d$ being a limit set, however, it would be quite elaborate to check it in any somewhat complicated case. It is more convenient to think about this result as a kind of reformulation of the original definition, whose significance lies in giving a somewhat new perspective, which helps in proving Theorem 2.5.5 through the construction given in the following lemma:

Lemma 2.5.10. *If $K \in \mathcal{K}_A^d$ is convex, then $K \in \overline{\mathcal{W}_A^d}$.*

Proof. It is well-known and easy to check that convex polytopes with rational vertices form a dense subset of convex sets in the Hausdorff metric, and the same argument shows that convex polytopes with rational vertices of $\mathcal{K}_A^d$ form a dense subset of convex sets of $\mathcal{K}_A^d$. Hence it suffices to prove the lemma for K convex polytopes with rational vertices. We can also assume that K has no boundary point in $\partial D_{\frac{1}{\sup A}}$. First we will further assume that A is an interval, or in other words we will allow the metric inducing functions to have values in $[\inf A, \sup A]$ instead of $\overline{A}$.

Fix $\varepsilon > 0$ rational with $\varepsilon < \frac{1}{\sup A}$ and take a finite ε-net $H = \{x_1, x_2, ..., x_k\}$ of the compact set ∂K. These points can be chosen so that each of them has rational coordinates. By definition and assumption we clearly have $\frac{1}{\sup A} < |x_i| \leq \frac{1}{\inf A}$. Now on the open line segment $I_i = (0, x_i)$ let $f(x) = |x_i|^{-1}$, and in the complement of these line segments let $f(x) = \sup A$. We claim that the closed unit ball B_f of d_f is in K. As this unit ball obviously contains the points of H (and also the segments connecting the origin to the points of H), this claim would simply imply that the Hausdorff distance of B_f and K is at most ε, which would yield $K \in \overline{\mathcal{W}_A^d}$.

First we show that if $x \in I_i$ for some i then $[0, x]$ is a d_f-optimal topological path. Proceeding towards a contradiction assume that there exists $x \in I_i$ for some i such that there is a shorter topological path Γ in d_f from 0 to x other than $[0, x]$. Choose x and Γ so that the number of linear pieces of Γ is minimal. Amongst such xs and Γs choose x and Γ so that the number of intersected intervals I_j is minimal. As $[0, x]$ is optimal in ℓ_1, we might assume that Γ has a common line segment with one of the intervals I_j, as otherwise we have $|\Gamma| \geq |x|$ and $f|_\Gamma \geq f|_{[0,x]}$. Besides that we can also assume that x is the first point of Γ in I_i. Now choose y to be the last point of Γ in one of the intervals I_j before x. By the choice of Γ we can immediately yield that the piece of Γ from 0 to y equals $\Gamma_1 = [0, y]$ in fact. Denote the second part of Γ by Γ_2. We know that in Γ_1 we have $f(t) = |x_j|^{-1}$ while in Γ_2 we have $f(t) = \sup A$. Moreover, in $[0, x]$ we have $f(t) = |x_i|^{-1}$. Consequently,

$$\int_\Gamma f(t)\mathrm{d}s = |y||x_j|^{-1} + |y - x| \sup A, \tag{2.6}$$

while

$$\int_{[0,x]} f(t)\mathrm{d}s = |x||x_i|^{-1}. \tag{2.7}$$

By assumption, we have an inequality between these quantities:

$$|y||x_j|^{-1} + |x - y| \sup A < |x||x_i|^{-1}. \tag{2.8}$$

Our aim right now is to get a contradiction, which we try to achieve by using the convexity of K. Note that by $K \in \mathcal{K}_A^d$ we know that in the direction of $x - y$ the shape K contains a segment of ℓ_1-length $(\sup A)^{-1}$ starting from the origin, and in the direction of y it contains a segment of ℓ_1-length $|x_j|$ by definition. Thus we have

$$\frac{x - y}{|x - y| \sup A}, \ \frac{y|x_j|}{|y|} \in K. \tag{2.9}$$

Let us express x as a positive linear combination of these vectors:

$$x = \frac{x - y}{|x - y| \sup A} \cdot (|x - y| \sup A) + \frac{y|x_j|}{|y|} \cdot \frac{|y|}{|x_j|}. \tag{2.10}$$

The sum of these coefficients might differ from 1, thus multiply both of them by the same scalar to get a convex combination of the original vectors. The convexity of K yields that this vector is also in K by (2.9):

$$\frac{x}{|x-y|\sup A + |y||x_j|^{-1}} \in K. \tag{2.11}$$

On the other hand, the furthest point of K in the direction of x is the endpoint of I_i, that is $x_i = \frac{x}{|x|}|x_i|$. It means

$$\frac{|x_i|}{|x|} \geq \frac{1}{|x-y|\sup A + |y||x_j|^{-1}}. \tag{2.12}$$

Taking the reciprocal of this inequality contradicts (2.8), thus it verifies the claim about the d_f-optimal topological paths to the points of the intervals $I_1, ..., I_k$.

This observation yields that to any $x \in K$ there exists a d_f-optimal topological path Γ: amongst the ones which do not share segments with any of the intervals $I_1, ..., I_k$ the $[0, x]$ line segment is optimal with d_f-length $|x| \sup A$. On the other hand, amongst the ones which hit any of these intervals we only have to consider the ones which are of the form $[0, y] \cup [y, x]$ for some $y \in I_j$ where $[x, y]$ does not intersect any of these intervals, as our previous observation implies. However, for any interval I_j there is an optimal Γ_j of these topological paths by a simple compactness argument. Hence the d_f-optimal topological path Γ to x arises as the optimal one of the path $[0, x]$ and $\Gamma_1, ...\Gamma_n$. This argument also shows that what is the closed unit ball B_f of d_f: if the optimal topological path Γ to x with d_f-length at most 1 does not hit any of the intervals $I_1, ..., I_k$, then we have $x \in D_{\frac{1}{\sup A}}$, that is in K. On the other hand, assume that $\Gamma = \Gamma_j = [0, y] \cup [y, x]$ for some $y \in I_j$. Here

$$\int_{[0,y]} f(t)\mathrm{d}s = |y||x_j|^{-1}$$

and

$$\int_{[y,x]} f(t)\mathrm{d}s = |x-y| \sup A.$$

As the d_f-length of the path Γ is at most 1, the sum of these quantities is also at most 1, consequently

$$|x-y| \leq \frac{1 - |y||x_j|^{-1}}{\sup A} =: r_y. \tag{2.13}$$

Thus we obtain that B_f contains an ℓ_1-ball of radius r_y around y. However, for $y = 0$ this ℓ_1 ball is contained by K as it equals $D_{\frac{1}{\sup A}}$ and $K \in \mathcal{K}_A^d$. On the other hand, for $y = x_j$ this ℓ_1 ball is trivial, hence it is also contained by K as $x_j \in K$. Consequently, as the function r_y is linear in $[0, x_j]$ and K is convex, we have that the ℓ_1-ball of radius r_y around y is in K for each $y \in I_j$.

What we conclude by this argument that B_f equals the union of $D_{\frac{1}{\sup A}}$ and all the ℓ_1-balls of radius r_y around y taken for each $y \in I_j$. Moreover, all the points of B_f are in K. We have already seen that it concludes the proof of the lemma in the case when A is an interval.

Now let us consider the case when A is arbitrary. Let $A' = [\inf A, \sup A]$. As $\mathcal{K}_A^d = \mathcal{K}_{A'}^d$, we have $K \in \mathcal{K}_{A'}^d$ as well. Hence we can consider the function f we constructed in the previous case, which induces a metric with unit ball B_f so that its Hausdorff distance from K is at most ε. We are going to replace f by $\tilde{f}$ so that $\tilde{f}$ induces a percolation metric $d_{\tilde{f}}$ which is almost the same as d_f, hence its unit ball $B_{\tilde{f}}$ is also close to K. As f has values differing from $\sup A$ only in the intervals I_i, it suffices to modify f there.

Let $s > 0$ be small, to be fixed later. We partition each of the intervals I_i into subintervals of equal length at most $s_i \leq s$, where ε is an integer multiple of each of the s_is. As all the coordinates of x_i and ε are rational, we can choose the s_is this way. Such a subinterval J_i will be cut into two further subintervals $J_{i,1}$ and $J_{i,2}$ with length $s_{i,1}$ and $s_{i,2}$ so that if $\tilde{f} = \sup A$ in $J_{i,1}$ and $\tilde{f} = \inf A$ in $J_{i,2}$, then f and $\tilde{f}$ has the same integral in J_i, that is

$$s_i |x_i|^{-1} = s_{i,1} \sup A + s_{i,2} \inf A.$$

Our long-term goal is to prove that the Hausdorff distance of $B_{\tilde{f}}$ and K can be arbitrarily small for sufficiently small s. There are two things to be checked: we need that K is contained by a small neighborhood of $B_{\tilde{f}}$, and the converse. The first one is simple: by construction it is obvious to see that the integral of f and $\tilde{f}$ equals on any line segment I_i, hence $B_{\tilde{f}}$ contains I_i. However, the intervals I_i form an ε-net of K which yields that K is contained by the ε-neighborhood of $B_{\tilde{f}}$. The second containment proves to be trickier.

By the choice of f we know that B_f is contained by the ε-neighborhood of K. Hence it would be sufficient to verify that $B_{\tilde{f}}$ is contained by a small neighborhood of B_f. Our first step in this direction is verifying the following claim: for small s, if $x \in I_i \setminus D_\varepsilon$, then there exists a $d_{\tilde{f}}$-optimal topological path from ∂D_ε to x, notably the line segment $J = [x_i', x] \subseteq I_i$, where $x_i' = [0, x_i] \cap \partial D_\varepsilon$. Proceeding towards a contradiction, assume there is a shorter piecewise linear topological path with finitely many pieces Γ in $d_{\tilde{f}}$ to some x. Let z be its starting point. Clearly we can assume that it is the last point of Γ in ∂D_ε. Moreover, by an argument similar to the first step of the case when A is an interval, we can assume that $\Gamma = [x_j', y] \cup [y, x]$ where $[x_j', y] \subseteq I_j$ for some $j \neq i$.

First we note that if $|J|$ is very short, that is x is sufficiently close to D_ε, then it cannot be possible. Indeed, the set of points x_i' form a discrete set, hence between such points there is a minimal ℓ_1-distance δ. Consequently, $|\Gamma| \geq \delta$. Hence if $|J| < \delta \cdot \frac{\inf A}{\sup A}$ then we surely have that the integral of $\tilde{f}$ on Γ exceeds its integral on J, a contradiction. Thus we can assume that $|J|$ is larger than this bound independent from s. Focus only on ss smaller than this bound.

By the definition of $\tilde{f}$ it is quite simple to give an upper bound on the integral of $\tilde{f}$ on J. Explicitly, in J we have that $\tilde{f}$ equals $\sup A$ and $\inf A$ alternately, and if we have consecutive segments with values $\sup A$ and $\inf A$, then the integral of $\tilde{f}$ on the union of these segments is the same as the integral of f. Hence the integral of $\tilde{f}$ might be larger than the integral of f due to the fact that there is one more line segment in which $\tilde{f}$ has value $\sup A$ while f has value $|x_i|^{-1}$ on the complete segment. This segment has length $s_{i,1}$. Consequently, we obtain the following bound

$$\int_J \tilde{f} \leq \int_J f + s_{i,1}(\sup A - |x_i|^{-1}). \tag{2.14}$$

Similarly, we can give a lower bound on the integral

$$\int_{[x_j{}', y]} \tilde{f} \geq \int_{[x_j{}', y]} f, \tag{2.15}$$

since s_j divides ε, and hence the alternating sequence of line segments with value $\sup A$ and $\inf A$ starts with a complete interval with value $\sup A$ from $x_j{}'$. Consequently, as $\tilde{f}$ equals f almost everywhere in $[y, x]$, we yield

$$\int_\Gamma \tilde{f} \geq \int_\Gamma f. \tag{2.16}$$

Combining (2.14) and (2.16) and the hypothetical inequality between $\int_J \tilde{f}$ and $\int_\Gamma \tilde{f}$ we conclude

$$\int_\Gamma f < \int_J f + s_{i,1}(\sup A - |x_i|^{-1}). \tag{2.17}$$

We distinguish three cases based on the relation between $|x_i|$ and $|x_j|$.

(i) Assume that $|x_i| < |x_j|$. As there are finitely many such js, for such js we can choose $r > 0$ such that $|x_i|^{-1} > |x_j|^{-1} + r$. Considering what we obtained in the first part, we see

$$\int_{[0,x_j{}']\cup\Gamma} f \geq \int_{[0,x_i{}']\cup J} f. \tag{2.18}$$

By (2.17) and (2.18), we have

$$\int_{[0,x_j{}']} f > \int_{[0,x_i{}']} f - s_{i,1}(\sup A - |x_i|^{-1}), \tag{2.19}$$

that is

$$\varepsilon|x_j|^{-1} > \varepsilon|x_i|^{-1} - s_{i,1}(\sup A - |x_i|^{-1}) > \varepsilon|x_j|^{-1} + \varepsilon r - s_{i,1}(\sup A - |x_i|^{-1}) \quad (2.20)$$

by the choice of r. However, for small enough s (and consequently, small enough $s_{i,1}$) it is impossible.

(ii) Assume that $|x_i| > |x_j|$. For such js we might choose $r > 0$ such that $|x_i|^{-1} + r < |x_j|^{-1}$ for all such j. Now the integral of $\tilde{f}$ on J is at most $(|x| - \varepsilon)|x_i|^{-1} + s_{i,1}(\sup A - |x_i|^{-1})$, while the integral of $\tilde{f}$ on Γ might be estimated from below by

$$(|x| - \varepsilon)|x_j|^{-1} > (|x| - \varepsilon)(|x_i|^{-1} + r).$$

Comparing these bounds we should have

$$(|x| - \varepsilon)r < s_{i,1}(\sup A - |x_i|^{-1}).$$

In this expression the left hand side has a fixed positive bound as we ruled out the possibility of $|J| = |x| - \varepsilon$ being too small. However, the right hand side can be arbitrarily small if we choose s small enough, a contradiction.

(iii) Finally, assume $|x_i| = |x_j|$. Now by assumption we might choose $r > 0$ such that $|x_i| < \sup A - r$. We know that $|y - x| \geq \delta'$ for some $\delta' > 0$ as the ℓ_1-distance between any two distinct segments $[x_j', x_j], [x_i', x_i]$ is positive, and there are only finitely many such segments. Hence we can deduce $\int_\Gamma \tilde{f} \geq |J||x_i|^{-1} + \delta' r$, as $|\Gamma| \geq |J|$, which is supposed to be smaller then $|J||x_i|^{-1} + s_{i,1}(\sup A - |x_i|^{-1})$. However, it cannot hold if we choose s small enough.

Thus we proved the claim: if $x \in I_i \setminus D_\varepsilon$, then there exists a $d_{\tilde{f}}$-optimal topological path from ∂D_ε to x, notably the line segment $J = [x_i', x] \subseteq I_i$, and hence $d_{\tilde{f}}(x_i', x) \leq d_f(x_i', x) + \varepsilon$, if s is sufficiently small. This observation quickly yields that for any $x \in K \setminus D_\varepsilon$ we have $d_{\tilde{f}}(\partial D_\varepsilon, x) \leq d_f(\partial D_\varepsilon, x) + \varepsilon$, as we have that the $d_{\tilde{f}}$-optimal topological path Γ from ∂D_ε to x might have a common line segment with at most one of the segments $[0, x_i]$, thus the integral of f and $\tilde{f}$ on Γ might only differ by ε as the two functions only differ in this piece of Γ.

Given this fact we can also say something about $d_{\tilde{f}}(0, x)$ for $x \notin D_\varepsilon$. (Other xs are contained by $B_{\tilde{f}}$ anyway.) A path from 0 to x must intersect ∂D_ε at some point y. Thus we may conclude

$$\begin{aligned} d_{\tilde f}(0,x) &\geq \inf_{y\in\partial D_\varepsilon} (d_{\tilde f}(0,y) + d_{\tilde f}(y,x)) \\ &\geq \inf_{y\in\partial D_\varepsilon} ((d_f(0,y) - \varepsilon(\sup A - \inf A)) + (d_f(y,x) - \varepsilon)) \\ &= d_f(0,x) - \varepsilon(1 + \sup A - \inf A), \end{aligned} \tag{2.21}$$

where the second inequality comes from the fact that for points with ℓ_1 distance ε we have that the difference of their d_f and $d_{\tilde f}$ distance is at most $\varepsilon(\sup A - \inf A)$.

Using (2.21), we might finish the proof swiftly. We have seen that it would be sufficient to verify that $B_{\tilde f}$ is contained by a small neighborhood of B_f. We state that it holds for the neighborhood of ℓ_1-radius $\frac{\varepsilon(1+\sup A-\inf A)}{\inf A}$. Indeed, consider x so that it is not in this neighborhood. Then for any $z \in \partial B_f$ we have $|z-x| > \frac{\varepsilon(1+\sup A-\inf A)}{\inf A}$. Now consider any piecewise linear topological path Γ from 0 to x with last point z in ∂B_f. By (2.21), as $|z| \geq \frac{1}{\sup A} > \varepsilon$ we obtain

$$\int_\Gamma \tilde f \geq d_{\tilde f}(0,z)+d_{\tilde f}(z,x) > d_f(0,z)-\varepsilon(1+\sup A-\inf A)+|z-x|\inf A > 1, \tag{2.22}$$

that is $x \notin B_{\tilde f}$. Thus we have that $B_{\tilde f}$ is in the neighborhood of B_f with radius $\frac{\varepsilon(1+\sup A-\inf A)}{\inf A}$. Consequently, it is in the neighborhood of K with radius $\frac{\varepsilon(1+\sup A-\inf A)}{\inf A} + \varepsilon$. Thus the Hausdorff distance of K and $B_{\tilde f}$ can be arbitrarily small, which verifies $K \in \mathcal{W}_A^d$. □

Using a construction similar to the one we have just seen, we can prove Theorem 2.5.5.

Proof of Theorem 2.5.5. As in the proof of Lemma 2.5.10, it is sufficient to consider convex polytopes with rational vertices in $\mathcal{K}_A^d$, and we can also assume that they have no boundary point in $\partial D_{\frac{1}{\sup A}}$. Let $\varepsilon > 0$ be rational and smaller than $\frac{1}{\sup A}$ and let $x_1, ..., x_k,\ \tilde f$ be as in the proof of the lemma with $d_H(K, B_{\tilde f}) < \varepsilon' := \frac{\varepsilon(1+\sup A-\inf A)}{\inf A}+\varepsilon$. This original $\tilde f$ had value $\sup A$ everywhere except for certain pieces of the line segments I_i, where it equaled $\inf A$. These pieces contained a certain λ_i ratio of $|I_i|$. By approximating λ_i with rational numbers $(\lambda_{i,j})_{j=1}^\infty$ and using them for this ratio, we obtain approximating functions $(\tilde f_j)_{j=1}^\infty$ so that $B_{\tilde f_j}$ converges to $B_{\tilde f}$ in the Hausdorff metric as the sum of the lengths of line segments where we modify $\tilde f_j \neq \tilde f$ can be arbitrarily small. Hence we can assume that $\tilde f$ is already defined such that these ratios are rational: each I_i is partitioned into line segments of equal length s_i, and each line segment is divided into line segments of rational length $s_{i,1}$ and $s_{i,2}$, and $\tilde f$ has value $\inf A$ in the latter ones. We will modify this function in two steps.

We know that for ε we have that there is a $d_{\tilde{f}}$-optimal path from ∂D_ε to any $x \in I_i \setminus D_\varepsilon$, notably the line segment $[x_i', x]$ for $x_i' = [0, x_i] \cap \partial D_\varepsilon$. Moreover, from (i)-(iii) it is simple to see that for small enough s there is a constant $c > 0$ such that this line segment is shorter in $d_{\tilde{f}}$ than any other topological path sharing a segment with another $[x_j', x_j]$ by at least c. We will capitalize on this fact by a bit technical, but necessary argument. Let $\theta > 0$ be small enough to be fixed later, and consider the cone C_i with vertex 0 and base $B_i = \{y : |y| = |x_i|, |y - x_i| \leq \theta\}$, that is the union of all the line segments $[0, y]$ for $\{y : |y| = |x_i|, |y - x_i| \leq \theta\}$. If θ is small enough, these cones are disjoint. We will define the g modification of $\tilde{f}$ the following way: if $y \in D_\varepsilon$, then let $g = \sup A$. If $y \in C_i \setminus D_\varepsilon$, then let $g(y) = \tilde{f}(x)$ where x is the unique point of I_i with $|x| = |y|$. Apart from these sets we simply let $g = \tilde{f} = \sup A$. By the existence of c we can easily deduce the following claim: for small enough θ we have that for any $x \in C_i \cap \partial D_\varepsilon$ there is a d_g-optimal topological path from ∂D_ε to x, and this path is completely contained by C_i. Indeed, as there is an ℓ_1-optimal topological path in C_i and apart from these cones we have $g = \sup A$, we know that if there is a shorter topological path Γ in d_g from the boundary to some $x \in C_i$ then it must hit another C_j. We can assume that Γ starts from some $y \in C_j$ with $i \neq j$. However, by choosing θ small we can have an arbitrarily small bound on the difference of the integrals $\int_\Gamma \tilde{f}$ and $\int_\Gamma g$. Combining this remark with the existence of c yields our claim, which can be used to give a bound on how large B_g can be compared to $B_{\tilde{f}}$ in a manner similar to the concluding step of the proof of Lemma 2.5.10. After these technical manipulations we obtain using small enough θ that $d_H(K, B_g) < 2\varepsilon'$. We will use a further modified version $\tilde{g}$ of g, which is obtained as follows: focus on a certain $I_i \setminus D_\varepsilon$ and $C_i \setminus D_\varepsilon$. This line segment is divided into pieces $I_{i,1,1}, I_{i,1,2}, I_{i,2,1}, I_{i,2,2}, ..., I_{i,l,1}, I_{i,l,2}$ with lengths alternating between $s_{i,1}$ and $s_{i,2}$. Denote the corresponding slices of C_i by $C_{i,1,1}, C_{i,1,2}, C_{i,2,1}, C_{i,2,2}, ..., C_{i,l,1}, C_{i,l,2}$. Consider now any of these line segments, for example $[a, b] = I_{i,1,1}$. (For any other line segment we can proceed the same way.) Here $a, b \in \mathbb{Q}^d$, hence for all the mt multiples with $t \in \mathbb{N}$ of a certain m we have $a, b \in \frac{\mathbb{Z}^d}{mt}$. As $C_{i,1,l}$ contains an open cylinder with height $[a, b]$, if we choose a large enough such mt we have an ℓ_1 optimal path from a to b which is a path in the graph $\frac{\mathbb{Z}^d}{mt}$. Now if we choose m to be a large enough common multiple M of all the (finitely many) ms appearing this way, we arrive at a $\frac{\mathbb{Z}^d}{M}$ for which all the $[a, b]$s can be replaced by a path in the graph $\frac{\mathbb{Z}^d}{M}$ the above way. After all we obtain paths $\Gamma_i \subseteq C_i$ for each i which are ℓ_1-optimal from x_i' to x_i. Now in each Γ_i we will define $\tilde{g}$ to have the same value as g, but apart from that we let $\tilde{g} = \sup A$. This way we obtain $\tilde{g} \geq g$, hence $B_{\tilde{g}} \subseteq B_g$ holds, which yields that the $2\varepsilon'$-neighborhood of K necessarily contains $B_{\tilde{g}}$. However, by definition along the path Γ_i the integral of $\tilde{g}$ is the same as the integral of $\tilde{f}$ along $[x_i', x_i]$, which can be arbitrarily close to 1. Consequently, we have that an arbitrarily large piece of Γ_i is contained by $B_{\tilde{g}}$. Hence as the $\varepsilon' + \theta$-neighborhood of $\bigcup_{i=1}^k \Gamma_i$

contains K, we have the same for the $\varepsilon' + \theta$-neighborhood of $B_{\tilde{g}}$. Consequently, if $\theta < \varepsilon'$, we have $d_H(K, B_{\tilde{g}}) < 2\varepsilon'$. In the following we will use this $\tilde{g}$ which is defined based on the parameters ε, M which are to be fixed later.

Let us return to the statement of the theorem. By separability arguments it suffices to prove that for any polytope K with rational vertices and no boundary points in $D_{\frac{1}{\sup A}}$, in a residual subset of Ω there exists a suitable sequence $t_n \to \infty$ with $\frac{B(t_n)}{t_n} \to K$. Denote the set of configurations for which it does not hold by $F(K)$. Then by the definition of convergence, $F(K)$ is expressible as a countable union as follows:

$$F(K) = \bigcup_{i=1}^{\infty} \bigcup_{l=1}^{\infty} F\left(K, \frac{1}{i}, l\right),$$

where $F(K, \delta, \mu_0)$ stands for the set of configurations in which for any $\mu > \mu_0$ we have

$$d_H\left(K, \frac{\tilde{B}(\mu)}{\mu}\right) > \delta.$$

Consequently, verifying that $F(K, \delta, l)$ is nowhere dense for each δ, l would conclude the proof. Clearly it suffices to prove it for sufficiently small δ and sufficiently large l. Having this purpose in mind fix a cylinder set U in Ω with nontrivial projections $U_{e_1}, ...U_{e_j}$. We try to find a smaller cylinder set $V = V_t$ such that it is disjoint from $F(K, \delta, l)$. Now choose l so large that the edges $\frac{e_i}{l} \subseteq D_\varepsilon$. Consider the function $\tilde{g}$ defined in the first step of the proof for some $M > l$ and ε to be fixed later. This function is constant by definition on any edge of $\frac{\mathbb{Z}^d}{Mt}$ for $t \in \mathbb{N}$, denote this value by $\tilde{g}\left(\frac{e}{Mt}\right)$. Now for any edge $e \notin \{e_1, ..., e_j\}$, but intersecting $D_{\frac{Mt}{\inf A}}$, we will define $V_{t,e} = (\tilde{g}\left(\frac{e}{Mt}\right) - \varepsilon_e, \tilde{g}\left(\frac{e}{Mt}\right) + \varepsilon_e) \cap A$, where the sum of these ε_es is at most $\varepsilon \inf A$. Let V_t be defined by these projections and consider any configuration ω in it. As in the proof of Theorem 2.5.8, the passage times in this configuration determine a function $f_{t,\omega}$ by rescaling, such that the Hausdorff distance of $\frac{B(Mt)}{Mt}$ and $B_{f_{t,\omega}}$ converges to 0 as $t \to \infty$. (Here we use the fact we noted there that $B_{f_{t,\omega}}$ depends only on the values of $f_{t,\omega}$ in $D_{\frac{1}{\inf A}}$). However, by definition the function $f_{t,\omega}$ is almost the same as $\tilde{g}$ in $D_{\frac{1}{\inf A}}$: the only differences arise due to the existence of the edges $e_1, ..., e_j$ and the error term ε_e for each edge. However, these minor differences cannot imply a significant deviation of the integral on any relevant path in the definition of the sets $B_{\tilde{g}}$ and $B_{f_{t,\omega}}$: as relevant topological paths has ℓ_1-length at most $\frac{1}{\inf A}$ the error terms may not yield a difference larger than ε in the integral of $f_{t,\omega}$ and $\tilde{g}$. On the other hand, the edges $\frac{e_1}{l}, ..., \frac{e_j}{l}$ are all in D_ε, which is a set with ℓ_1-diameter 2ε, hence they cannot yield a difference larger than $2\varepsilon(\sup A - \inf A)$. As depending on the choice of ε all these quantities can be arbitrarily small, we have that the Hausdorff distance of $B_{\tilde{g}}$ and $B_{f_{t,\omega}}$ can be arbitrarily small. Consequently, for small enough ε, and large enough t we have $d_H(B_{\tilde{g}}, \frac{B(Mt)}{Mt}) < \frac{\delta}{2}$. But for small enough ε we also have $d_H(B_{\tilde{g}}, K) < \frac{\delta}{2}$. That is, we have by triangle inequality

$$d_H\left(K, \frac{\tilde{B}(Mt)}{Mt}\right) < \delta,$$

for some $Mt > l$. It means that for large enough t we have that V_t is necessarily disjoint from $F\left(K, \delta, l\right)$, which yields that this latter set is nowhere dense. We have seen that it concludes the proof. □

Chapter 3

Hilbert first passage percolation

3.1 Motivation and definition

The line of research appearing in this chapter was inspired by a question of Kornélia Héra after a talk of the author given about [10]. Roughly, her question was what can be said if we consider $A \subseteq \mathbb{C}$ instead of $A \subseteq [0, +\infty)$: to each edge, we assign a complex number instead of a nonnegative one, and a passage time of a path is defined to be the absolute value of the sum of the complex numbers along it. The definition of passage time between points is the same. Since we do not use the complex multiplicative structure, we can think of $\mathbb{C}$ as $\mathbb{R}^2$, or to treat a more general case, we can work with a real Hilbert space $\mathcal{H}$ and $A \subseteq \mathcal{H}$. The passage vector $v(e)$ of any edge will be an element of A, while the passage time of the edge is $\tau(e) = \|v(e)\|_{\mathcal{H}}$. The passage vector $v(\Gamma)$ of a path Γ is defined as the sum of passage vectors of the contained edges, and the passage time $\tau(\Gamma)$ of the path is the norm of the passage vector of the path. The passage set between any two points $x, y \in \mathbb{Z}^d$ is the set $S(x, y)$ of passage vectors of paths connecting them. Finally, the passage time between any two points $x, y \in \mathbb{R}^d$ is

$$T(x, y) = \inf_{\Gamma} \tau(\Gamma),$$

where the infimum is taken over all the paths connecting x' to y', where x' and y' are the unique lattice points such that $x \in x' + [0, 1)^d$, $y \in y' + [0, 1)^d$. In other words,

$$T(x, y) = \inf\{\|v\|_{\mathcal{H}} : v \in S(x, y)\}.$$

We call this setup Hilbert first passage percolation or simply Hilbert percolation.

We note here that various generalizations of the probabilistic setup also exist, originally motivated by the setup with non-i.i.d edge weights. In [4], the general stationary and ergodic case is considered, and an analogue of the Cox–Durrett theorem is proved for instance. For the most general version known see [3]. As it is quite involved and not of our direct interest, we do not discuss its details here.

The theory of Hilbert first passage percolation is obviously a generalization of the ordinary topological first passage percolation, as in the case $\mathcal{H} = \mathbb{R}$ and $A \subseteq [0, \infty)$ the new definition of passage times coincides with the old one. However, it is useful to note that if A contains negative values, we face a slight ambiguity as we do not have this coincidence. Anyway, we should not worry about this phenomenon as the original topological model collapses in that case and does not deserve much attention, as we have discussed it in Section 2.1.

When it comes to questions of residuality, it is usual to restrict ourselves to separable spaces for technical reasons. As a consequence, henceforth we assume that all the real Hilbert spaces involved are separable.

3.2 Strongly positively dependent case

Our first goal is to display that the Hilbert percolation is degenerate for a reasonably large family of As in the sense that generically most of the passage times are zeros. To this end, we introduce some notation and definitions. In this section, we will restrict further our scope to finite dimensional spaces, i.e. $\mathcal{H} = \mathbb{R}^k$.

Definition 3.2.1. *Let $H \subseteq \mathcal{H}$. The convex cone generated by H is the smallest set $\operatorname{cone}(H) \subseteq \mathcal{H}$ which is closed under linear combinations with nonnegative coefficients. The closed convex cone generated by H is the closure of $\operatorname{cone}(H)$ which we denote by $\overline{\operatorname{cone}(H)}$.*

Definition 3.2.2. *Let $H \subseteq \mathcal{H}$. We say that H is strongly positively dependent if for any $x \in H$ we have $-x \in \overline{\operatorname{cone}(H)}$.*

At first sight, one might believe that if A is strongly positively dependent then the passage time between any two points is 0 for a generic configuration. However, a very simple counterexample refutes this idea: if $d = 1$ and $A = \{-1, 1\} \subseteq \mathbb{R}$ for example, then clearly the passage time between integers of different parity is odd, thus cannot be 0. In general we must point out that we do not have paths of any ℓ_1-length between fixed lattice points, which causes technical inconveniences. We can only construct paths which have length with the same parity as $|x-y|$. Motivated by this remark it is useful to introduce the notation $M(A)$ for the submonoid in $\overline{\operatorname{cone}(A)}$ which contains those linear combinations with nonnegative integer coefficients in which the sum of coefficients is even.

The following theorem displays that in the generic case the model gets trivial if A is strongly positively dependent and we have enough degree of freedom, that is $d \geq 2$.

Theorem 3.2.3. *Let $\mathcal{H} = \mathbb{R}^k$ and assume that $A \subseteq \mathcal{H}$ is bounded and strongly positively dependent, moreover, let $d \geq 2$. Then generically we have that $\overline{S(x, y)} = \overline{M(A)}$ for any $x, y \in \mathbb{Z}^d$ for which $|x - y|$ is even, and $\overline{S(x, y)} = \overline{M(A) + A}$ for any $x, y \in \mathbb{Z}^d$ for which $|x - y|$ is odd.*

Corollary 3.2.4. *In the setting of Theorem 3.2.3, generically we have $T(x, y) = 0$ for any $x, y \in \mathbb{Z}^d$ for which $|x-y|$ is even, while $T(x, y) = \inf\{\|b\|_{\mathcal{H}} : b = m+a, m \in M(A), a \in A\}$ for any $x, y \in \mathbb{Z}^d$ for which $|x - y|$ is odd.*

Remark 3.2.5. If $d = 1$, it is easy to construct counterexamples to the theorem and to the corollary, due to the fact that for given x, y and $e \in E$ the parity of the times a path Γ from x to y crosses e is completely determined. Consequently, if for example $A = \{-1, 0, 1\}$, while $\tau([0, 1]) = 1$ and $\tau([1, 2]) = 0$, then the passage time of any Γ from 0 to 2 is odd, thus $T(0, 2) \geq 1$ for any configuration, while it should be 0 generically according to Corollary 3.2.4 as A is strongly positively dependent.

Proof of Theorem 3.2.3. Let us consider points with even ℓ_1-distance, the other case will simply follow from that. The $\overline{S(x, y)} \subseteq \overline{M(A)}$ containment clearly holds as all the passage vectors between points even distance apart are in $M(A)$. Hence it suffices to verify the other containment which follows from $M(A) \subseteq \overline{S(x, y)}$. For technical purposes choose a countable dense subset $M_0 \subseteq M(A)$. Then we can reduce the problem to the question whether $M_0 \subseteq \overline{S(x, y)}$ holds. As there are countably many pairs x, y and M_0 is also countable, it is enough to show that for any fixed $m \in M_0$ and pair x, y we have that $m \in \overline{S(x, y)}$ apart from a nowhere dense set.

In order to verify this claim fix a cylinder set $U \subseteq \Omega$ and $n \in \mathbb{N}$. It is clearly sufficient to find a smaller cylinder set $V \subseteq U$ such that for any configuration in V there exists $s \in S(x, y)$ such that $\|m - s\|_{\mathcal{H}} < \frac{1}{n} := \varepsilon$. Let us denote the set of edges belonging to nontrivial projections of U by $E_U = \{e_1, e_2, ..., e_N\}$. Fix now a large hypercube K centered at the origin which contains all the edges in E_U and also x and y. Now we can define a cylinder set U' which has very small projections to the edges in K. More precisely, we require these projections to have sufficiently small diameter to guarantee that in U' the sum of passage vectors on these edges is in an open set of diameter $\frac{\varepsilon}{2}$, regardless of which configuration we consider. Choose now a vertex z on ∂K and let Γ be a path crossing each of its edges precisely once from x to y such that it does not leave K but contains z once. The point z cuts it into two parts Γ_1, Γ_2. By the previous note about the diameters, we have that the passage vector of Γ is in an open set of diameter $\frac{\varepsilon}{2}$. Let us denote one of these passage vectors by α for the sake of specificity, all the others that may arise in another configuration have distance at most $\frac{\varepsilon}{2}$ from it. Our aim is to define nontrivial projections on further edges of a cycle Γ' starting from z such that the passage vector of $\Gamma^* = \Gamma_1 \cup \Gamma' \cup \Gamma_2$ is in a neighborhood of m with radius ε, which would conclude the proof of the first part. As z is a vertex, we will be able to choose Γ' so that it does not contain any of its edges twice and $\Gamma' \cap K = \{z\}$.

As α is a passage vector between x and y for some configuration, we clearly have $\alpha \in \text{cone}(A)$. As A is strongly positively dependent, it clearly implies $-\alpha \in \overline{\text{cone}(A)}$. Hence for any $Q \in \mathbb{N}$ there exists $\beta \in \text{cone}(A)$ such that

$$\|\beta - (-\alpha)\|_{\mathcal{H}} < \frac{\varepsilon}{8Q} = \varepsilon^*, \tag{3.1}$$

where Q is to be fixed later. By a simple consequence of Carathéodory's theorem about convex hulls we have that

$$\beta = \sum_{i=1}^{k} r_i a_i, \tag{3.2}$$

where each coefficient $r_i > 0$, while $a_i \in A$ and k is fixed. By (3.2), we can rewrite (3.1) as

$$\left\| \sum_{i=1}^{k} r_i a_i + \alpha \right\|_{\mathcal{H}} < \varepsilon^*. \tag{3.3}$$

By the simultaneous version of Dirichlet's approximation theorem we can choose $p_1, ..., p_k \geq 0$ integers and $1 \leq q \leq Q$ such that

$$\left| \frac{p_i}{q} - r_i \right| < \frac{1}{qQ^{\frac{1}{k}}}. \tag{3.4}$$

Using (3.4) and the triangle inequality we can rewrite the estimate in (3.3) as

$$\left\| \sum_{i=1}^{k} \frac{p_i}{q} a_i + \alpha \right\|_{\mathcal{H}} < \varepsilon^* + \frac{1}{qQ^{\frac{1}{k}}} \sum_{i=1}^{k} \|a_i\|_{\mathcal{H}}. \tag{3.5}$$

Multiplying by $2q$ and using $q \leq Q$ and the definition of ε^* yields

$$\left\| \sum_{i=1}^{k} 2p_i a_i + 2q\alpha \right\|_{\mathcal{H}} < \frac{\varepsilon}{4} + \frac{2}{Q^{\frac{1}{k}}} \sum_{i=1}^{k} \|a_i\|_{\mathcal{H}} < \frac{\varepsilon}{3} \tag{3.6}$$

for well-chosen Q, as A, and hence $\sum_{i=1}^{k} \|a_i\|_{\mathcal{H}}$ is bounded.

As $m \in M_0 \subseteq M(A)$, by definition it is expressible as an even sum of elements in A, that is $m = \sum_{j=1}^{2l} a_{m(j)}$, where each $a_{m(j)}$ is in A. Now choose the cycle Γ' starting from z such that it does not hit K until eventually returning to z, and contains all its edges precisely once. Moreover, $|\Gamma'| = 2l + \sum_{i=1}^{k} 2p_i + (2q-1)|\Gamma|$.

(It is an even number, so it can be done.) On this cycle we can define nontrivial projections the following way: on the first $2l$ edges, we define nontrivial projections centered at each $a_{m(j)}$ respectively with sufficiently small diameter to be precised later. On the next $2p_1, 2p_2, ..., 2p_k$ edges we define nontrivial projections centered at $a_1, a_2, ...a_k$ respectively with sufficiently small diameter again. Finally, we think of the last $(2q-1)|\Gamma|$ edges as $2q-1$ consecutive copies of Γ, that is we define nontrivial projections as sufficiently small open subsets of the projections belonging to the corresponding edges of Γ. These projections together with the ones in the definition of U' define V. If we choose the above neighborhoods small enough, we can guarantee that the passage vector of Γ' for any configuration has distance at most $\frac{\varepsilon}{6}$ from $m + \sum_{i=1}^{k} 2p_i a_i + (2q-1)\alpha$. As a consequence, by (3.6) and triangle inequalities we can deduce

$$\|v(\Gamma_1 \cup \Gamma' \cup \Gamma_2) - m\|_{\mathcal{H}} < \left\| \sum_{i=1}^{k} 2p_i a_i + 2q\alpha \right\|_{\mathcal{H}} + \frac{\varepsilon}{6} + \frac{\varepsilon}{2} < \varepsilon \tag{3.7}$$

for any configuration in V, which concludes the case when $|x-y|$ is even with the choice $s = v(\Gamma_1 \cup \Gamma' \cup \Gamma_2) \in S(x,y)$.

In the other case the previous argument might be copied with one essential change. In this case we want to have passage vectors near vectors of the type $m + a \in M(A) + A$. Given this, upon defining Γ and the projections to the edges of K, we will proceed the same way as previously, except for this time we will separate an edge $e \in \Gamma \setminus E_U$ and on that we will define the projection of V to be a very small neighborhood of a. Apart from this edge, Γ uses an even number of edges, thus we can define Γ' and V as previously in order to have that $v(\Gamma_1 \cup \Gamma' \cup \Gamma_2) - a$ is very close to m in V. Consequently, $v(\Gamma_1 \cup \Gamma' \cup \Gamma_2)$ is very close to $m + a$. The technical details are left to the reader. □

It would be nice to say something about how common are the strongly positively dependent sets for example amongst the compact sets equipped with the Hausdorff metric, whose family shall be denoted by $\mathcal{K}^d$. This is the aim of the following proposition, which roughly states that the set of such As is not too small, but not too large either:

Proposition 3.2.6. *The set of strongly positively dependent compact sets contains nontrivial open sets in $\mathcal{K}^d$, and so does its complement.*

Proof. For the complement it is very simple to verify the claim: we can consider the ball of radius $\frac{1}{2}$ centered at the singleton $\{\xi_1\}$. It is good indeed as any set A in this neighborhood exclusively contains vectors with positive first coordinate.

For the set of strongly positively dependent compact sets, our construction relies on the following remark: if the convex hull $\operatorname{conv}(A)$ contains 0 in its interior, then

A is strongly positively dependent. Indeed, in this case for any $a \in A$ we have that $\lambda(-a) \in \text{conv}(A)$ for sufficiently small $\lambda > 0$. Consequently, $\lambda(-a)$ can be written as a finite linear combination of elements of A with positive coefficients, which yields that $-a \in \text{cone}(A)$, as stated.

Now consider $A = \{\pm\xi_1, \pm\xi_2, ..., \pm\xi_k\}$ in $\mathcal{K}^d$. Then $\text{conv}(A)$ contains 0 in its interior, as it contains D_1, the unit ball centered at the origin in the ℓ_1-metric. In other words, the distance of 0 from $\partial(\text{conv}(A))$ is 1, and 0 is contained by $\text{conv}(A)$. Now consider a small neighborhood G_A of A in $\mathcal{K}^d$ and an element K of it. We would like to show that for a sufficiently small neighborhood we have that 0 is in the interior of $\text{conv}(K)$. We know that K contains at least one point very close to each $\pm\xi_i$, and if G_A is small enough, these points must be distinct. If we replace K by a subset of it formed by $2k$ such points, we shrink $\text{conv}(K)$, hence it would be sufficient to verify our claim for the convex hulls of such finite sets. But such a convex hull is a polytope, which depends continuously on its vertices. Moreover, the boundary also depends continuously on the vertices. Consequently, as we know that the distance of 0 from $\partial(\text{conv}(A))$ is 1, and 0 is in A, we have that the distance of 0 from $\partial(\text{conv}(K))$ is also positive and 0 is in K if K is chosen from a sufficiently small neighborhood G_A. Thus the set of strongly positively dependent compact sets contains nontrivial open sets in $\mathcal{K}^d$, indeed. □

3.3 Optimal paths and geodesics in Hilbert percolation

In this section we would like to examine the geometric properties of the Hilbert percolation. In the ordinary topological first passage percolation we called a path geodesic if its passage time equals the passage time between its endpoints, which was appropriate in the sense that subpaths of geodesics were also of minimal length. However, as even very simple examples might display, it is not the case anymore: for instance, let $A = \{-1, 0, 1\}$ and $d = 2$, and consider the configuration in which we have two neighboring edges with passage vectors -1 and 1 respectively, while the passage vector of all other edges is 0. In this case, the passage time of the path of these two edges is 0, hence it is optimal, while its subpaths of length 1 have passage time 1. However, between any two points the passage time is 0, hence these subpaths are not optimal. It motivates a separation of definitions:

Definition 3.3.1. *A path in $\mathbb{Z}^d$ is an optimal path, if its passage time equals the passage time between its endpoints. Moreover, a path is a geodesic, if all of its subpaths are optimal.*

If we would like to have a somewhat tame geometry on the discrete lattice, it is natural to expect from the model that optimal paths are not self intersecting, and in general longer paths have higher passage time. The following lemma gives a necessary and sufficient condition guaranteeing this property. (We denote by (a, b) the inner product of $a, b \in \mathcal{H}$).

Lemma 3.3.2. *We have $\tau(\Gamma_1) \leq \tau(\Gamma_2)$ for each paths $\Gamma_1 \subseteq \Gamma_2$ and each configuration if and only if for any $a, b \in A$ we have $(a, b) \geq 0$. In this case, we call A positive.*

Proof. First assume the existence of $a, b \in A$ with $(a, b) < 0$. Choose $n \in \mathbb{N}$ so that $2n(a, b) + \|b\|_{\mathcal{H}}^2 < 0$. Now consider a configuration in which there are $n + 1$ consecutive edges so that the passage time of the first n is a, while the last one has passage time b. Let the first n edges form Γ_1, and let the union of all these edges be Γ_2. Then the square of the passage time of Γ_1 is

$$\tau(\Gamma_1)^2 = \|na\|_{\mathcal{H}}^2 = n^2\|a\|_{\mathcal{H}}^2,$$

which implies by the choice of n

$$\tau(\Gamma_2)^2 = \|na + b\|_{\mathcal{H}}^2 = n^2\|a\|_{\mathcal{H}}^2 + 2n(a, b) + \|b\|_{\mathcal{H}}^2 < n^2\|a\|_{\mathcal{H}}^2 = \tau(\Gamma_1)^2.$$

This denies $\tau(\Gamma_1) \leq \tau(\Gamma_2)$, hence we proved one of the directions.

For the other direction assume that for any $a, b \in A$ we have $(a, b) \geq 0$, and $\Gamma_1 \subseteq \Gamma_2$. As we can add edges one by one, it suffices to prove the claim for $\Gamma_2 = \Gamma_1 \cup e$ for an edge e. Now the square of the passage time of Γ_1 in any configuration is

$$\tau(\Gamma_1)^2 = \left\| \sum_{i=1}^{|\Gamma_1|} a_i \right\|_{\mathcal{H}}^2.$$

for some $a_i \in A$, while the square of the passage time of Γ_2 is

$$\tau(\Gamma_2)^2 = \left\| a^* + \sum_{i=1}^{|\Gamma_1|} a_i \right\|_{\mathcal{H}}^2 = \|a^*\|_{\mathcal{H}}^2 + 2\sum_{i=1}^{|\Gamma_1|}(a, a_i) + \tau(\Gamma_1)^2 \geq \tau(\Gamma_1)^2,$$

where the last inequality holds by assumption. It concludes the proof. □

Now if A is not positive, then in a small neighborhood of the configuration constructed in the proof there exist paths $\Gamma_1 \subseteq \Gamma_2$ with $\tau(\Gamma_1) > \tau(\Gamma_2)$. Thus if Ω is a Baire space, then we have $\tau(\Gamma_1) \leq \tau(\Gamma_2)$ in the generic case for each paths $\Gamma_1 \subseteq \Gamma_2$ if and only if A is positive. Thus in the following we will restrict ourselves to positive As.

Another natural question motivated by geometry is when we have that all the optimal paths are geodesics. If $d = 1$, we clearly have this property as optimal paths are not self-intersecting, and if $d = 1$ there is a unique path with no self-intersections between any two points. Things get interesting when $d \geq 2$, however, we must realize that such As are terrifyingly rare:

Lemma 3.3.3. *Assume that A is positive and $d \geq 2$. We have that all the optimal paths are geodesics in all the configurations if and only if A is contained by a ray, that is for any $a, b \in A$ we have $a = \lambda b$ or $b = \lambda a$ for some $\lambda \geq 0$. Moreover, if this condition does not hold, there are configurations in which there are $x, y \in \mathbb{Z}^d$ such that there is no geodesic between x and y at all.*

Proof. If A is contained by a ray we obviously have this property as the passage time of a path is simply the sum of the passage times of the edges. On the other hand, assume that A contains a, b so that they are not contained by the same ray. We can clearly assume $\|a\|_{\mathcal{H}} \geq \|b\|_{\mathcal{H}}$. Moreover, by the assumption we have $(a, b) \leq \mu \|a\|_{\mathcal{H}} \|b\|_{\mathcal{H}}$ for some $0 < \mu < 1$, and $a, b \neq 0$. The proof from this point is a construction in which we have an optimal path which is not a geodesic.

First consider the case $\|a\|_{\mathcal{H}} = \|b\|_{\mathcal{H}}$. For this case we can give a very simple construction, see Figure 3.1. Notably, amongst the paths from X to Y there is a unique one with optimal ℓ_1-length and passage vector $a + a + b + b$, notably one of the paths through Z. Consequently, this path is the unique optimal path from X to Y. However, it cannot be a geodesic, as there is a path from X to Z with passage vector $b + a$, hence it has smaller passage time than the path with passage vector $a + a$. Thus there are no geodesics from X to Y at all.

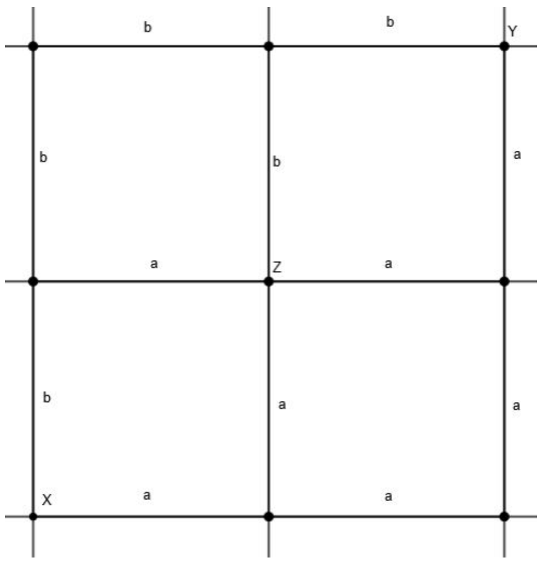

Figure 3.1: The case when $\|a\|_{\mathcal{H}} = \|b\|_{\mathcal{H}}$.

Consider the other case, when there is a strict inequality between the norms of

a and b, that is $\|a\|_{\mathcal{H}} > \|b\|_{\mathcal{H}}$. For the norm of a and b we can define $n_a < n_b$ such that both of them are even and the inequalities

$$n_a\|a\|_{\mathcal{H}} > n_b\|b\|_{\mathcal{H}},\ n_a\|a\|_{\mathcal{H}} \leq n_b(\|b\|_{\mathcal{H}} + \varepsilon)$$

simultaneously hold for some $\varepsilon > 0$ to be fixed later. We consider the following configuration: on the line segment $\Gamma_1 = [0, n_a\xi_1]$ let all the passage vectors be a, while in the line segment $\Gamma_3 = [n_a\xi_1, n_a\xi_1 + 3n_b\xi_2]$ let the passage vectors be b. Moreover, on the line segments

$$[0, -\frac{n_b - n_a}{2}\xi_2],\ [-\frac{n_b - n_a}{2}\xi_2, n_a\xi_1 - \frac{n_b - n_a}{2}\xi_2],\ [n_a\xi_1 - \frac{n_b - n_a}{2}\xi_2, n_a\xi_1]$$

let all the passage vectors be b. (The union of these line segments will be denoted by Γ_2.) Concerning the remaining edges we separate two cases. (See Figure 3.2 for the discussion belonging to one of them.) Notably, we know that if we consider the line $L = \{ta + ((3n_b + n_a) - t)b : t \in \mathbb{R}\} \subseteq \mathcal{H}$, then there is a unique $t = t_0 \in \mathbb{R}$, for which the norm of $p_{t_0} = t_0a + ((3n_b + n_a) - t_0)b$ is minimal, that is the orthogonal projection of the origin to L. If we increase or decrease t gradually from t_0, then

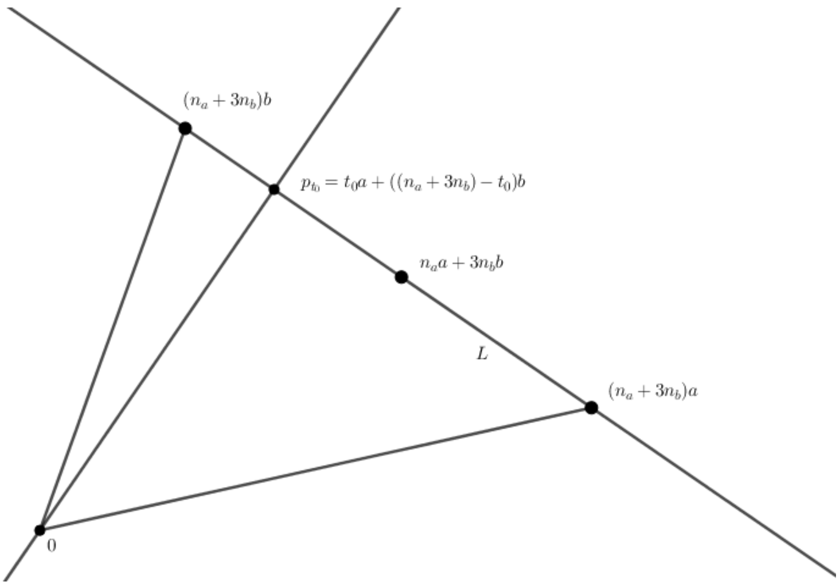

Figure 3.2: The definition of L, example for the first case.

due to the orthogonality of $[0, p_{t_0}]$ and L, the norm of p_t is strictly increasing in both directions. From this observation we infer that if we consider $t = n_a$, then if we move t towards one of the directions, the norm will strictly increase. Now if this increase shows up in the direction of $(3n_b + n_a)a$, then we place the passage vector a to all the remaining edges. On the other hand, if this increase appears in the direction of $(3n_b + n_a)b$, then we place the passage vector b to all the remaining edges.

Now let us calculate the square of the passage times of Γ_1 and Γ_2:

$$\tau(\Gamma_1)^2 = n_a^2 \|a\|_{\mathcal{H}}^2, \ \tau(\Gamma_2)^2 = n_b^2 \|b\|_{\mathcal{H}}^2.$$

Consequently, we have $\tau(\Gamma_1) > \tau(\Gamma_2)$ by the choice of n_a, n_b, hence Γ_1 cannot be optimal. Thus if we have that $\Gamma_1 \cup \Gamma_3$ is the only optimal path from the origin to $n_a\xi_1 + 3n_b\xi_2$, that concludes the proof, as one of its subpaths is not optimal, consequently, it cannot be a geodesic. In the following we will examine whether there are other optimal paths between these points and can they be geodesics.

First consider the case in which we placed the passage vector b to all the remaining edges, that is the norm of $n_a a + 3n_b b$ is smaller than the norm of $ta + ((3n_b + n_a) - t)b$ for any $t < n_a$. In this case any path $\Gamma \neq \Gamma_1 \cup \Gamma_3$ uses at least $n_a + 3n_b$ edges, and at least $3n_b + 1$ of them has passage vector b. Thus by the assumption of this case concerning the norms, we certainly have that the norm of the passage vector of Γ is larger than the norm of $n_a a + 3n_b b = v(\Gamma_1 \cup \Gamma_3)$. (At this point we also use the positivity of A: if Γ uses more edges, than $n_a + 3n_b$, then we reduce the passage time by forgetting about edges with passage vector a.) Thus this case is concluded, $\Gamma_1 \cup \Gamma_3$ is the only optimal path from the origin to $n_a\xi_1 + 3n_b\xi_2$.

Consider now the other case, that is the norm of $n_a a + 3n_b b$ is smaller than the norm of $ta + ((3n_b + n_a) - t)b$ for any $t > n_a$. Assume that there is another path $\Gamma \neq \Gamma_1 \cup \Gamma_3$ from the origin to $n_a\xi_1 + 3n_b\xi_2$ which is optimal. It must hit the line segment $[-\frac{n_b - n_a}{2}\xi_2, n_a\xi_1 - \frac{n_b - n_a}{2}\xi_2]$: indeed, otherwise Γ would contain at least n_a edges with passage vector a. Now if $|\Gamma| > n_a + 3n_b$, it immediately implies that the passage time of Γ exceeds the passage time of $\Gamma_1 \cup \Gamma_3$, a contradiction. On the other hand, if $|\Gamma| = n_a + 3n_b$, the number of edges with passage vector a surely exceeds n_a. Hence by the starting assumption of this case concerning the norms, the norm of the passage vector of Γ is larger than the norm of $n_a a + 3n_b b = v(\Gamma_1 \cup \Gamma_3)$, a contradiction. Hence Γ hits the line segment $[-\frac{n_b - n_a}{2}\xi_2, n_a\xi_1 - \frac{n_b - n_a}{2}\xi_2]$ at some point x indeed. Denote its first part from the origin to x by Γ' and its second part from x to $n_a\xi_1 + 3n_b\xi_2$ by Γ''. Assume that Γ' has an edge which is not contained by Γ_2, and hence its passage vector is a. In this case Γ' has an entire subpath Γ_0' connecting points of Γ', and with edges of passage vector a. However, Γ_0' can be replaced with an ℓ_1-optimal path such that all of its edges have passage vector b. Hence its passage time is lower than $\tau(\Gamma_0')$, yielding that Γ_0' cannot be optimal. Consequently, Γ has a subpath which is not optimal, while Γ is optimal, thus Γ cannot be a geodesic. We can proceed similarly if Γ'' has an edge which is not contained by $\Gamma_2 \cup \Gamma_3$. Thus the only case remaining is that Γ contains all the edges of $\Gamma_2 \cup \Gamma_3$, and hence $\tau(\Gamma)$ is at least $\tau(\Gamma_2 \cup \Gamma_3)$. However, in this case the square of its passage time equals

$(4n_b)^2\|b\|^2_{\mathcal{H}}$, while the square of passage time of $\Gamma_1 \cup \Gamma_3$ can be estimated by

$$\begin{aligned}\tau(\Gamma_1 \cup \Gamma_3)^2 &= n_a^2\|a\|^2_{\mathcal{H}} + 6n_a n_b(a,b) + 9n_b^2\|b\|^2_{\mathcal{H}} \\ &\leq n_b^2(\|b\|_{\mathcal{H}} + \varepsilon)^2 + 6\mu n_b^2\|b\|_{\mathcal{H}}(\|b\|_{\mathcal{H}} + \varepsilon) + 9n_b^2\|b\|^2_{\mathcal{H}}.\end{aligned} \tag{3.8}$$

Thus by $\tau(\Gamma_1 \cup \Gamma_3)^2 > \tau(\Gamma_2 \cup \Gamma_3)^2 = (4n_b)^2\|b\|^2_{\mathcal{H}}$ we conclude the following inequality with $\varepsilon = \frac{\|b\|_{\mathcal{H}}}{M}$:

$$6n_b^2\|b\|^2_{\mathcal{H}} < 6\mu n_b^2\|b\|^2_{\mathcal{H}} + (2+6\mu)n_b^2\|b\|_{\mathcal{H}}\varepsilon + n_b^2\varepsilon^2 = \left(6\mu + \frac{(2+6\mu)}{M} + \frac{1}{M^2}\right) n_b^2\|b\|^2_{\mathcal{H}}.$$

However, for sufficiently large M this inequality cannot hold. This is a contradiction, Γ cannot be optimal in this case. Hence we found that there can be no optimal path from 0 to $n_a\xi_1 + 3n_b\xi_2$ which is a geodesic. It concludes the proof. □

Given these lemmas we can deduce the following aesthetic result roughly stating that if the geometric structure is tame in the above sense then the Hilbert percolation we consider is essentially ordinary:

Theorem 3.3.4. *Let $A \subseteq \mathcal{H}$ such that Ω is a Baire space and let $d \geq 2$. Assume that generically for any $\Gamma' \subseteq \Gamma$ we have that $\tau(\Gamma') \leq \tau(\Gamma)$ and there is a geodesic between any pair of lattice points. Then A is contained by a ray, that is it is linearly isomorphic to a subset of $[0, +\infty)$.*

Remark 3.3.5. We note that if for example $A \subseteq \mathcal{H}$ is G_δ, then Ω is a Baire space, hence the above theorem holds in quite natural cases. Indeed, $\mathcal{H}$ is obviously Polish as $\mathcal{H}$ is separable, hence A is also Polish due to Alexandrov's theorem. Consequently, Ω is also Polish, as a countable product of Polish spaces. Thus Ω is Baire due to Baire's category theorem. (For details, see [9] for instance.)

Proof of Theorem 3.3.4. From the remark following Lemma 3.3.2 we know that A is positive by the assumption of the theorem. Moreover, from the proof of Lemma 3.3.3 it follows very quickly that if A is not contained by a ray then there are nontrivial open sets in which there are no geodesics between certain points, as we can consider sufficiently small neighborhoods of the configurations constructed there. However, as Ω is a Baire space, nontrivial open sets cannot be disjoint from a residual subset of Ω. Thus A is contained by a ray in fact. It concludes the proof.

□

3.4 Geodesic rays in Hilbert percolation

In the ordinary topological first passage percolation we proved that in the generic configuration for any point x there exists a geodesic ray such that x is its starting point, and the proof was based on the fact that there exists a finite geodesic between any pair of lattice points in the generic configuration. Given the previous theorem, it is natural to ask how many distinct geodesic rays may exist generically in the Hilbert percolation, and whether it is possible that there are no geodesic rays at all. The following theorem is naturally analogous to Theorem 2.3.3

Theorem 3.4.1. *Assume $d \geq 2$. If $A \subseteq \mathcal{H}$ is nontrivial, that is it has cardinality larger than 1, then generically there exists at most one geodesic ray.*

Proof of Theorem 3.4.1. The case when A is contained by a ray is already covered by Theorem 2.3.3. Hence we can assume that A is not linearly isomorphic to a subset of the nonnegative reals. Still, the proof will be quite similar to the proof of Theorem 2.3.3, hence we will focus on the differences this time.

Let $a, b \in A$ so that we cannot get one from another by multiplying with a nonnegative scalar and $\|a\|_{\mathcal{H}} \leq \|b\|_{\mathcal{H}}$. Consequently, we might choose $0 < \mu < 1$ satisfying $(a, b) \leq \mu \|b\|_{\mathcal{H}}^2$. First we will prove that the origin is the starting point of distinct geodesics in a meager subset only. Let $U, U', K_1, K_2, E_U, E^*$ be as in the proof of Theorem 2.3.3 defined in terms of the parameters p, q, q', r to be fixed later. We will define $V \subseteq U'$ as a cylinder set which has nontrivial projections to the edges in $E_U \cup E^*$. The underlying concept is the same as in that proof: for the configurations in V we would like to have essentially one (and the same) geodesic from the boundary ∂K_1 to the boundary ∂K_2, notably the line segment connecting $p\xi_1$ and $q\xi_1$. By this we mean that for any lattice points $x_1 \in \partial K_1$ and $x_2 \in \partial K_2$, a geodesic Γ from x_1 to x_2 eventually arrives in $p\xi_1$, and then it goes along the line segment $[p\xi_1, q\xi_1]$. It would be sufficient exactly as it was earlier.

We will obtain this property following the same strategy: our purpose is to define V so that sufficiently many paths in $\partial K_1 \cup [p\xi_1, q\xi_1] \cup \partial K_2$ are cheap while other paths in $K_2 \setminus K_1$ are expensive for configurations in V. To this end, we use a slightly more complicated construction this time, in which we will guarantee paths to be cheap by having edges with passage vectors a and b alternatingly. To obtain this, consider the connected subgraph G of $\mathbb{Z}^d$ in $\partial K_1 \cup [p\xi_1, q\xi_1] \cup \partial K_2$ and its vertex $p\xi_1$. Let G_0 be the breadth-first search tree rooted at vertex $p\xi_1$. By definition, in G_0 the unique path from $p\xi_1$ to any other vertex $x \in G$ is optimal in the graph distance of G, which is equivalent to being optimal in ℓ_1 inside $\partial K_1 \cup [p\xi_1, q\xi_1] \cup \partial K_2$. Moreover, as the points of $[p\xi_1, q\xi_1]$ are cut vertices of G, that is the deletion of any of them cuts G into two distinct connected components, it obviously implies that for any lattice points $x \in \partial K_1 \cup [p\xi_1, q\xi_1)$ and $y \in (p\xi_1, q\xi_1] \cup \partial K_2$, the unique path from x to y contained by G_0 is optimal in G. Now we can define projections to edges of G_0 such that they equal small neighborhoods of a and b and for any such G-optimal path these projections appear alternatingly. Indeed, for branches rooted at $p\xi_1$ and proceeding towards ∂K_2 let us define the projection to the first edge to be a neighborhood of b, and the later ones to be alternatingly neighborhoods of a

and b. On the other hand, for branches rooted at $p\xi_1$ and contained by ∂K_1, let us define the projection to the first edge to be a neighborhood of a, and the later ones to be alternatingly neighborhoods of b and a. It yields indeed that for any lattice points $x \in \partial K_1 \cup [p\xi_1, q\xi_1)$ and $y \in (p\xi_1, q\xi_1] \cup \partial K_2$, on the G-optimal path from x to y contained by G_0 the projections equal to neighborhoods of a and b alternatingly. On any other edge inside $K_2 \setminus \operatorname{int} K_1$ let the projection be a small neighborhood of b. At this point, we think of all of these projections being the singletons $\{a\}$ and $\{b\}$ respectively, which are not necessarily open in A, but might be fattened suitably later. Then the proof relies on the fact that for any G-optimal path Γ' of the above type with $|\Gamma'| \geq 2$ we have

$$\begin{aligned}
\tau(\Gamma')^2 &\leq \left\| \frac{|\Gamma'|+1}{2} b + \frac{|\Gamma'|-1}{2} a \right\|_{\mathcal{H}}^2 \\
&\leq \left(\frac{|\Gamma'|+1}{2} \right)^2 \|b\|_{\mathcal{H}}^2 + \left(\frac{|\Gamma'|-1}{2} \right)^2 \|a\|_{\mathcal{H}}^2 + \left(\frac{(|\Gamma'|+1)(|\Gamma'|-1)}{2} \right) (a,b) \\
&\leq \frac{|\Gamma'|^2+1}{2} \|b\|_{\mathcal{H}}^2 + \frac{|\Gamma'|^2-1}{2} (a,b) \leq \frac{3}{4} |\Gamma'|^2 \|b\|_{\mathcal{H}}^2 + \frac{1}{4} |\Gamma'|^2 (a,b) \\
&\leq |\Gamma'|^2 \left(\frac{3}{4} + \frac{1}{4} \mu \right) \|b\|_{\mathcal{H}}^2 .
\end{aligned} \tag{3.9}$$

Consequently, for $\lambda = \sqrt{\frac{3}{4} + \frac{1}{4}\mu} < 1$ we have

$$\tau(\Gamma') \leq \lambda |\Gamma'| \|b\|_{\mathcal{H}}. \tag{3.10}$$

Now proceeding towards a contradiction assume that there is a geodesic Γ_0 from ∂K_1 to ∂K_2 which does not contain the line segment $[p\xi_1, q\xi_1]$. Similarly to arguments in the proof of Theorem 2.3.3, by passing to a subpath we can infer the existence of a geodesic Γ from $x_1 \in \partial K_1 \cup [p\xi_1, q\xi_1)$ to $x_2 \in (p\xi_1, q\xi_1] \cup \partial K_2$ which uses only edges not contained by $\partial K_1 \cup [p\xi_1, q\xi_1] \cup \partial K_2$. If $|x_1 - x_2| = 1$, it is clearly impossible, thus we may assume $|x_1 - x_2| > 1$. Our aim is to show a cheaper path Γ' contained by $\partial K_1 \cup [p\xi_1, q\xi_1] \cup \partial K_2$ as that would yield a contradiction. To this end, we construct Γ' by a similar method as in the proof of Theorem 2.3.3: in this case, let it be the unique path from x_1 to x_2 in G_0, which is optimal in G. For this path, we can use (3.10). Indeed, by this bound, if we want to show $\tau(\Gamma') < \tau(\Gamma)$ to obtain a contradiction, we can do a little trick and replace all the passage vectors of edges in $\partial K_1 \cup [p\xi_1, q\xi_1] \cup \partial K_2$ by λb. As $|\Gamma'| \geq 2$ necessarily, it does not decrease the passage time of Γ', and the passage time of Γ does not change at all. Consequently, all the passage vectors in $K_2 \setminus \operatorname{int} K_1$ are multiples of b, and the ones in $\partial K_1 \cup [p\xi_1, q\xi_1] \cup \partial K_2$ are cheaper than the others. Hence the situation we face is linearly isomorphic to the one in the proof of Theorem 2.3.3. Thus for a suitable choice of the parameters p, q, q', r we get a contradiction for this specific configuration, $\tau(\Gamma') < \tau(\Gamma)$, as Γ' is a path which is optimal in ℓ_1 amongst the paths contained by $\partial K_1 \cup [p\xi_1, q\xi_1] \cup \partial K_2$, while Γ does not have any edge in this set, which is sufficient by Remark 2.3.5. Finally, by taking small enough neighborhoods to be the projections instead of the singletons this inequality will not fail as there are only finitely many pairs Γ, Γ' to consider. It concludes the proof of the claim that the origin is the starting point of distinct geodesics only in a meager subset. The statement of the theorem is obtained from this claim the same way as in the proof of Theorem 2.3.3. □

To conclude this section, we provide an example for a closed set A so that there are no geodesic rays at all generically. (As Ω is a Baire space in this case, it means indeed that in a large subset of Ω there are no geodesic rays.) Let $d = 2$, and let $A = \{a, b, 2b\} = \{(1,0), (0,1), (0,2)\} \subseteq \mathcal{H} = \mathbb{R}^2$. It suffices to prove that there is no geodesic ray starting from the origin generically. Fix a cylinder set U. Let us use a construction similar to the one in the proof of Theorem 2.3.3: fix a cylinder set V such that there exists K_1, K_2 as earlier so that any geodesic ray starting from the origin eventually reaches $q\xi_1$ and do not enter $\operatorname{int} K_2$ again. More explicitly, in V the passage vectors of edges in $\partial K_1 \cup [p\xi_1, q\xi_1] \cup \partial K_2$ equal b while of other edges in $K_2 \setminus K_1$ they equal $2b$. Now it suffices to fix a finite number of further passage vectors so that in such configurations there is no geodesic from $q\xi_1$ not entering $\operatorname{int} K_2$: that would mean that there is no geodesic ray starting from the origin. We do so by fixing some passage vectors in $q\xi_1 + D_7$ to be a or b as shown in Figure 3.3 (D_7 denotes the closed ball with ℓ_1 radius 7 centered at the origin): in the first quadrant, the red, decorated edges correspond to passage vector a, while the blue, simple ones to passage vector b. The passage vectors in the negative half-plane are obtained by reflection to the horizontal axis.

Now we claim that there is no geodesic from $q\xi_1$ to points with $q\xi_1 + x$ where $|x| = 6$ and has nonnegative first coordinate. As we gained the passage vectors in the negative halfplane by reflection, it suffices to prove this claim for x with nonnegative second coordinate, too. Now it is easy to check that if $x = (0,6)$ or $x = (6,0)$ then there are paths with passage vector $4a + 4b$ from $q\xi_1$ to $q\xi_1 + x$. Meanwhile all the paths with minimal ℓ_1-length 6 has passage vector $6a$ or $6b$. But $\|4a + 4b\| = 4\sqrt{2} < 6 = \|6a\| = \|6b\|$. Consequently, the paths with passage vector $4a + 4b$ are the optimal ones, while none of them is a geodesic as their first 2 or first 4 steps cannot be optimal. Thus there is no geodesic to such points. On the other hand, if both of the coordinates of x are positive, then there are optimal paths with passage vector $3a + 3b$ from $q\xi_1$ to $q\xi_1 + x$. However, their first 2 or first 4 steps are not optimal as they have not been before. Thus there are no geodesics to such points either. Consequently, there is no geodesic from $q\xi_1$ to points with $q\xi_1 + x \notin \operatorname{int} K_2$ where $|x| = 6$. Thus the origin cannot be the starting point of a

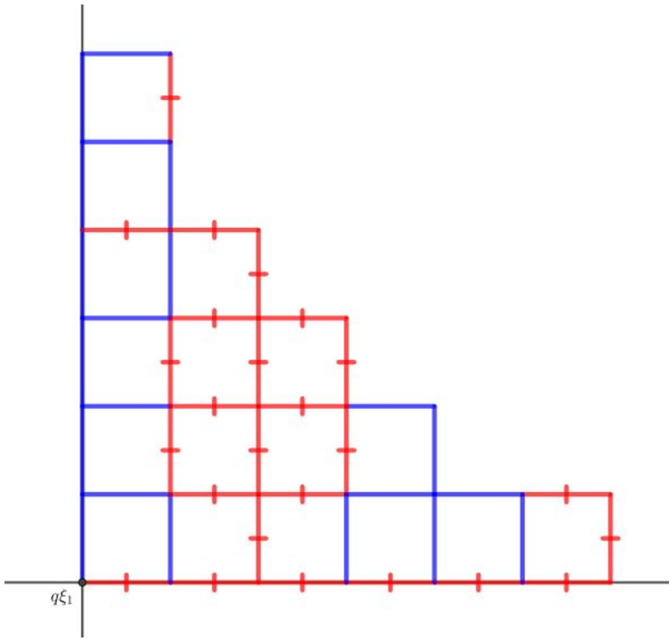

Figure 3.3: Red decorated edges have passage vector a, while blue simple ones have passage vector b.

geodesic ray as any such geodesic ray eventually reaches $q\xi_1$ and should continue for an infinite number of steps as a geodesic outside of $\operatorname{int} K_2$, which is impossible by the previous observation. It yields that there are no geodesic rays at all generically as we claimed.

Chapter 4

Concluding remarks and open problems

In the above chapters we answered several questions, but in the meantime new ones arose. In the following, we list a few of them.

In Section 2.5 we made considerable progress concerning the asymptotic behavior of $\frac{B(t)}{t}$: we gave complete characterization of the limit sets $K = \lim_{n\to\infty} \frac{B(t_n)}{t_n}$ in the cases $\inf A = 0$ or $\sup A = \infty$. Moreover, we gave a necessary condition concerning such limit sets for general A, more explicitly we proved that such a set must be in $\overline{\mathcal{W}_A^d}$. The point of view provided by this recognition helped us in proving that all the convex sets of $\mathcal{K}_A^d$ arise as limit sets. However, on one hand, we could not prove that this condition is sufficient, and on the other hand, the definition of the family $\mathcal{W}_A^d$ is quite unhandy and does not really extend our understanding about the limit sets in its own right as it is somewhat just a continuous and more general rephrasing of the definition of a limit set.

Question 4.0.1. *Is it true that all the sets in $\mathcal{W}_A^d$ arise as limit sets? If not, find what is the good family to consider, if yes, try to define it more conveniently.*

In Section 3.2 we understood the nature of passage times in the strongly positively dependent case of the Hilbert first passage percolation in the finite dimensional case for bounded A. Our argument relied on Dirichlet's approximation theorem, which has variants for infinite dimensional spaces, too, but they are insufficient for our purposes (see [6]). It would be nice to understand the behavior of the percolation in these cases as well. Moreover, we also capitalized on the boundedness of A, hence it should also be examined what happens if this condition is dropped.

Question 4.0.2. *What can be said about the Hilbert percolation in the strongly positively dependent case if the space is infinite dimensional or A is not bounded?*

In Section 3.4 we provided an example for a set A such that there are no geodesic rays at all generically. The proof heavily relied on the simple structure of A: roughly we used the fact that there is a configuration in which $q\xi_1$ is not a starting point of geodesics longer than 5 edges. Now it is reasonable to ask if it is always the case when A is not linearly isomorphic to a subset of the nonnegative reals.

Question 4.0.3. *Is it true that there exists a geodesic ray generically if and only if A is linearly isomorphic to a subset of $[0, +\infty)$?*

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