Characterizations and Properties of Graphs of Baire Functions

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Contents

Chapter 1

Introduction

The notion of Baire functions was introduced by René-Louis Baire in his doctoral thesis (see [2]). This theory proved to be an interesting subject and has been studied subsequently in real analysis, general topology, and descriptive set theory. First, let us recall the definition of Baire classes and take a glance at a few examples.

If X is a topological space and Y is a metric space, we say that a function $f : X \to Y$ is of Baire class 1, if it is the pointwise limit of a sequence of continuous functions.

For example, let us fix $x \in \mathbb{R}$. If we consider the sequence $(f_n)_{n=1}^{\infty}$, where each $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_n(t) = e^{-n|x-t|}$, we can easily see that $\lim_{n\to\infty} f_n(t) = 0$ for any $t \neq x$ and $\lim_{n\to\infty} f_n(x) = 1$. It yields that the characteristic function of any real number is a Baire-1 function.

Thus we defined functions of Baire class 1. Higher Baire classes obtained recursively: a function is of Baire class α for some countable ordinal α if it is the pointwise limit of a sequence of functions of lower Baire classes. It is useful to note that if $Y = \mathbb{R}$, then the functions of each Baire class form a linear space. According to this remark and using the simple fact seen in the example above, we can immediately deduce that the Dirichlet function, that is the characteristic function of the set of rational numbers, is Baire-2, as the sum of a series of Baire-1 functions.

In this thesis, we will examine a few questions concerning graphs of Baire functions. The main mofivation of this research was the article of E. S. Thomas and the article of Agronsky, Ceder, and Pearson (see [1] and [9]): in the former one a characterization of bounded real-valued Baire-1 functions was given using their graph, in the latter this result was generalized for the case of not necessarily bounded real-valued Baire-1 functions defined on a metric space.

In the second section of this work, we will present our research published in [6]. We show an application of the aforementioned results by investigating a property of graphs of Baire-1 and Baire-2 functions. The problem is the following: if T is a given subset of $[0,1] \times \mathbb{R}$, when does there exist a Baire-1 or Baire-2 function $f : [0,1] \to \mathbb{R}$ such that the accumulation points of its graph are exactly the points of T? We answer these questions in two steps in both cases. It is easier to understand the theorems and the proofs if we also require f to be bounded, thus we start with this case. During this process we will mostly use elementary methods, which will end up in a bit complicated proofs.

In the third section, we will present our results published in [7], in which we directly focused on the results of [1] and generalized them in two senses. On one hand, we state analogous theorems concerning higher Baire classes. On the other hand, we attempt to do so with less strict conditions about the domain and codomain. In order to prove these results, we will use selection theorems such as Michael's selection theorem. Finally, we will show that the theory of selection theorems provide a simple and elegant way for handling some of the problems seen in the second section, even in much more general settings.

Chapter 2

The Accumulation Problem

2.1 Notation

Throughout this part we use the following notation: the graph of the real function f is denoted by $gr(f)$. If f is a real function, the set of accumulation points of $gr(f)$ is L_f. The vertical line given by the equation $x = r$ is denoted by v_r. If H is a subset of $\mathbb{R}^2$, and r is a real number, the intersection of v_r and H is denoted by $H(r)$. For simplicity, if $(r, y) \in H$, we say that $y \in H_r$. The open ball with center r and radius ε is $B(r, \varepsilon)$. We use this notation for one-dimensional neighborhoods in $\mathbb{R}$, and also for two-dimensional neighborhoods in $\mathbb{R}^2$. We clarify this ambiguity by making clear if the center is a point of $\mathbb{R}$ or of $\mathbb{R}^2$. The interval $[0, 1]$ is denoted by I. The cardinality of a set H is $\#(H)$. The diameter of a set H is $\mathrm{diam}(H)$. Finally, if a set $A \subseteq I$ is the subset of the domain of f, and $a \in A$, sometimes we refer to the point $(a, f(a))$ as a point of $gr(f)$ above A.

2.2 Preliminary Results

In the introduction we have already mentioned the result of Agronsky, Ceder, and Pearson. This theorem will be a very useful tool for us, so it is appropriate to recall it. We need the following definition:

Definition 2.2.1. *An open set $S \subseteq \mathbb{R}^2$ is an open strip if for every $r \in R$ the set $S(r)$ is an open interval.*

In [1], a characterization of Baire-1 functions was given in Theorem 2.2 by using this definition. At this point, we only formulate a special case of this theorem:

Proposition 2.2.2. *Let $f : I \to \mathbb{R}$ be a function. It is Baire-1 if and only if there is a sequence (S_n) of open strips such that $\cap_{n=1}^{\infty} S_n = gr(f)$.*

As we will see, this theorem is a truly useful tool if our goal is to show that a certain function is Baire-1. Besides that we will also apply the following lemma, which handles a variant of the accumulation problem.

Lemma 2.2.3. *For a given closed set $T \subseteq I \times \mathbb{R}$, there exists a countable set $A \subseteq I$ such that there is a function $f : A \to \mathbb{R}$ satisfying $L_f = T$.*

Proof. Let $T_i = (I \times [-i, i]) \cap T$ for all $i \in \mathbb{N}$. Then every T_i is compact. Let us consider an open ball of radius one around each point of T_1. These open balls cover T_1, hence it is possible to choose a finite covering. Let us take a point in each chosen open ball such that the x coordinates of these points are pairwise different. Let us denote the set of these points by H_1, and the set of their x coordinates by A_1.

Now, similarly, let us consider open balls with radius $\frac{1}{2}$ around each point of T_2 and choose a finite covering, then finally take points in these chosen neighborhoods and define H_2 and A_2 analogously. We can continue this procedure by induction: in the n^{th} step we consider the $\frac{1}{n}$-neighborhoods of the points of T_n, and we define the finite sets H_n and A_n using these open balls.

Let $A = \cup_{n=1}^{\infty} A_n$ and $H = \cup_{n=1}^{\infty} H_n$. These are countable sets. Let f be the function that assigns to every $x \in A$ the y coordinate of the chosen point above x. Then this point of the graph is clearly a point of H. We would like to prove that $L_f = T$ for this function f. We do this by verifying two containments.

(i) $T \subseteq L_f$. Let us consider any point P of T. By definition, $P \in T_k$ for a suitable k positive integer. Thus for every n larger than k there exists a point $x_n \in A_n$ such that the distance of $(x_n, f(x_n))$ and P does not exceed $\frac{1}{n}$. Therefore, there exists a sequence of distinct points in $gr(f)$ that converges to P, hence $T \subseteq L_f$.

(ii) $L_f \subseteq T$. Let us consider any point P of L_f. Since it is an accumulation point of $gr(f)$, there exists a sequence (p_n) in $gr(f)$ converging to P and containing each of its terms only once. Now if k is given, for sufficiently large n the point p_n is in H_m with $m \geq k$. It means that the distance of p_n and T does not exceed $\frac{1}{k}$. Thus there are points of T arbitrarily close to the sequence (p_n). Therefore, the limit of (p_n) is in T, since T is closed. Hence $P \in T$ and $L_f \subseteq T$.

□

Remark 2.2.4. The above proof shows that there are only finitely many points of the graph $gr(f)$ that are more than ε apart from T for a given $\varepsilon > 0$. Later we will use this slightly stronger result.

2.3 Functions of Baire class 2

As we have promised, we consider the bounded case first. It is obvious that if $L_f = T$, then T must be a compact set, being bounded and closed. There is another condition needed: $T(x)$ is never empty for $x \in I$. Indeed, if (x_n) is a sequence that converges to x, $(x_n \neq x)$, the sequence formed by the points $(x_n, f(x_n))$ is a bounded sequence in $\mathbb{R}^2$, and its limit is in T, thus $T(x) \neq \emptyset$.

We point out that until this point we have not used the Baire-2 property of the function f. Despite that, as we will see, these conditions are also sufficient:

Theorem 2.3.1. *Suppose $T \subseteq I \times \mathbb{R}$. There exists a bounded Baire-2 function $f : I \to \mathbb{R}$ such that $L_f = T$ if and only if*

- *T is compact,*
- *$T(x)$ is nonempty for $x \in I$.*

Proof. Before beginning the formal proof, we give a short sketch. First, we construct a function f_0 such that $f_0(x) \in T(x)$ for every $x \in I$. After this step, we apply Proposition 2.2.2 to prove that f_0 is a Baire-1 function. Finally, we use Lemma 2.2.3 to modify f_0 on a countable set A to obtain a bounded Baire-2 function f such that $L_f = T$.

Put $f_0(x) = \max(T(x))$ for every $x \in I$. Since $T(x)$ is nonempty, this definition makes sense. The function f_0 is Baire-1; this is a well-known fact since f_0 is upper semicontinuous and every upper semicontinuous function is Baire-1. Nevertheless, it is useful to find a direct proof which uses Proposition 2.2.2 to understand better how this theorem works.

We define a nested sequence of open strips, (S_n). First, we construct a subset S_n' of S_n, that is the union of certain neighborhoods of points of $gr(f_0)$. Let the radius of such an open ball be $\varepsilon_{x,n}$, where $\varepsilon_{x,n}$ satisfies the following three conditions: $\varepsilon_{x,n} \leq \frac{1}{n}$ and $\varepsilon_{x,n} \leq \varepsilon_{x,n-1}$ for every $n \geq 2$. It is obviously possible. Moreover, we have a bit more complicated so-called overlapping condition related to the projection of the open balls $B((x, f_0(x)), \varepsilon_{x,n})$ to the x-axis. Specifically:

$$\forall x \in I, \forall n \in \mathbb{N}, \forall r \in \mathbb{R}, r \in B(x, \varepsilon_{x,n}) \text{ we have } f_0(r) - f_0(x) < \frac{1}{n}\ .$$

Such $\varepsilon_{x,n}$ can be chosen. If not, then there is a sequence (x_k) that converges to x and $f_0(x_k) \geq f_0(x) + \frac{1}{n}$ for every k. In this case $(f_0(x_k))$ is a bounded sequence, so it has a convergent subsequence. As a consequence, the sequence $(x_k, f_0(x_k))$ has a limit point in the plane whose first coordinate is x, and whose second coordinate is larger than $f_0(x) = \max(T(x))$ by at least $\frac{1}{n}$. Since T is closed, it is a contradiction. Thus for every $n \in \mathbb{N}$ and $x \in I$, we can choose some $\varepsilon_{x,n}$ satisfying all three of our conditions. By taking the union of the neighborhoods $B((x, f_0(x)), \varepsilon_{x,n})$, we obtain an open set S_n' containing $gr(f_0)$ for every n. Also $S_n' \subseteq S_{n-1}'$ for every $n \geq 2$, since S_n' is the union of open balls with the same centers and smaller radii. However, it is not sufficient for us: our aim is to construct open strips. But this problem can be solved easily. Specifically, there is a simple way to extend an arbitrary open set H' to an open strip H: for every x, let $H(x) = (\inf(H'(x)), \sup(H'(x))$. Figure 2.1 demonstrates such an extension, in a case where H' is the union of a few open disks: H is the open set bounded by the dashed lines. It is plain to see that the set H made this way is an open strip which contains H'. We also use this method to construct $S_n(x)$ by extending $S_n'(x)$. The property $S_n \subseteq S_{n-1}$ is obviously preserved during the extension.

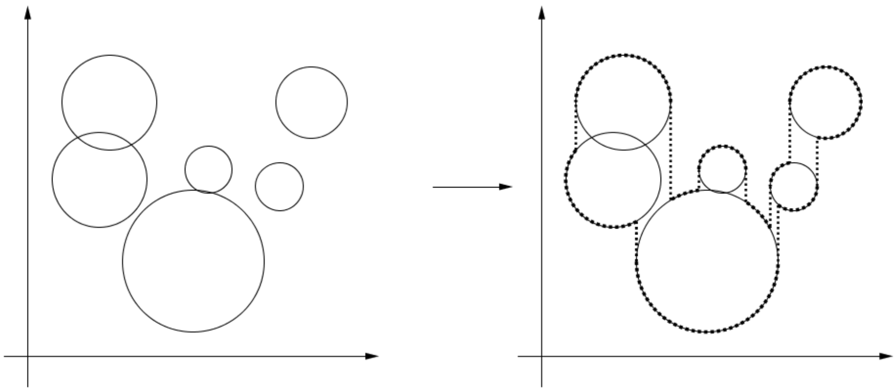

Figure 2.1: Extending an open set into an open strip

To apply Proposition 2.2.2, we have to verify that $S = \cap_{n=1}^{\infty} S_n = gr(f_0)$. It is clear that S contains $gr(f_0)$ since S_n' contains every point of $gr(f_0)$ for all n. We have to show that S has no other points. Proceeding towards a contradiction, let us assume that there exists a point $x \in I$ and $y \neq f_0(x)$ such that $(x, y) \in S$. We distinguish two cases.

a) The case $y > f_0(x)$. Since $(x, y) \in S_n$ for every n, the set S_n' has a point (x, z_n) above (x, y). The sequence (z_n) is obviously bounded, hence it has a limit point $z \geq y$. But S_n' is formed by open balls whose centers are the points of $gr(f_0) \subseteq T$ and whose radii are not larger than $\frac{1}{n}$. Thus $(x, z) \in T$ as T is closed. So T has a point whose first coordinate is x and whose second coordinate is larger than $f_0(x) = \max(T(x))$, a contradiction.

b) The case $y < f_0(x)$. By a similar argument to the previous one, we might notice that S_n' has a point (x, z_n) below (x, y) for every n. Let $k \in \mathbb{N}$ satisfy

$y < f_0(x) - \frac{1}{k}$. Then if $n \geq 2k$, amongst the open balls forming S'_n we might find a ball that intersects v_x and for its center $(x_n, f_0(x_n))$ the inequality $f_0(x_n) < f_0(x) - \frac{1}{2k}$ holds. But by definition, it is impossible: this neighborhood must satisfy the overlapping condition, thus it cannot intersect v_x, a contradiction. Hence f_0 is a function of Baire class 1.

Using Lemma 2.2.3, we modify f_0 on a countable set A, so that the accumulation set of the new points above A is T. We denote this altered function by f. Then it is a bounded Baire-2 function. Nevertheless, if we consider now the whole graph, $L_f = T$ remains true, since every point of the graph above $I \setminus A$ is in T. Therefore other accumulation points cannot occur. □

In the following, we turn our attention to the not necessarily bounded Baire-2 functions. In this case the conditions are more complicated and the proof is a bit more difficult. However, we give a similar characterization.

We approach the problem by finding out some necessary conditions. During that process, we use only that $f : I \to \mathbb{R}$, as we did earlier in our previous theorem. It is easy to see that T must be closed in this case, too. But it is not true at all that $L_f(x) = T(x)$ must be nonempty for every $x \in I$. For instance, let f be the function that vanishes in 0, and elsewhere its value is $\frac{1}{x}$. Then $L_f(0)$ is empty. Nevertheless, we may suspect that $T(x)$ cannot be empty in any set C. Our lemma is the following:

Lemma 2.3.2. *If $f : I \to \mathbb{R}$ and $C = \{x \in I : L_f(x) = \emptyset\}$, then C is countable.*

Proof. Proceeding towards a contradiction, let us assume that C is uncountable. Put $C_n = \{x \in C : |f(x)| < n\}$ for every $n \in \mathbb{N}$. Then $C = \cup_{n=1}^{\infty} C_n$, and there exists an uncountable C_n. As a consequence, it contains one of its limit points, c. Thus there exists a sequence (c_i) in C_n $(c_i \neq c)$ that converges to c. Since $(f(c_i))$ is bounded, it has a convergent subsequence, therefore $L_f(c)$ cannot be empty, a contradiction. □

We state that these necessary conditions are also sufficient, namely:

Theorem 2.3.3. *Suppose $T \subseteq I \times \mathbb{R}$. There is a Baire-2 function $f : I \to \mathbb{R}$ such that $L_f = T$ if and only if*

- *T is closed,*
- *there is a countable $C \subseteq I$ such that $T(x)$ is nonempty for $x \in I \setminus C$.*

Proof. The concept of the proof is similar to our proof given for the bounded case. We begin by the construction of a function f_0 and then we prove that it is a Baire-1 function. The desired function f will be obtained by modifying f_0 on a countable set using Lemma 2.2.3.

We start by observing that C is a G_δ set. Suppose $c \in C$. Since T is closed, it has a $B_{c,n}$ neighborhood for every $n \in \mathbb{N}$ such that for all $x \in B_{c,n}$ distinct from c, the absolute value of every element of $T(x)$ is larger than n. Otherwise $T(c)$ would not be empty. Then for a given n, the set $B_n = \cup_{c \in C} B_{c,n}$ is an open set containing C. On the other hand, clearly $\cap_{n=1}^{\infty} B_n = C$. Hence the set C is G_δ, as we wanted to show.

Now, we begin the construction of our function. The easier part is its definition on C. We consider an enumeration of the countable set $C = \{c_1, c_2, ...\}$ and we let $f_0(c_n) = n$ for every n. However, the definition of f_0 in $I \setminus C$ cannot be as straightforward as it was in our previous proof. Namely, it is possible that $T(x)$ has no maximum. Therefore we have to be more careful.

For every $n \in \mathbb{N}$, let

$$U_n = \{x \in I : \exists r \in T(x), |r| \leq n\}.$$

As T is closed, it is easy to see that each U_n is closed, too. It is also obvious that $U_n \subseteq U_{n+1}$ and $\cup_{n=1}^{\infty} U_n = I \setminus C$. Thus, for every $x \in I \setminus C$ there is a smallest n_x such that $x \in U_{n_x}$. Using this property, we may define $f_0(x)$ as the largest element of $T(x)$, whose absolute value does not exceed n_x. We can do so since $T(x)$ is closed and it has such an element. The inequalities $n_x - 1 < |f_0(x)| \leq n_x$ are also true, as otherwise x would be the element of U_m for some $m < n_x$. (Or, if $n_x = 1$, then $0 = n_x - 1 \leq |f_0(x)| \leq n_x = 1$.)

Now, we have defined f_0 on I. We would like to use Proposition 2.2.2 to show that f_0 is Baire-1. In order to do this, we construct the open strip S_n for every n. First, we define the open set S'_n constisting of some balls $B((x, f_0(x)), \varepsilon_{x,n})$. We select $\varepsilon_{x,n}$ so that $\varepsilon_{x,n} \leq \frac{1}{n}$ and $\varepsilon_{x,n} \leq \varepsilon_{x,n-1}$ for every $n \geq 2$, as we did earlier. Nevertheless, as we defined f_0 differently in certain sets, our further conditions should be case-specific: we handle separately the case $x \in C$ and the case $x \in I \setminus C$.

(i) The case $x \in C$. It means that $x = c_k$ for some k. Let

$$E_n = \cup_{x \in C} B((x, f_0(x)), \varepsilon_{x,n}),$$

and F_n be its projection onto the x-axis, that is $F_n = \cup_{x \in C} B(x, \varepsilon_{x,n})$. Let us choose these neighborhoods such that $\cap_{n=1}^{\infty} F_n = C$. It is possible since C is a G_δ set. Furthermore, we also demand that $B(c_k, \varepsilon_{c_k,n})$ does not contain the points $c_1, ..., c_n$, with the exception of c_k. We remark that these conditions imply $\cap_{n=1}^{\infty} E_n$ equals the graph of $f_0|C$.

(ii) The case $x \in I \setminus C$. Let us make some remarks concerning this complementary set. Let $V_1 = U_1$, and for $n \geq 2$, let $V_n = U_n \setminus U_{n-1}$. Then the set V_n is F_σ for every n, as the difference of closed sets. Consequently, there exist closed sets $V_{n,i}$ for every n and i such that $V_n = \cup_{i=1}^{\infty} V_{n,i}$. We can take an enumeration $W_1, W_2, ...$ of the sets $V_{n,i}$. Let $x \in V_k$. We can suppose that the $\varepsilon_{x,n}$ are chosen so that $B(x, \varepsilon_{x,n})$ does not contain the points $c_1, c_2, ..., c_n$. Furthermore, we can suppose that $B(x, \varepsilon_{x,n})$ does not intersect the sets $W_1, W_2, ..., W_n$, except for those which contain x. Finally, we have a special overlapping condition, namely that $f_0(r) - f_0(x) < \frac{1}{n}$ for every $r \in B(x, \varepsilon_{x,n}) \cap V_k$. One can prove that this condition can be satisfied as we proved it last time, in the bounded case. It is worth mentioning that if $f_0(x) < 0$, then $(x, -(k-1))$ cannot be a limit point of a sequence of points in $gr(f_0)$ above $I \setminus C$. Since T is closed, if such a sequence would exist, then $(x, -(k-1)) \in T$. But it means that $x \in U_{k-1}$, hence $x \notin V_k$.

Now the open set S'_n is defined for each n. As in the bounded case, our next step is making strips of these open sets: let $S_n(x) = (\inf(S'_n(x)), \sup(S'_n(x)))$ for every $x \in I$. Set $\cap_{n=1}^{\infty} S_n = S$ and similarly $\cap_{n=1}^{\infty} S'_n = S'$. We are going to show that $S = gr(f_0)$. Since $gr(f_0) \subseteq S$ is obvious, we can focus on proving $S \subseteq gr(f_0)$, or equivalently, proving that S has no point outside of $gr(f_0)$. We examine the relation of these sets independently for every $x \in I$: our goal is $S(x) \subseteq gr(f_0)(x)$. We distinguish the same cases which we distinguished during the construction of $S'_n(x)$:

a) The case $x \in C$, that is, $x = c_k$ for some k. Let us consider the set $S'_n(x)$. If $n \geq k$, amongst the open neighborhoods forming S'_n there can be only one that intersects v_x: the neighborhood of $(x, f_0(x))$. Thus for sufficiently large n the equality $S'_n(x) = S_n(x)$ holds, and $S'_n(x)$ contains only one open interval whose radius is $\frac{1}{n}$. Hence if n converges to infinity, we find that the only element of $S(x)$ is $f_0(x)$. Therefore $S(x) \subseteq gr(f_0)(x)$.

b) The case $x \in I \setminus C$. It means $x \in V_k$ and $x \in W_m$ for some k and m. Let us consider $S'_n(x)$. We would like to find out for which r the open ball $B((r, f_0(r)), \varepsilon_{r,n})$ can intersect v_x. It is clear that for sufficiently large n a neighborhood around a $(c_i, f_0(c_i))$ cannot do so as the intersection of these open balls are exactly the graph of $f_0|C$. Furthermore, if $n \geq m$, then the neighborhood chosen around $(r, f_0(r))$ can intersect v_x if and only if $r \in W_m$. Indeed, we have chosen these neighborhoods such that they do not intersect $W_1, W_2, ..., W_n$, unless those which are containing r. Thus if n is large enough, v_x can be intersected by a certain $B((r, f_0(r)), \varepsilon_{r,n})$ only if $r \in W_m$. Only these places are relevant if we want to find out what $S(x)$ is. But how did we define W_m? It is a subset of V_k thus the values of f_0 in W_m are between $k-1$ and k. It is important to us that f_0 is bounded here, and $f_0(x) = \max(T_k(x))$ for each element of W_m, where $T_k = (I \times [-k, k]) \cap T$, as in Lemma 2.2.3. Therefore, in the relevant places we defined f_0 as we would have done in Theorem 2.3.1, if we had regarded T_k instead of T. Consequently, in this case one can conclude the proof of $S(x) \subseteq gr(f_0)(x)$ as it was done there.

After these observations, the conclusion of the proof is clear. We use Lemma 2.2.3 as we did just before and alter the function on a countable set A, such that $L_f = T$ for the resulting function f. Then f is obviously a Baire-2 function. □

By proving this theorem we finished our characterization of accumulation points of Baire-2 functions. On the other hand, our proofs clarified that for any ordinal number α larger than 2 the Baire-α functions are not interesting concerning our question. Namely, the accumulation set of the graph of a Baire-α function is also the accumulation set of a Baire-2 function. This fact explains why we examine only the Baire-1 and Baire-2 functions.

2.4 Functions of Baire class 1

First, we focus again on the bounded case. Since Baire-1 functions are also Baire-2 functions, the conditions we found earlier recur in this case: T should be compact and $T(x)$ should be nonempty, if $x \in I$. Nevertheless, it is clear, that these conditions are not sufficient. Namely, if $L_f = T$ and for a given x the set $T(x)$ has multiple elements, then f is discontinuous at x. But a Baire-1 function cannot have an arbitrary set of discontinuities: it must be a meager F_σ set. Thus if $D = \{x : \#(T(x)) > 1\}$, then D should be a meager F_σ set. As we will see, these conditions suffice. However, before the statement of the actual theorem, let us notice that if we require T to be closed, then it is redundant to require D to be F_σ. Indeed, let $D_n = \{x : \operatorname{diam}(T(x)) \geq \frac{1}{n}\}$ for each $n \in \mathbb{N}$. Then it is easy to see that these sets are closed and their union is D. (Moreover, each D_n is nowhere dense, otherwise some of them would contain an interval, and D cannot do so.) Consequently, D is an F_σ set. Using this fact, our theorem is simply the following:

Theorem 2.4.1. *Suppose $T \subseteq I \times \mathbb{R}$. There is a bounded Baire-1 function $f : I \to \mathbb{R}$ such that $L_f = T$ if and only if*

- *T is compact,*
- *$T(x)$ is nonempty, if $x \in I$,*
- *the set $D = \{x : \#(T(x)) > 1\}$ is meager.*

Proof. Let us begin the proof by the construction of f. First, we use Lemma 2.2.3 to define f on a countable set A such that the accumulation set of the graph of $f|A$ coincides with T. We can suppose that A is disjoint from D. Indeed, in any neighborhood of any point $x \in I$ there are infinitely many points of $I \setminus D$, since D is meager. Thus we have defined f on A. On the other hand, on $I \setminus A$ let us define f as we did it in the bounded Baire-2 case: let $f(x) = \max(T(x))$. For this f, we have $L_f = T$, and obviously f is bounded.

We would like to apply Proposition 2.2.2 to f. We use the usual method: we define the open set S'_n for each n, which is the union of open balls around points of the graph with $\varepsilon_{x,n}$ radius, and then we extend these sets to open strips. The conditions concerning $\varepsilon_{x,n}$ will be case-specific, except for the usual size conditions.

(i) The case $x \in A = \{a_1, a_2, ...\}$. Then $x = a_k$ for some k. Our first condition on $\varepsilon_{x,n}$ is that $B(x, \varepsilon_{x,n})$ must not contain the points $a_1, a_2, ..., a_n$, except for a_k. The second condition is related to the overlapping of D. Since D is a meager F_σ set, we can choose $D_1, D_2, ...$ nowhere dense closed sets such that $D = \cup_{n=1}^{\infty} D_n$. Moreover, none of these sets contains x since $x \in A$ and the sets A and D are disjoint. Therefore, the condition "$B(x, \varepsilon_{x,n})$ and $\cup_{i=1}^{n} D_i$ are disjoint" can also be satisfied.

(ii) The case $x \in I \setminus A$. First, in order to stay away from the set A, the open ball $B(x, \varepsilon_{x,n})$ must not contain the points $a_1, a_2, ..., a_n$. The second condition is identical to the overlapping condition of the bounded Baire-2 case: if $r \in B(x, \varepsilon_{x,n}) \setminus A$, then $f(r) - f(x) < \frac{1}{n}$.

We have finished the construction of the open set S'_n, and now, we can extend it to obtain the open strip S_n by taking the infimum and the supremum along each v_x. Our goal is to prove that the intersection S of the sets S_n is $gr(f)$. Of course, the challenging part is the verification of $S \subseteq gr(f)$. Let us consider $S(x)$ for each x. We separate three cases by the location of x:

a) The case $x \in A$, that is $x = a_k$. If $n \geq k$, then amongst the neighborhoods forming S'_n there can be only one that intersects v_x, namely, the open ball centered at $(x, f(x))$. Therefore, $S_n(x) = S'_n(x)$, and

$$S_n(x) = (f(x) - \varepsilon_{x,n}, f(x) + \varepsilon_{x,n}) \subseteq \left(f(x) - \frac{1}{n}, f(x) + \frac{1}{n} \right).$$

This fact immediately implies that the only element of $S(x)$ is $f(x)$.

b) The case $x \in D$. It means that $x \in D_k$ for some k. Thus if $n \geq k$, the neighborhoods $B((a_k, f(a_k)), \varepsilon_{a_k,n})$ cannot intersect v_x. Therefore, if n is sufficiently large, if we want to describe $S_n(x)$, we have to deal only with the points in $I \setminus A$. But above $I \setminus A$ we defined f and the neighborhoods forming S'_n as we defined f_0 and S'_n in the proof of Theorem 2.3.1. Consequently, the proof given there for $S(x) = gr(f_0)(x)$ for any $x \in I$ works.

c) The case $x \in I \setminus (A \cup D)$. Proceeding towards a contradiction, we assume that $S(x)$ has an element y distinct from $f(x)$. Then $S'_n(x)$ has a point z_n for each n such that $|f(x) - z_n| \geq |f(x) - y|$. By definition, the set $gr(f)$ is bounded, thus it is obvious that there exists some $K \in \mathbb{R}$ such that for any n and x, the $S'_n(x)$ has no element larger than K. It implies that the sequence (z_n) is bounded. Therefore, it has a convergent subsequence whose limit is some $z \in \mathbb{R}$. For this limit z the inequality $|f(x) - z| \geq |f(x) - y|$ also holds, thus $f(x) \neq z$. Since there is a point of $gr(f)$ whose distance from (x, z_n) does not exceed $\frac{1}{n}$, the point (x, z) is also an accumulation point of $gr(f)$, thus $(x, z) \in L_f$, a contradiction. Namely, for our f the equation $L_f = T$ holds, however, the only element of $T(x)$ is $f(x) \neq z \in L_f(x)$.

Therefore $S = gr(f)$, thus we can apply Proposition 2.2.2. Hence f is a bounded Baire-1 function, such that $L_f = T$. □

As we have characterized the bounded Baire-1 functions, now we might focus on the most challenging problem appearing in this part: the characterization of the not necessarily bounded Baire-1 functions. However, as we will see, during the proof we will apply the same ideas. Following the usual scheme, we begin by thinking about necessary conditions concerning T.

The conditions we found during the examination of the general Baire-2 case obviously recur: T is a closed set and $T(x) = \emptyset$ can hold only on a countable subset of I. As T is closed, this subset is G_δ. Of course we need more than these simple conditions. We have to pay attention to the fact that a Baire-1 function cannot have an arbitrary set of discontinuities: it must be a meager F_σ set, and at points of continuity, $\#(L_f(x)) = 1$, thus $\#(T(x)) = 1$. However, we must be careful. In the bounded case, the property $\#(L_f(x)) = 1$ already guaranteed that f is continuous at x, or f has a removable discontinuity at x. But in this case, it is not true at all: for instance, if $f(x) = \frac{1}{2x-1}$ for $x > \frac{1}{2}$, and $f(x) = 0$ for $x \leq \frac{1}{2}$, then although $L_f\left(\frac{1}{2}\right) = 0$, it does not imply that f is continuous at $\frac{1}{2}$ or it has a removable discontinuity there. Therefore, we must pay attention to the infinite limits. If we embed T into $I \times \overline{\mathbb{R}}$ and take its closure $\overline{T}$, then we have to demand that this $\overline{T}$ can intersect the extended vertical lines in multiple points only above a meager F_σ set. However, the additional F_σ condition is unnecessary since we supposed that T is closed. Indeed, if $D_n = \{x : \operatorname{diam}(\overline{T}(x)) \geq \frac{1}{n}\}$, then these sets are nowhere dense closed sets and their union is D, hence D is F_σ.

If we collect all of these remarks, we gain a more complicated system of conditions than the ones in the previous cases. We show that it is sufficient.

Theorem 2.4.2. *Suppose $T \subseteq I \times \mathbb{R}$. There is a Baire-1 function $f : I \to \mathbb{R}$ such that $L_f = T$ if and only if*

- *T is closed,*
- *there is a countable $C \subseteq I$, such that $T(x)$ is nonempty for $x \in I \setminus C$,*
- *the set $D = \{x : \#(\overline{T}(x)) > 1\}$ is meager.*

Proof. We define f on a countable set A, such that the accumulation set of the graph of f restricted to A equals T. We do so using the method given in Lemma 2.2.3. It is easy to see that we can construct such a set A disjoint from C and D.

Now let us focus on $I \setminus A$. We define f on this set as we defined f_0 in the proof of Theorem 2.3.3. First, if $C = \{c_1, c_2, ...\}$, then $f(c_n) = n$ for each $n \in \mathbb{N}$. Besides that we also define U_n as we did it in (**??**). These are closed sets in this case, too, though not necessarily disjoint from A. At places which are not in A let us define f as we defined f_0 after (**??**): if $x \in U_n$, let $f_0(x)$ be the largest element of $T(x)$ which has absolute value not exceeding n. Now we are ready with the construction of f and $L_f = T$ clearly holds: if we consider only the points of the graph above A, it is true by definition, furthermore, sequences containing infinitely many points of the graph above C cannot converge, and points of the graph above $I \setminus (A \cup C)$ are in T. Thus every accumulation point of $gr(f)$ is also the accumulation point of the graph of $f|A$, and the set of these accumulation points is T. (We note that C might intersect D, a concern that we will address later.)

We would like to apply Proposition 2.2.2 to f by giving the open sets S'_n formed by neighborhoods of points of $gr(f)$ and extending them to open strips. Again, we separate some cases. We also use our familiar notation: $A = \{a_1, a_2, ...\}$, $C = \{c_1, c_2, ...\}$, and $D = \cup_{n=1}^{\infty} D_n$, where D_n is a nowhere dense, closed set for each n.

(i) The case $x \in C$, $x = c_k$. Here, we define our neighborhoods with $\varepsilon_{x,n}$ radius quite comfortably, namely, we can define the sets E_n and F_n as we did it in (i) of the proof of Theorem 2.3.3 and repeat the conditions used there. Hence we can choose these open balls such that $\cap_{n=1}^{\infty} F_n = C$, and $B(c_k, \varepsilon_{c_k,n})$ does not contain the points $c_1, ..., c_n$, with the exception of c_k. We also require that this neighborhood is disjoint from $\{a_1, a_2, ..., a_n\}$. We remark that these conditions imply $\cap_{n=1}^{\infty} E_n$ equals the graph of $f_0|C$.

(ii) The case $x \in A$. We evoke the conditions of (i) of the proof of Theorem 2.4.1. Namely, $B(x, \varepsilon_{x,n})$ does not intersect the closed sets $D_1, D_2, ..., D_n$, and it does not contain $a_1, a_2, ..., a_n$, with the exception of x. Furthermore we give the following additional condition: these neighborhoods have to stay away from C, thus they must not contain $c_1, c_2, ..., c_n$.

(iii) The case $x \in I \setminus (A \cup C)$. We evoke the condition system of (ii) of the proof of Theorem 2.3.3. We define the sets V_n and W_n as we did there: $V_1 = U_1$, and $V_n = U_n \setminus U_{n-1}$ for $n \geq 2$. Then any set V_n is F_σ. Let $W_1, W_2, ...$ be an enumeration of the closed sets forming them. Now if $x \in V_k$, we require $B(x, \varepsilon_{x,n})$ to be disjoint from $c_1, c_2, ..., c_n$, and also disjoint from the sets $W_1, W_2, ..., W_n$, except for those containing x. Furthermore, of course, we give an overlapping condition: $f_0(r) - f_0(x) < \frac{1}{n}$ for each $r \in B(x, \varepsilon_{x,n}) \cap V_k$. These are exactly the conditions we used in (ii) of the proof of Theorem 2.3.3. The only additional condition is the following: $B(x, \varepsilon_{x,n})$ must not contain the points $a_1, a_2, ..., a_n$.

Thus we have constructed the open set S'_n for each n. We extend it in the usual way to form the open strip S_n. Our goal is to verify that their intersection S equals $gr(f)$. The challenging part is to show that S contains no points distinct from $gr(f)$. Let us consider $S(x)$ and $S'(x)$ for each x. We separate four cases by the location of x:

a) The case $x \in C$, $x = c_k$. This is obvious: if $n \geq k$, the only chosen neighborhood that intersects v_x amongst the ones forming $S'_n(x)$ is the neighborhood of $(x, f(x))$, and thus $S'_n(x) = S_n(x)$. Therefore, $S_n(x)$ is an interval whose diameter does not exceed $\frac{2}{n}$ and contains $f(x)$. Thus the only element of $S(x)$ is $f(x)$, as we wanted to show.

b) The case $x \in A$, $x = a_k$. We can simply repeat our previous argument: for sufficiently large n, there is only one chosen neighborhood that intersects v_x, and since the diameters of these neighborhoods converge to 0, the only element of $S(x)$ is $f(x)$.

c) The case $x \in D \setminus C$. It means $x \in D_k$ for some $k \in \mathbb{N}$. Now, if $n \geq k$, the neighborhood $B((x', f(x')), \varepsilon_{x',n})$ for $x' \in A$ cannot intersect v_x. It is also true that for sufficiently large n, the neighborhood $B((x', f(x')), \varepsilon_{x',n})$ for $x' \in C$ cannot intersect v_x, since these neighborhoods are nested and their intersection is the graph of $f|C$. Hence it is enough to consider the graph of f above $I \setminus (A \cup C)$. At these places we defined f and the open balls forming S'_n as we defined f_0 and the open balls forming S'_n during the proof of Theorem 2.3.3. Consequently, case b) of the proof of Theorem 2.3.3 can be used to prove $S(x) = gr(f)(x)$.

d) The case $x \in I \setminus (A \cup C \cup D)$. Proceeding towards a contradiction, let us suppose that $S(x)$ contains some $y \in \mathbb{R}$, where $y \neq f(x)$. It means that for every n we can choose a point z_n in $S'_n(x)$, such that $|f(x) - z_n| \geq |f(x) - y|$. Since

$z_n \in S'_n(x)$, the point (x, z_n) is in one of the open balls forming S'_n. Here, if n is sufficiently large, then this ball is centered at a point of the graph above $I \setminus C$. Indeed, if n is large enough, the neighborhoods around points of the graph above C cannot intersect v_x by definition. Now, the sequence (z_n) has a limit point z in $\overline{\mathbb{R}}$. Obviously, for this z the inequality $|f(x) - z| \geq |f(x) - y|$ also holds, thus $f(x) \neq z$. However, if n is sufficiently large, there is a point of the graph not above C whose distance from (x, z_n) does not exceed $\frac{1}{n}$. Consequently, there is a sequence (p_n) of points of the graph above $I \setminus C$ such that (p_n) converges to (x, z). Without loss of generality, we might assume that the elements of this sequence are all distinct. Since these points are not above C, they are above A or they are also elements of T. Nevertheless, if n is sufficiently large, for any given $\varepsilon > 0$, a point p_n that is above A cannot be farther than ε from a point of T, as we noted in Remark 2.2.1. This fact immediately implies $(x, z) \in \overline{T}$, a contradiction, since the only element of $\overline{T}(x)$ is $f(x)$ by our assumptions.

Hence $S = gr(f)$, therefore we might apply Proposition 2.2.2. Thus f is a Baire-1 function satisfying $L_f = T$. □

2.5 A remark about $\overline{T}$

Before the end of the second section, we would like to point out something in connection with our theorems about the not necessarily bounded functions. Namely, amongst the conditions of the last theorem there was one condition about $\overline{T}$. However, $\overline{T} = \overline{L_f}$ does not necessarily hold for the function we constructed.

For instance let T be the following closed set: let $C = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, $c_1 = 0$, and for $n \geq 2$, let $c_n = \frac{1}{n-1}$. For each point x in $I \setminus C$ let $T(x) = \{-\frac{1}{d(x,C)}\}$, where $d(x, C)$ is the distance of x from C. Then it is easy to see that this set T satisfies the conditions of Theorem 2.4.2 with regards to the not necessarily bounded Baire-1 functions. It is also true, that $\overline{T}(0) = \{-\infty\}$. Now, let us consider f, specifically $\overline{L_f}(0)$. We recall that in our construction $f(c_n) = n$. It implies $\overline{L_f}(0) = \{-\infty, +\infty\}$. It means that although $L_f(0) = T(0) = \emptyset$, $\overline{T}(0) \neq \overline{L_f}(0)$. Thus the sets we examined earlier are equal, but these extended sets are not.

This example raises two new questions: if we regard our theorems about the not necessarily bounded Baire-1 and Baire-2 functions and we do not change the conditions, is it possible to construct a function f in each of these cases that satisfies $L_f = T$ and $\overline{L_f} = \overline{T}$ simultaneously? However, we might answer these questions easily:

Proposition 2.5.1. *Suppose $T \subseteq I \times \mathbb{R}$.*

- *If there exists a Baire-2 function satisfying $L_f = T$, then it can be chosen such that $\overline{L_f} = \overline{T}$ also holds.*

- *If there exists a Baire-1 function satisfying $L_f = T$, then it can be chosen such that $\overline{L_f} = \overline{T}$ also holds.*

Proof. We will appropriately modify the functions we have constructed in the proofs of Theorem 2.3.3 and Theorem 2.4.2. It is clear that for those functions $\overline{T} \subseteq \overline{L_f}$ holds. Indeed, for any point $t \in T$ there are points of $gr(f)$ arbitrarily close to t. Thus if we consider a point (x, ∞) of $\overline{T}$, then it is also an accumulation point of $gr(f)$. Hence if $\overline{L_f} \neq \overline{T}$, then $\overline{T}$ is a proper subset of $\overline{L_f}$.

For those functions it is also clear that if $\overline{L_f}$ has a point p which is not in $\overline{T}$, then it is an accumulation point of the graph of $f|C$. Namely, if we take a sequence (p_n) in $gr(f)$ which converges in $I \times \overline{\mathbb{R}}$ and contains only finitely many points of $gr(f)$ above C, then after a while every term of this sequence is above A or in T. The terms above A will get arbitrarily close to a point of T if n is sufficiently large. Thus if we have a point in $\overline{L_f}$ which is a limit point of such a sequence, then it is also a point of $\overline{T}$. Hence if $\overline{L_f}$ has a point outside $\overline{T}$, then there exists a sequence in the graph of $f|C$ converging to this point.

It is a problem we can easily handle in both cases by modifying f on C: if $C = \{c_1, c_2, ...\}$, then let $|f(c_n)| = n$. The sign is determined by whether $\overline{T}$ contains $(c_n, +\infty)$ or $(c_n, -\infty)$. If both of them occurs, then let $f(c_n) = n$. If we define the function f on C this way, then L_f clearly does not change, the equality $L_f = T$ still holds. Indeed, if a sequence of points of $gr(f)$ above C converges to a point in $I \times \overline{\mathbb{R}}$, then the second coordinate of this point is $+\infty$ or $-\infty$. By symmetry, we can consider the $+\infty$ case. For a subsequence (c_{n_k}) the sequence $(c_{n_k}, f(c_{n_k}))$ converges to some $(x, +\infty) \in I \times \mathbb{R}$. We can suppose that all the numbers $f(c_{n_k})$ are positive. Then by definition, in the $\frac{1}{n_k}$ neighborhood of c_{n_k} we might choose a point a_k such that $T(a_k)$ has an element larger than n_k. We denote this element of T by t_k. Now it is clear that the sequence (t_k) is in T and it also converges to $(x, +\infty)$. Hence all the elements of $\overline{L_f}$ are in $\overline{T}$, too. Thus we constructed a function of the corresponding Baire class satisfying $L_f = T$ and $\overline{L_f} = \overline{T}$ simultaneously. $\square$

Chapter 3

Characterizations of Baire Functions

3.1 Motivation and a classical result

First, we would like to recall the complete form of the already cited theorem of [1]. One can easily see that the open strips of $X \times \mathbb{R}$ can be defined the same way as we did it previously for an arbitrary topological space X. Using this extension, the authors of [1] proved the following generalized version of Proposition 2.2.2 in fact:

Proposition 3.1.1. *Let X be a metric space and let $f : X \to \mathbb{R}$ be a function. It is Baire-1 if and only if there is a sequence $(G_n)_{n=1}^{\infty}$ of open strips such that $\cap_{n=1}^{\infty} G_n = gr(f)$.*

This is the theorem we would like to generalize in the two aforementioned senses.

As it is well-known, the Baire functions and Borel sets have a strong relationship. In this work, we use the following classification of the Borel sets of a topological space X:

Definition 3.1.2. *A set A is of additive class 1, ($A \in \Sigma_1$), if and only if it is open. For any countable ordinal greater than zero, A is of multiplicative class α, ($A \in \Pi_\alpha$), if and only if its complement is in Σ_α. Finally, A is of additive class α, ($A \in \Sigma_\alpha$), for $\alpha > 1$ if and only if there is a sequence of sets $A_1, A_2, \ldots$ such that each A_i is in Π_{α_i} for some $\alpha_i < \alpha$ and $\bigcup_{i=1}^{\infty} A_i = A$.*

It is useful to remark that the behaviour of the Borel hierarchy can be a bit chaotic in general topological spaces. To be more precise, we prefer if the higher Borel classes contain the lower ones, that is for $0 < \beta < \alpha < \omega_1$, every set in Π_β or Σ_β is also in Π_α and Σ_α. However, this property does not hold necessarily: for example if we regard the cofinite topology over any uncountable set, we can immediately see that none of the nontrivial open sets is in Σ_2. The following result is well-known and can be easily obtained by transfinite induction: if X has the property that any open set is in Σ_2 (or equivalently, any closed set is in Π_2) then every set in Π_β or Σ_β is also in Π_α and Σ_α for any $0 < \beta < \alpha < \omega_1$. The spaces satisfying this requirement are called G_δ or perfect spaces and their defining property can be regarded as a separation axiom: the closed sets can be separated from their complements using only countably many open sets. It can be easily checked that all the metrizable spaces are perfect spaces, which is a fact we will use in this work.

Another important remark is that in the later referred [3], [4], and [5], and in several further articles and books other types of notation are used for Borel classes, causing a subtle ambiguity with our recent work. In particular, in many papers the elements of Σ_0 are the open sets instead of the elements of Σ_1, and the higher Borel classes are defined from this starting point the same way we did in Definition 3.1.2. It is worth mentioning that this translation of the indices only leads to a difference in the case of finite ordinals as in the definition of Σ_ω we consider the same unions.

Let us return to the aforementioned relationship of Borel sets and Baire classes. The following fact is well-known (see [4]), however, as it is short and useful to prove, we will not omit the proof and formulate it as a proposition:

Proposition 3.1.3. *Let $f : X \to Y$ be a Baire-α function where X is a topological space, Y is a metric space, and α is a countable ordinal. Then for any open set $G \subseteq Y$ the set $f^{-1}(G) \subseteq X$ is a $\Sigma_{\alpha+1}$ set, or in other words, f is a Borel-$(\alpha + 1)$ mapping.*

Proof. We proceed by transfinite induction. For $\alpha = 0$ the proposition states that for continuous functions the inverse image of an open set is open which is true by definition. What remains to discuss is the inductive step. Let us assume $\alpha \geq 1$ and we already know the statement for smaller ordinals, and let $(f_k)_{k=1}^\infty$ be a sequence of functions from lower Baire classes whose pointwise limit is f, namely let f_k be Baire-α_k where $\alpha_k < \alpha$. If α is a successor ordinal, we might assume $\alpha_k = \alpha - 1$. Let us denote the neighborhood of radius $\varepsilon > 0$ of a closed set F by $B(F, \varepsilon)$, which is clearly an open set. Then we may construct the following decomposition of G into closed sets $(F_n)_{n=1}^\infty$:

$$G = \bigcup_{n=1}^{\infty} Y \setminus B\left(Y \setminus G, \frac{1}{n}\right) = \bigcup_{n=1}^{\infty} F_n.$$

One can easily check that our decomposition implies that $f(x) = \lim_{k\to\infty} f_k(x) \in G$ holds if and only if there is an n such that $f_k(x) \in F_n$ for all large enough k. Indeed, as F_n is closed, if there is such an n, then the sequence $(f_k(x))$ cannot converge out of $F_n \subseteq G$ hence $f(x) \in G$. Conversely, if $f(x) \in G$, it has a neighborhood of radius ε for suitable positive ε in G. By convergence, for large enough k the point $f_k(x)$ is in the neighborhood of $f(x)$ of radius $\frac{\varepsilon}{2}$, thus $f_k(x) \in B\left(Y \setminus G, \frac{\varepsilon}{2}\right)$ for large enough

k. Choosing n such that $\frac{1}{n} < \frac{\varepsilon}{2}$ gives us a suitable n in our statement, thus it proves the other direction of our equivalence.

This equivalence yields the following equation:

$$f^{-1}(G) = \{x : f(x) \in G\} = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \{x : f_k(x) \in F_n\}.$$

Now, for any set $\{x : f_k(x) \in F_n\}$ the inductive hypothesis can be used: $f_k(x)$ is Baire-α_k thus the inverse image of any open set is Σ_{α_k+1}, hence the inverse image of the closed set F_n is in Π_{α_k+1}. Indeed, the inverse image of the complement is the complement of the inverse image, and the complement of F_n is open, while the complement of its inverse image is in Σ_{α_k+1} whose complement is in Π_{α_k+1}. Now if α is a successor ordinal, these sets in Π_{α_k+1} are in Π_α as $\alpha_k + 1 = \alpha$. Otherwise, if α is a limit ordinal the sets in Π_{α_k+1} are in Π_α by definition: the same unions can be regarded. Hence if we take the intersection of sets of these type, for all $k \geq m$, we will still have a Π_α set for any ordinal. Finally if we take the countable union of such sets (that is, for all n and m) we will obtain a $\Sigma_{\alpha+1}$ set as the inverse image of the open set G. $\square$

We will use this theorem several times in the followings.

3.2 The characterization of Baire-1 functions

In order to generalize the result of [1], which we referred to as Proposition 3.1.1, let us revisit the proof given there. In one of the directions of the equivalency, we have to construct open strips to a given function. The key of its proof is Proposition 3.1.3, more precisely its special case about $X \to \mathbb{R}$ Baire-1 functions. As we have seen, it holds in much more general settings. What about the other direction? The concept is that through each open strip, one can "lead" a continuous function, thus their pointwise limit is hopefully the graph lying in the intersection. Hence these continuous functions should be constructed. For the $X = [0, 1]$ case, the authors of [1] used a somewhat elementary method. However, for the case when X was an arbitrary metric space, they used Michael's Selection Theorem (see [8], stated as Proposition 3.2.3 here). In order to draw up this classical result of Michael, we need two definitions:

Definition 3.2.1. *A topological space X is paracompact if it is Hausdorff and each of its open coverings admits a locally finite refinement.*

Definition 3.2.2. *A multifunction $F : X \to Y$ is lower hemicontinuous, if the inverse image of any open set $G \subseteq Y$ under F, that is the set of points $x \in X$ for which $F(x)$ has a nontrivial intersection with G, is also open.*

Michael's selection theorem states the following:

Proposition 3.2.3. *Let X be a paracompact topological space, Y a Banach space, and $F : X \to Y$ a multivalued, lower hemicontinuous multifunction wih nonempty, closed, convex values. Then there exists a continuous selection $f : X \to Y$ of F.*

This fact might lead us to the idea to regard the Baire-1 functions defined on a paracompact topological space X with values from a Banach space Y instead of $\mathbb{R}$ as this selection theorem holds in this more general situation. Furthermore, if Y is a Banach space, we can easily find a natural counterpart of the notion of open strips in $X \times Y$:

Definition 3.2.4. *We say that an open set $G \subseteq X \times Y$ is an open strip if the vertical section $G(x) = G \cap (\{x\} \times Y)$ is convex for all $x \in X$.*

Thus we can formulate the following generalization of Proposition 3.1.1:

Theorem 3.2.5. *Let $f : X \to Y$ be a function where X is a paracompact topological space and Y is a Banach space. Then f is Baire-1 if and only if there is a sequence $(G_n)_{n=1}^{\infty}$ of open strips such that $\cap_{n=1}^{\infty} G_n = gr(f)$ and $diam(G_n(x)) \to 0$ for each $x \in X$ as n tends to infinity.*

We might realize that we have a previously unseen condition concerning the diameter of the vertical sections. Its importance is to be discussed later, by the time we have seen in the proof where we used it.

Proof of Theorem 3.2.5. The proof is similar to the one given in [1] for the more specific case, with some suitable modifications. First, let us assume that f is Baire-1, hence there is a sequence of continuous functions $(f_n)_{n=1}^{\infty}$ with pointwise limit f. Let us notice that the set $\left\{x : \|f_n(x) - f(x)\| < \frac{1}{k}\right\}$ is in Σ_2. Indeed, if we let $g_n(x) = f_n(x) - f(x)$, it is also a Baire-1 function, and the set we are interested in is $g_n^{-1}\{y : \|y\| < \frac{1}{k}\}$, which is the inverse image of an open ball. Applying Proposition 3.1.3 yields that $\left\{x : \|f_n(x) - f(x)\| < \frac{1}{k}\right\}$ is in Σ_2 as we stated. As a consequence, it can be written as the countable union of closed sets $A(n,k,i) \subseteq X$:

$$\left\{x : \|f_n(x) - f(x)\| < \frac{1}{k}\right\} = \bigcup_{i=1}^{\infty} A(n,k,i).$$

We will define the subsets $H(n,k,i)$ of $X \times Y$ as follows:

$$H(n,k,i) = \left\{(x,y) : x \in A(n,k,i), \|y - f_n(x)\| \geq \frac{1}{k}\right\}.$$

We show that $H(n,k,i)$ is closed. In order to prove it, let us write it as an intersection of two sets which are easier to handle:

$$H(n,k,i) = [A(n,k,i) \times Y] \cap \left\{ (x,y) : \|y - f_n(x)\| \geq \frac{1}{k} \right\}.$$

The first one of these sets on the right hand side is clearly closed in $X \times Y$ as $A(n,k,i)$ was closed in X, hence it suffices to prove that the second set on the right hand side is also closed. Let us define the following function $h_n : X \times Y \to \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the nonnegative halfline:

$$h_n(x,y) = \|y - f_n(x)\|.$$

Our claim is that the continuity of f_n implies the continuity of h_n. To prove this, we need to show that the inverse image of an open set $G \subseteq \mathbb{R}_+$ under h_n is open in $X \times Y$. Thus let us assume $h_n(x_0, y_0) \in G$ for some $(x_0, y_0) \in X \times Y$, which yields for some $\varepsilon > 0$ its neighborhood of radius ε is the subset of G, that is $B\left(h_n(x_0,y_0), \varepsilon\right) \subseteq G$. We need that $h_n(x,y)$ is also in G if (x,y) is an element of a suitable neighborhood U of (x_0, y_0). We state this holds if we regard the following neighborhood:

$$U = f_n^{-1}\left(B\left(f_n(x_0), \frac{\varepsilon}{2}\right)\right) \times B\left(y_0, \frac{\varepsilon}{2}\right)$$

By the continuity of f_n it is indeed a neighborhood of (x_0, y_0) as $f_n^{-1}\left(B\left(f_n(x_0), \frac{\varepsilon}{2}\right)\right)$ is an open subset of X. Furthermore, if $(x,y) \in U$, by the triangle inequality we have

$$\|y - f_n(x)\| \leq \|y - y_0\| + \|y_0 - f_n(x_0)\| + \|f_n(x_0) - f_n(x)\| < \varepsilon + \|y_0 - f_n(x_0)\|,$$

and

$$\|y - f_n(x)\| \geq -\|y - y_0\| + \|y_0 - f_n(x_0)\| - \|f_n(x_0) - f_n(x)\| > -\varepsilon + \|y_0 - f_n(x_0)\|,$$

which implies $h_n(x,y) \in B\left(h_n(x_0,y_0), \varepsilon\right) \subseteq G$.
Thus h_n is continuous indeed, yielding $\left\{(x,y) : \|y - f_n(x)\| \geq \frac{1}{k}\right\} = h_n^{-1}\left(\left[\frac{1}{k}, \infty\right)\right)$ is a closed set of $X \times Y$. By our previous remarks it implies that $H(n,k,i)$ is also closed.

The set of such sets $H(n,k,i)$ is countable thus we can take an enumeration $H_1, H_2, \ldots$ of them. Let us denote by G_j^* the complement of H_j in $X \times Y$, that is an open set. Furthermore, one can easily check that G_j^* is an open strip, that is the $G_j^*(x)$ vertical section is convex for each $j \in \mathbb{N}$ and $x \in X$. Indeed, by the construction of G_j^*, this vertical section is either the complete space Y or the ball of radius $\frac{1}{k}$ centered at $f_n(x)$ for some $k \in \mathbb{N}$ and $x \in X$. However, balls are convex in Banach spaces, hence G_j^* is an open strip. It implies $G_j = \bigcap_{l=1}^{j} G_l^*$ is also an open strip. Furthermore, the sequence $(G_j)_{j=1}^{\infty}$ is nested and $diam(G_j(x))$ tends to 0 for each $x \in X$. Indeed, when we constructed G_j, we took the intersection of the complements of some sets $H(n,k,i)$. A vertical section of this complement is either the entire Y or a ball with diameter $\frac{2}{k}$. But as all $x \in X$ appears in $A(n,k,i)$ for any k, for some i and large enough n, this implies that $diam(G_j(x)) \leq \frac{2}{k}$ for large enough j. As a consequence, $diam(G_j(x)) \to 0$. Hence if we could verify that the intersection of the open strips $(G_j^*)_{j=1}^{\infty}$ equals $gr(f)$, that would conclude the proof. But the proof of this fact is quite straightforward, we can check two inclusions. First, $(x, f(x)) \in G_j^*$ for any $j \in \mathbb{N}$ and $x \in X$, implying $gr(f) \subseteq \bigcap_{j=1}^{\infty} G_j$. In order to show this, let us recall that the complement of G_j^* is $H_j = H(n,k,i)$ for some $n,k,i \in \mathbb{N}$. We need $(x, f(x)) \notin H(n,k,i)$. Proceeding towards a contradiction, let us assume $(x, f(x)) \in H(n,k,i)$, yielding $x \in A(n,k,i)$. Then by the definition of $A(n,k,i)$, the inequality $\|f_n(x) - f(x)\| < \frac{1}{k}$ holds. However, $\|f(x) - f_n(x)\| \geq \frac{1}{k}$ by the definition of $H(n,k,i)$, a contradiction. Thus $gr(f) \subseteq \bigcap_{j=1}^{\infty} G_j$. For the other inclusion, it suffices to prove that for any $x \in X$ and $y \in Y$ distinct from $f(x)$, we have $(x,y) \in H(n,k,i)$ for suitable $n,k,i \in \mathbb{N}$. In order to verify this, choose k such that $\frac{1}{k} < \frac{\|y - f(x)\|}{2}$ and n such that $\|f_n(x) - f(x)\| < \frac{1}{k}$. As f is the pointwise limit of $(f_n)_{n=1}^{\infty}$, it is possible. Then by definition there exists i such that $x \in A(n,k,i)$. Furthermore, $\|y - f_n(x)\| \geq \|y - f(x)\| - \|f_n(x) - f(x)\| > \frac{1}{k}$ by the triangle-inequality, implying $(x,y) \in H(n,k,i)$, which concludes the proof. Thus $\bigcap_{j=1}^{\infty}(G_j) = gr(f)$, we finished the proof of this direction.

For the other direction, let us assume $gr(f) = \bigcap_{j=1}^{\infty} G_j$ where for each j the set G_j is an open strip. We can also assume that their sequence is nested as the finite intersection of open sets is open and any intersection of convex sets is convex. Thus $G_{j+1} \subseteq G_j$ for any j. Let us define F_j as it follows:

$$F_j = \bigcup_{x \in X} \overline{G_j(x)},$$

where the overline means the closure. Hence F_j stands for the closure by coordinates. Regard it as a multivalued function defined on X with range 2^Y, whose values are naturally the vertical sections of the set. Then this multivalued function has nonempty closed, convex values. Furthermore, we can easily show that F_j is lower hemicontinuous: let us assume $V \cap F_j(x)$ is nonempty for some open set V of Y and $x \in X$. Since $F_j(x)$ is the closure of the open set $G_j(x)$, we have $V \cap G_j(x)$ is nonempty. Let $y \in Y$ be one of its elements. As G_j is open, it contains a neighborhood of (x,y). This neighborhood intersects $X \times \{y\}$ in a set whose projection to X is open and suitable for us in the definition of lower hemicontinuity as one can easily check. Thus $F_j : X \to 2^Y$ is a lower hemicontinuous function with nonempty closed, convex values. By the Michael selection theorem there exists a continuous selection $f_j : X \to Y$ in F_j. Furthermore, as the intersection of the sets $F_j(x)$ is only $f(x)$ and their diameter tends to 0, we obtain $f_j(x) \to f(x)$. Hence f is the pointwise limit of continuous functions, meaning f is Baire-1. $\square$

Let us return to the condition concerning the diameters of the vertical sections. This condition was used in the proof of the second direction, where we desired to verify the pointwise convergence of the sequence $(f_n)_{n\in\mathbb{N}}$. What we would like to emphasize that this diameter condition is vital and can be found implicitly in the more specific form of the theorem, too. We formulate the relevant fact as a proposition, since the author of this thesis firmly believes this result has been published already but has yet to see a source:

Proposition 3.2.6. *Let E be a finite dimensional Banach space and $(C_n)_{n=1}^\infty$ is a nested sequence of closed convex sets such that $\bigcap_{n=1}^\infty C_n$ equals a point p. Then $diam(C_n) \to 0$.*

Proof. Proceeding towards a contradiction, let us assume $diam(C_n) > d > 0$ for all $n \in \mathbb{N}$. Then C_n must contain a point p_n such that $\|p - p_n\| > \frac{d}{2}$ as it easily follows from the triangle inequality. As C_n is convex and it contains p and p_n, it also contains the $[p, p_n]$ segment, and on this segment a point x_n satisfying $\|p - x_n\| = \frac{d}{2}$. Now the points x_n all lie on the boundary of the ball with centre p and radius $\frac{d}{2}$. By a consequence of Riesz's lemma, in our finite dimensional space this set is compact, hence the sequence $(x_n)_{n=1}^\infty$ has an accumulation point on this boundary. Let us denote it by x. As C_n is closed for each $n \in \mathbb{N}$ and their sequence is nested, it implies $x \in C_n$. As a consequence, $x \in \bigcap_{n=1}^\infty C_n = \{p\}$, which is clearly a contradiction as the distance of x and p is $\frac{d}{2}$. $\square$

This proposition implies that if we work with a finite dimensional Banach space, it is unnecessary to have the additional limit condition concerning the diameters of the vertical sections. However, one can easily construct counterexamples to Proposition 3.2.6 if we permit infinite dimensional spaces. For instance, let us consider ℓ^1 which is a Banach space, and let C_n be the subspace of those sequences whose first n coordinates equal zero. Then these sets are closed, convex, and their intersection contains only the zero vector, meanwhile the diameters are not even finite.

By this, we have given the characterization of Baire-1 functions in more general settings, thus we have achieved one of our goals.

3.3 Higher Baire classes

We wish to give a similar characterization for higher Baire classes. We hope that by suitably generalizing the concept of open strips to higher Borel classes we can obtain an analogous result.

Let us introduce Fréchet spaces:

Definition 3.3.1. *A topological vector space Y is a Fréchet space if it is locally convex and completely metrizable with a translation invariant metric.*

Briefly, it is a generalization of Banach spaces: the metric does not have to be induced by a norm. However, convexity makes sense in Fréchet spaces, too. Thus the open strips of $X \times Y$, where Y is Fréchet space, can be defined word by word the same way as they have been when Y was a Banach space. The concept of open strips can be extended in a straightforward way:

Definition 3.3.2. *Let X be a topological space and Y be a Fréchet space. We say that a Σ_α set $S \subseteq X \times Y$ is a Σ_α-strip if the vertical section $S(x) = S \cap (\{x\} \times Y)$ is convex for all $x \in X$.*

Concerning the characterization of higher Baire classes, we might have the conjecture that a function $f : X \to Y$ is Baire-α for a countable ordinal α if and only if its graph is the intersection of a sequence of Σ_α-strips. However, we still need to find out what further conditions we need about X and Y. In order to have an idea, let us focus on how we could start the proof. We have to regard two directions: on one hand, we have to construct strips to a given function, on the other hand, we would like to "thread" functions from lower Baire classes through each strip. Of these directions, the first one was based on a much more general fact. This motivates the conception that first, we should consider this direction independently: maybe we need fewer restrictions concerning X and Y. Our next theorem reflects on this question:

Theorem 3.3.3. *Let $f : X \to Y$ be a Baire-α function where X is a topological space, Y is a metric space and let α be a countable ordinal. Then there exists a sequence $(G_n)_{n=1}^{\infty}$ of Σ_α sets in $X \times Y$ such that $\cap_{n=1}^{\infty} G_n = gr(f)$ and $diam(G_n(x)) \to 0$ for each $x \in X$ as n tends to infinity. Furthermore, if Y is a Fréchet space, these Σ_α sets can be chosen to be Σ_α strips.*

Proof. The proof has a similar structure to the proof of Theorem 3.2.5, we just have to be more careful with the sets in higher Borel classes and make some slight, but necessary changes. As f is Baire-α, there is a sequence of functions $(f_n)_{n=1}^{\infty}$ with pointwise limit f, where f_n is Baire-α_n for some $\alpha_n < \alpha$, and if α is a successor ordinal, we can assume $\alpha_n = \alpha - 1$. Proposition 3.1.1 easily yields that the set $\left\{x : d_Y\left(f_n(x), f(x)\right) < \frac{1}{k}\right\}$ is in $\Sigma_{\alpha+1}$. To verify this, we show that if the functions $g_1, g_2 : X \to Y$ are Baire-α, then the function $\rho_{g_1,g_2} : X \to \mathbb{R}_+$ defined by $\rho_{g_1,g_2}(x) = d_Y\left(g_1(x), g_2(x)\right)$ is also Baire-α. We proceed by transfinite induction: if $\alpha = 0$, that is our functions are continuous, then our claim can be proven as the similar statement in the proof of Theorem 3.2.5. Furthermore, if we have $\alpha > 0$, then g_1

is the pointwise limit of the functions $(g_{1,n})_{n=1}^{\infty}$ and g_2 is the pointwise limit of the functions $(g_{2,n})_{n=1}^{\infty}$, such that these functions are in lower Baire classes. Thus by the continuity of the metric d_Y, we have

$$\rho_{g_1,g_2}(x) = \lim_{n\to\infty} d_Y\left(g_{1,n}(x), g_{2,n}(x)\right) = \lim_{n\to\infty} \rho_{g_1,g_2,n}(x).$$

However, the induction hypothesis easily yields that each of the functions $\rho_{g_1,g_2,n}$ are in lower Baire classes than Baire-α. Thus ρ_{g_1,g_2} is a Baire-α function, as we stated. As a consequence,

$$\left\{x : d_Y\left(f_n(x), f(x)\right) < \frac{1}{k}\right\} = \rho_{g_1,g_2}^{-1}\left(\left[0, \frac{1}{k}\right)\right)$$

is in $\Sigma_{\alpha+1}$ by Proposition 3.1.3, as we consider the inverse image of an open set in $\mathbb{R}_+$ under a Baire-α function. Thus it can be written as the countable union of Π_α sets $A(n,k,i) \subseteq Y$:

$$\left\{x : d_Y\left(f_n(x), f(x)\right) < \frac{1}{k}\right\} = \bigcup_{i=1}^{\infty} A(n,k,i).$$

We define the subsets $H(n,k,i)$ of $X \times Y$ as follows:

$$H(n,k,i) = \left\{(x,y) : x \in A(n,k,i), d_Y\left(y, f_n(x)\right) \geq \frac{1}{k}\right\}.$$

We state $H(n,k,i)$ is in Π_α. The proof of this claim starts with the same reformulation, that is we write $H(n,k,i)$ as the intersection of two simpler sets:

$$H(n,k,i) = \left[A(n,k,i) \times Y\right] \cap \left\{(x,y) : d_Y\left(y, f_n(x)\right) \geq \frac{1}{k}\right\}.$$

The first one of these sets on the right hand side is clearly in Π_α in $X \times Y$ as $A(n,k,i)$ was in Π_α in X, hence it suffices to prove that the second set on the right hand side is also in Π_α. Let us define the following function $h_n : X \times Y \to \mathbb{R}_+$:

$$h_n(x,y) = d_Y\left(y, f_n(x)\right).$$

One can easily prove by transfinite induction on α_n that if f_n is Baire-α_n then h_n is also Baire-α_n: the initial case $\alpha_n = 0$, where f_n is continuous, can be verified exactly as we did it in the proof of Theorem 3.2.5, we only have to replace the norms of the differences in the inequalities with the respective distances. Now if f_n is the pointwise limit of a sequence of functions $(\phi_{n,m})_{m=1}^{\infty}$ from lower Baire classes, then

h_n is the pointwise limit of the sequence of functions $d_Y\left(y, \phi_{n,m}(x)\right)_{m=1}^{\infty}$, and for these functions the inductive hypothesis can be used. Hence h_n is Baire-α_n, yielding $\left\{(x, y) : d_Y\left(y, f_n(x)\right) \geq \frac{1}{k}\right\}$ is in Π_{α_n+1}, and as a consequence, it is also in Π_α, as we can separate the cases of successor and limit ordinals as in the proof of Proposition 3.1.3. Thus $H(n, k, i)$ is in Π_α.

At this point, we can proceed exactly as we did in the proof of Theorem 3.2.5. We can take an enumeration $H_1, H_2, ...$ of the sets $H(n, k, i)$ and define G_j as $(X \times Y) \setminus \bigcup_{l=1}^{j} H_l$. Then these sets are in Σ_α and their intersection is $gr(f)$. Furthermore, if Y is a Fréchet space, these sets are also Σ_α strips as balls are convex sets in Fréchet spaces. □

3.4 The other direction

Our goal is to verify the converse of the previous theorem in certain settings, thus giving the characterization of Baire functions between as general spaces as it is possible. As we will see, it is a more difficult task.

A possible line of thought has already been sketched in the previous section: we would like to "thread" functions from lower Baire classes through Σ_α-strips. Let us observe that it is another selection problem, just like the one handled by the Michael selection theorem! The only change is that we are interested in Baire-α selections instead of continuous ones. This problem is surprisingly complicated, giving such topological classification of Baire classes seems rather difficult. What is a more "popular" topic, that is the theory of measurable selections. A classical result concerning this problem is the theorem of Kuratowski and Ryll-Nardzewski published in [5]. We cite only a special case:

Proposition 3.4.1. *Let X be a metric space and let Y be a separable complete metric space. Assume $\alpha \geq 1$ is a countable ordinal and let $\Psi : X \to 2^Y$ be a multifunction with nonempty closed values such that $\Psi^{-1}(G)$ is in Σ_α for each open subset G of Y. Then Ψ admits a Borel class α selection, that is a mapping $f : X \to Y$ such that the inverse image of any open set of Y is in Σ_α.*

This theorem is a truly powerful tool if our goal is to construct Borel mappings. However, it is not exactly what we wish to do: we need Baire functions. We have already noted that these classes of functions have a strong relationship, for example Proposition 3.1.3 guarantees that a Baire-α function defined on a topological space with metric codomain is a Borel-α mapping. We have to be careful though, as the converse does not hold in general: for instance if X is a connected topological space with at least two points, and Y is the two point discrete space $\{0, 1\}$, then the characteristic function of a single point of X is Borel-2, but not Baire-1, as all the continuous functions from X to Y are constants. As our aim is to use Proposition

3.4.1 in as general setting as it is possible, it would be beneficial to know some results concerning conditions yielding the equivalence of Baire-α functions and Borel-($\alpha+1$) mappings. We can recall a special form of Theorem 8 of [3] (in that paper, every space is assumed to be perfect):

Proposition 3.4.2. *Let X be a perfect Suslin space and Y be a metric space. If X is metrizable and Y is a locally convex topological linear space then the family of Baire-α functions coincides with the family of Borel-$(\alpha+1)$ mappings.*

Thus by applying Propositions 3.4.1 and 3.4.2 simultaneously, we can have a sufficient condition for the existence of Baire-β selections for each β countable ordinal, as long as the domain and codomain satisfy the conditions of both of these theorems. However, the existence of these selections is only a tool for us: we use it in order to verify that a given function is Baire-α, as the pointwise limit of Baire functions from lower classes. Hence this concept can be directly applied only if α is a successor ordinal. This fact leads to the main result of this thesis:

Theorem 3.4.3. *Let $f : X \to Y$ be a function where X is a metrized Suslin space, Y is a separable Fréchet space. Then f is Baire-α for some successor countable ordinal α if and only if there is a nested sequence $(G_n)_{n=1}^{\infty}$ of Σ_α strips in $X \times Y$ such that*

- $\cap_{n=1}^{\infty} G_n = gr(f)$,
- *$diam(G_n(x)) \to 0$ for each $x \in X$ as n tends to infinity,*
- *the projection of $(X \times U) \cap G_n$ to X is in Σ_α for each open subset U of Y.*

Remark 3.4.4. : In other words, the last condition states that for the multifunction naturally assigned to G_n by taking its vertical sections, we have that the inverse image of each open set is in Σ_α. Practically, this condition lets us apply Proposition 3.4.1. The main difficulty of the proof is the direction in which we construct strips, namely the verification of this criterion.

Proof of Theorem 3.4.3. For the direction in which we assume that f is Baire-α, we can refer to the proof of Theorem 3.3.3, the only detail we have to check that is the third condition is also satisfied. In general, let us denote the projection of a set $C \subseteq X \times Y$ to X by $\pi(C)$, and let us denote the projection of $(X \times U) \cap G_n$ to X for the sake of simplicity by $\pi_n^*(U)$. Let us define the sets $A(n,k,i)$ and $H(n,k,i)$, and then the sequences $(H_j)_{j=1}^{\infty}$ and $(G_j)_{j=1}^{\infty}$ as we did in that proof. Namely, if $(f_n)_{n=1}^{\infty}$ is the sequence of functions from lower Baire classes with pointwise limit f, then

$$\left\{x : d_Y(f_n(x), f(x)) < \frac{1}{k}\right\} = \bigcup_{i=1}^{\infty} A(n,k,i), \text{where } A(n,k,i) \in \Pi_\alpha,$$

$$H(n,k,i) = [A(n,k,i) \times Y] \cap \left\{ (x,y) : d_Y(y, f_n(x)) \geq \frac{1}{k} \right\},$$

$(H_j)_{j=1}^{\infty}$ is the enumeration of these sets $H(n,k,i)$, and

$$G_j = (X \times Y) \setminus \bigcup_{l=1}^{j} H_l = \bigcap_{l=1}^{j} (X \times Y) \setminus H_l.$$

Our goal is to prove that $\pi_j^*(U)$ is in Σ_α for each open subset U of Y. Assume that G_j can be decomposed as the following:

$$G_j = (X \times Y) \setminus \bigcup_{l=1}^{j} H(n_l, k_l, i_l) = \bigcap_{l=1}^{j} (X \times Y) \setminus H(n_l, k_l, i_l).$$

Now we can divide each $(X \times Y) \setminus H(n_l, k_l, i_l)$ into two parts with disjoint projections to X:

$$\begin{aligned}(X \times Y) \setminus H(n_l, k_l, i_l) &= [(X \setminus A(n_l, k_l, i_l)) \times Y] \cup [(A(n_l, k_l, i_l) \times Y) \setminus H(n_l, k_l, i_l)] \\ &= V_{l,1} \cup V_{l,2},\end{aligned}$$

yielding

$$G_j = \bigcap_{l=1}^{j} (X \times Y) \setminus H(n_l, k_l, i_l) = \bigcap_{l=1}^{j} (V_{l,1} \cup V_{l,2}).$$

By distributivity, we can replace this intersection of unions by a union of intersections:

$$\bigcap_{l=1}^{j} (V_{l,1} \cup V_{l,2}) = \bigcup_{(\theta_1, \ldots \theta_j) \in \{1,2\}^j} \bigcap_{l=1}^{j} V_{l,\theta_l}.$$

What is intriguing about this expression, that is the projections of the sets $\bigcap_{l=1}^{j} V_{l,\theta_l}$ to X are clearly disjoint as two such intersection differs in at least one θ-coordinate, and the projections $\pi(V_{l,1})$ and $\pi(V_{l,2})$ are disjoint. As a consequence, the projection of the union

$$\bigcup_{(\theta_1, \ldots \theta_j) \in \{1,2\}^j} \bigcap_{l=1}^{j} V_{l,\theta_l}$$

to X equals the union of the projections, hence

$$\pi_j^*(U) = \pi\left((X \times U) \cap G_j\right) = \bigcup_{(\theta_1, \ldots \theta_j) \in \{1,2\}^j} \pi\left((X \times U) \cap \bigcap_{l=1}^{j} V_{l,\theta_l} \right). \tag{3.1}$$

We would like to show that this set is in Σ_α. Let us consider one of these sets

$$\pi\left((X \times U) \cap \bigcap_{l=1}^{j} V_{l,\theta_l}\right)$$

and take a closer look at $\bigcap_{l=1}^{j} V_{l,\theta_l}$. Amongst these sets, certain ones are of the type $V_{l,1}$, others are of the type $V_{l,2}$. Let us denote the set of indices belonging to the first type by J_1, and the set of indices belonging to the second type by J_2, yielding

$$\pi\left((X \times U) \cap \bigcap_{l=1}^{j} V_{l,\theta_l}\right) = \pi\left((X \times U) \cap \bigcap_{l \in J_1} V_{l,1} \cap \bigcap_{l \in J_2} V_{l,2}\right).$$

In this expression, $V_{l,1} = [(X \setminus A(n_l, k_l, i_l)) \times Y]$ for $l \in J_1$, meaning $V_{l,1}$ contains the whole space Y above $X \setminus A(n_l, k_l, i_l)$. As a consequence, one can easily verify that

$$\pi\left((X \times U) \cap \bigcap_{l \in J_1} V_{l,1} \cap \bigcap_{l \in J_2} V_{l,2}\right) = \bigcap_{l \in J_1} (X \setminus A(n_l, k_l, i_l)) \cap \pi\left((X \times U) \cap \bigcap_{l \in J_2} V_{l,2}\right).$$

Let us recall the definiton of $V_{l,2}$:

$$\pi\left((X \times U) \cap \bigcap_{l \in J_2} V_{l,2}\right) = \left\{x : x \in \bigcap_{l \in J_2} A(n_l, k_l, i_l), U \cap \bigcap_{l \in J_2} B_Y\left(f_{n_l}(x), \frac{1}{k_l}\right) \neq \emptyset\right\}.$$

Let us denote $U \cap \bigcap_{l \in J_2} B_Y\left(f_{n_l}(x), \frac{1}{k_l}\right)$ by $M(U, J_2, x)$. Using the previous identities, we can reformulate (3.1), yielding $\pi_j^*(U)$ equals the following:

$$\bigcup_{J_1, J_2}\left(\bigcap_{l \in J_1} (X \setminus A(n_l, k_l, i_l)) \cap \left\{x : x \in \bigcap_{l \in J_2} A(n_l, k_l, i_l), M(U, J_2, x) \neq \emptyset\right\}\right). \quad (3.2)$$

Now we will show that the condition $x \in \bigcap_{l \in J_2} A(n_l, k_l, i_l)$ might be omitted from this expression for each J_1, J_2 without changing the union. This omission extends each of the sets

$$\bigcap_{l \in J_1} (X \setminus A(n_l, k_l, i_l)) \cap \left\{x : x \in \bigcap_{l \in J_2} A(n_l, k_l, i_l), M(U, J_2, x) \neq \emptyset\right\}$$

to

$$\bigcap_{l\in J_1}(X\setminus A(n_l,k_l,i_l))\cap\{x:M(U,J_2,x)\neq\emptyset\}\,, \tag{3.3}$$

however, as we will show the increment is contained by other sets of the union in (3.2), yielding this union remains the same. Indeed, as J runs over the subsets of J_2, the sets

$$\bigcap_{l\in J}(X\setminus A(n_l,k_l,i_l))\cap\bigcap_{l\in J_2\setminus J}A(n_l,k_l,i_l)$$

give a natural partition of X. As a consequence, the set in (3.3) can be expressed as it follows, by taking the intersection with each of the elements of this partition and then forming their union:

$$\bigcup_{J\subseteq J_2}\left(\bigcap_{l\in J\cup J_1}(X\setminus A(n_l,k_l,i_l))\cap\bigcap_{l\in J_2\setminus J}A(n_l,k_l,i_l)\cap\{x:M(U,J_2,x)\neq\emptyset\}\right). \tag{3.4}$$

Replacing $M(U,J_2,x)$ by $M(U,J_2\setminus J,x)$ is equivalent to taking the intersection of the sets $B_Y\left(f_{n_l}(x),\frac{1}{k_l}\right)$ only for $l\in J_2\setminus J$ instead of J_2. Thus this replacement clearly extends the set in (3.4), and the condition $\bigcap_{l\in J_2\setminus J}A(n_l,k_l,i_l)$ can be moved inside $\left\{x:U\cap\bigcap_{l\in J_2}B_Y\left(f_{n_l}(x),\frac{1}{k_l}\right)\neq\emptyset\right\}$, yielding the set in (3.4) is contained by

$$\bigcup_{J\subseteq J_2}\left(\bigcap_{l\in J_1\cup J}(X\setminus A(n_l,k_l,i_l))\cap\left\{x:x\in\bigcap_{l\in J_2\setminus J}A(n_l,k_l,i_l),M(U,J_2\setminus J,x)\neq\emptyset\right\}\right).$$

Now we may notice that each of the unioned sets in this expression appears in the union in (3.2), which verifies our statement: we can make the omissions for any J_1 and J_2 without changing the union there. In other words, $\pi_j^*(U)$ is also the union of these modificated sets, that is

$$\pi_j^*(U)=\bigcup_{J_1,J_2}\left(\bigcap_{l\in J_1}(X\setminus A(n_l,k_l,i_l))\cap\{x:M(U,J_2,x)\neq\emptyset\}\right).$$

As it is a finite union, it suffices to prove about each of the unioned sets that they are in Σ_α, that is

$$\bigcap_{l\in J_1}(X\setminus A(n_l,k_l,i_l))\cap\{x:M(U,J_2,x)\neq\emptyset\}\in\Sigma_\alpha.$$

The sets $X\setminus A(n_l,k_l,i_l)$ are also in Σ_α, therefore it would be sufficient to prove the same about $M(U, J_2, x) = \left\{x : U \cap \bigcap_{l \in J_2} B_Y\left(f_{n_l}(x), \frac{1}{k_l}\right) \neq \emptyset\right\}$. The intersection which we regard in this set is the intersection of a finite collection of open sets, hence it is also open. Furthermore, U is separable as a subspace of the separable space Y. Thus it contains a countable dense set $\{u_1, u_2, ...\}$. As a consequence, $U \cap \bigcap_{l \in J_2} B_Y\left(f_{n_l}(x), \frac{1}{k_l}\right) \neq \emptyset$ holds if and only if there exists some u_t for $t \in \mathbb{N}$ such that $u_t \in \bigcap_{l \in J_2} B_Y\left(f_{n_l}(x), \frac{1}{k_l}\right)$, thus

$$\left\{x : U \cap \bigcap_{l \in J_2} B_Y\left(f_{n_l}(x), \frac{1}{k_l}\right) \neq \emptyset\right\} = \bigcup_{t=1}^{\infty}\left\{x : u_t \in \bigcap_{l \in J_2} B_Y\left(f_{n_l}(x), \frac{1}{k_l}\right)\right\}.$$

For some $x \in X$, the relation $u_t \in B_Y\left(f_{n_l}(x), \frac{1}{k_l}\right)$ holds if and only if $f_{n_l}(x) \in B_Y\left(u_t, \frac{1}{k_l}\right)$ by symmetry. Hence

$$\begin{aligned}\bigcup_{t=1}^{\infty}\left\{x : u_t \in \bigcap_{l \in J_2} B_Y\left(f_{n_l}(x), \frac{1}{k_l}\right)\right\} &= \bigcup_{t=1}^{\infty}\bigcap_{l \in J_2}\left\{x : f_{n_l}(x) \in B_Y\left(u_t, \frac{1}{k_l}\right)\right\} \\ &= \bigcup_{t=1}^{\infty}\bigcap_{l \in J_2} S(t, l).\end{aligned}$$

On the right hand side, each set $S(t, l)$ is the inverse image of an open set under f_{n_l} which is Baire-α_{n_l}, where $\alpha_{n_l} < \alpha$. Thus each $S(t, l)$ is in Σ_α by Proposition 3.1.3 as $X \times Y$ is metrizable, yielding that it is perfect. Hence if we take the finite intersection for $l \in J_2$ and then the countable union for $t = 1, 2, ...$, we will still have a set in Σ_α and as we have already seen it concludes the proof of the first direction.

For the other direction, let us assume $gr(f) = \bigcap_{j=1}^{\infty} G_j$ where the set G_j is in Σ_α for each j, their sequence is nested, and they satisfy the three conditions of the theorem. Let us define F_j as it follows:

$$F_j = \bigcup_{x \in X} \overline{G_j(x)},$$

thus F_j is the closure by coordinates. If we regard it as a multivalued function defined on X with range 2^Y, whose values are naturally the vertical sections of the set, we can easily verify that it satisfies the conditions of Proposition 3.4.1. Indeed, it has clearly nonempty, closed values, and as the projection of $(X \times U) \cap G_n$ to X is in Σ_α for each open subset U of Y, the inverse image $F_j^{-1}(U)$ is in Σ_α for the open subsets of Y. Hence F_j has a Borel-α selection f_j. As α is a successor ordinal, $\alpha - 1$ makes sense and Proposition 3.4.2 can be applied, yielding f_j is Baire-$(\alpha - 1)$. The conclusion is the same as it was in the proof of Theorem 3.2.5: as the intersection of the sets $F_j(x)$ is only $\{f(x)\}$ and their diameter tends to 0, $f_j(x) \to f(x)$ must hold, and as a consequence, f is the pointwise limit of Baire-$(\alpha - 1)$ functions, meaning f is Baire-α. □

3.5 Another glance at the accumulation problem

We desire to generalize some of the results presented in the second section using selection theorems, more precisely the theorem of Kuratowski and Ryll-Nardzewski. As this theorem is still not able to produce Baire functions directly, we also wish to use the theorem of Hansell. This idea already restricts the choice of the spaces, however, we go even further. In the case of $[0,1] \to \mathbb{R}$ functions, the starting point of our argument was Lemma 2.2.3, which practically stated that we can define a function on a countable subset satisfying $L_f = T$, which is a very comfortable assumption if we wish to manage Borel classes. Roughly speaking, it is based on the fact that if we modify a Borel set in countably many points, the Borel class of the altered set is easy to handle. As a consequence, we would like to draw up generalizations only in those cases when analogous results to Lemma 2.2.3 hold. We say that a topological space is σ-compact if it is the countable union of compact subspaces. We might state the following:

Lemma 3.5.1. *Let X and Y be σ-compact metric spaces such that X has no isolated points. For a given closed set $T \subseteq X \times Y$ there exists a countable set $A \subseteq X$ such that there is a function $f : A \to Y$ satisfying $L_f = T$.*

Proof. The product space $X \times Y$ is also σ-compact, hence there is an increasing sequence of compact sets $(C_n)_{n=1}^{\infty}$ with limit $X \times Y$. Then $T_n = T \cap C_n$ is also a compact set. We will construct A and f by induction. Let us consider an open ball of radius one around each point of T_1. These balls give an open cover of the compact set T_1 hence it is possible to choose a finite cover. Let us take a point in each ball of the finite cover such that the x coordinates of these points are pairwise different. As none of the points of X is isolated, it is clearly possible. Denote the set of these points by F_1, and the set of their x coordinates by A_1. In the following step, let us take open balls of radii $\frac{1}{2}$ around each point of T_2, choose a finite cover, and take points in each of these balls with pairwise different x coordinates, which are also distinct from the points in A_1. Let us define A_2 and F_2 analogously, and continue this procedure: in the $n^{\mathbf{th}}$ step regard the $\frac{1}{n}$-neighborhoods of the points of T_n, and define the finite sets F_n and A_n using these open balls. Now if we let $A = \bigcup_{n=1}^{\infty} A_n$ and $F = \cup_{n=1}^{\infty} F_n$, these are countable sets, and we may define f to be the function that assigns to every $x \in A$ the y coordinate of the chosen point in F above x. The equality $L_f = T$ can easily be checked, as in the proof of Lemma 2.2.3. □

We will give the generalization of Theorem 2.3.1 and 2.3.2 using this lemma. We formalize it in one theorem:

Theorem 3.5.2. *Let $T \subseteq X \times Y$, where X is a σ-compact metrizable Suslin space with no isolated points, and Y is a compact Fréchet space. There exists a Baire-2 function f satisfying $L_f = T$ if and only if T is closed and $T \cap (\{x\} \times Y) \neq \emptyset$ for each $x \in X$.*

Furthermore, if Y is σ-compact, but not compact, there exists a Baire-2 function f satisfying $L_f = T$ if and only if T is closed and there is a countable set $D \subseteq X$ such that $T \cap (\{x\} \times Y) \neq \emptyset$ for each $x \in X \setminus D$.

Before the proof, we would like to remark that the conditions concerning T are clearly necessary, even if we do not require f to be Baire-2:

Proposition 3.5.3. *In the setting of the first case of Theorem 3.5.2, if a subset T of $X \times Y$ equals L_f for a function $f : X \to Y$, then T is closed and $T \cap (\{x\} \times Y) \neq \emptyset$ for each $x \in X$.*

Furthermore, in the setting of the second case of Theorem 3.5.2, if a subset T of $X \times Y$ equals L_f for a function $f : X \to Y$, then T is closed and there is a countable set $D \subseteq X$ such that $T \cap (\{x\} \times Y) \neq \emptyset$ for each $x \in X \setminus D$.

Proof. As L_f is the set of the accumulation points of $gr(f)$, it must be closed in both cases. On the other hand, in the first case, if we consider any $x \in X$, by our conditions there is a sequence $(x_n)_{n=1}^{\infty}$ with elements from X distinct from x and with limit x. Thus by the compactness of Y, for any f the sequence $(f(x_n))_{n=1}^{\infty}$ has a limit point, implying the sequence $(x_n, f(x_n))_{n=1}^{\infty}$ has a limit point in $\{x\} \times Y$. Hence if $T = L_f$, the set T has to intersect any vertical line in the first case.

In the second case, proceeding towards a contradiction, let us assume the set D of points in X satisfying $T \cap (\{x\} \times Y) = \emptyset$ is uncountable and there exists a function $f : X \to Y$ for which $T = L_f$ holds. As both X and Y are σ-compact, it implies the existence of compact sets $C_X \subseteq X$ and $C_Y \subseteq Y$ such that $C_X \cap D$ is uncountable and the cardinality of $D^* = \{x : x \in C_X \cap D, f(x) \in C_Y\}$ is also uncountable. Thus by the separability of X, the set D^* contains one of its accumulation points, d. Therefore there exists a sequence (d_i) in D^*, $(d_i \neq d)$ with limit d. Since all the elements of the sequence $(f(d_i))$ are in the compact set C_Y, it has a convergent subsequence, therefore $L_f(d)$ cannot be empty, while $T(d)$ is, a contradiction. □

However, Theorem 3.5.2 states for such a set T we have a Baire-2 function satisfying $L_f = T$, yielding the following:

Corollary 3.5.4. *In the setting of any case of Theorem 3.5.2, if a subset T of $X \times Y$ equals L_f for a function $f : X \to Y$, then there exists a Baire-2 function such that $L_f = T$.*

Proof of Theorem 3.5.2. Let us regard the first case. Consider a metric on X. By Lemma 3.5.1, there exists a countable set $A \subseteq X$ and a function $f_0 : A \to Y$ satisfying $L_{f_0} = T$. We wish to extend this function to $f : X \to Y$ such that f is Baire-2 without making L_f larger. In order to do this, define a multifunction $F : X \to 2^Y$ the following way:

$$F(x) = \begin{cases} \{f_0(x)\} & \text{if } x \in A \\ T \cap (\{x\} \times Y) & \text{if } x \in X \setminus A. \end{cases}$$

As T is closed and its vertical sections are nonempty, F has nonempty closed values. Furthermore, $F^{-1}(G)$ is in Σ_3 for each open subset G of Y. Indeed, $T \cap (X \times G)$ is a set in Σ_2. Next we show that $\pi(T \cap (X \times G))$ is also in Σ_2. Let us recall that as $X \times Y$ is σ-compact, any closed set is the union of countably many compact sets, implying any set in Σ_2 is also the union of countably many compact sets. However, the projection of a compact set is obviously compact, thus closed. Hence $\pi(T \cap (X \times G))$ is in Σ_2 as we stated. Furthermore, one can easily verify that $F^{-1}(G)$ and $\pi(T \cap (X \times G))$ can differ only in the points of A, because if we regard T as a multifunction whose values are its vertical sections, T and F differ only in A. Thus $F^{-1}(G)$ differs only in a countable set from a set in Σ_2, yielding it is in Σ_3: indeed, a countable set is always in Σ_2, thus if we add a countable set to $\pi(T \cap (X \times G))$ we obtain another set in Σ_2, while removing a countable set is equivalent to intersecting with its complement, which is in Π_2. Hence the set we are interested in is the intersection of a set in Σ_2 and a set in Π_2, which are both in Σ_3 as X is metrizable. As a consequence, the intersection is also in Σ_3, as we stated. Hence F satisfies all the conditions of Proposition 3.4.1, yielding it admits a Borel-3 selection f. This function f is also Baire-2 since the conditions of Theorem 3.5.2 satisfy the conditions of Proposition 3.4.2. What remains to show that is $L_f = T$. We have already seen $T \subseteq L_f$ as $T = L_{f_0}$ by the construction of f_0 and $L_{f_0} \subseteq L_f$ clearly holds. For the other inclusion, we only have to verify that there is no sequence in $gr(f)$ with limit outside of T. However, in that case there would be such a sequence in $gr\,(f_0)$ as every point of $gr(f)$ is in the closed set T, except for the ones in $gr\,(f_0)$. Nevertheless that would imply L_{f_0} is already larger than T, which is a contradiction.

In the second case, we can proceed almost the same way. Let us define f_0 on a countable set A provided by Lemma 3.5.1. As Y is not compact, there exists a sequence $(y_1, y_2, ...)$ in Y without any accumulation point. Furthermore, as $D \setminus A$ is a countable set, we can enumerate its elements, possibly finitely: $(d_1, d_2, ...)$. Let us define the multifunction $F : X \to 2^Y$ as it follows:

$$F(x) = \begin{cases} \{f_0(x)\} & \text{if } x \in A \\ \{y_i\} & \text{if } x = d_i \\ T \cap (\{x\} \times Y) & \text{if } x \in X \setminus (A \cup D). \end{cases}$$

The steps of the previous case can be repeated to show that we can apply Proposition 3.4.1 to F without any difficulty, yielding the existence of a Borel-3 selection f, which is also Baire-2 by Proposition 3.4.2. What is a difference from the previous case, that in the proof of $L_f = T$ we have to take into account those sequences of points of $gr(f)$ which contain infinitely many points above $D \setminus A$. However, as the sequence $(y_1, y_2, ...)$ has no accumulation point in Y, such a sequence cannot have an accumulation point in $X \times Y$, thus $L_f = T$, indeed. $\square$

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