Budapest Semesters in Mathematics, Spring 2012

TOPICS IN GEOMETRY, Homework #1

Due February 17, 2012

1/1 Use the Cauchy inequality

$$(x_1y_1 + x_2y_2 + \ldots + x_ny_n)^2 \le (x_1^2 + x_2^2 + \ldots + x_n^2)(y_1^2 + y_2^2 + \ldots + y_n^2)$$

to prove that the triangle inequality holds for the distance function $d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|$ in $\mathbf{R}^n$.
(Hint: Show $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ first.)

1/2 Give an example of a finite metric space (e. g., a finite subset of Euclidean plane) whose isometry group is nontrivial and has odd order.

1/3 Prove that a square matrix is orthogonal if and only if its row vectors form an orthonormal basis.

1/4 Find all 2×2 orthogonal matrices with integer entries. Show that the number of $n \times n$ orthogonal matrices with integer entries is $2^n n!$.

Challenge / For Fun

1/A Given $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$, consider the quadratic polynomial function (of one variable t)

$$p(t) = \big(t\mathbf{a} + (1-t)\mathbf{b}\big) \cdot \big(t\mathbf{a} + (1-t)\mathbf{b}\big)\,.$$

What can be said about the discriminant of p? Deduce the Cauchy inequality.

1/B Prove that any distance-preserving map of a closed and bounded subset of $\mathbf{R}^n$ into itself is surjective (i. e., onto).
(For those who are familiar with the concept of compactness:) Prove the same for compact metric spaces.

1/C Can an isometry of Euclidean plane map a bounded set B into a proper subset of B?

1/D True or false (explain why):
(**a**) Any metric space consisting of 3 points admits a distance-preserving map into $\mathbf{R}^2$.
(**b**) Any metric space consisting of 4 points admits a distance-preserving map into $\mathbf{R}^3$.