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Flows in networks with dynamic ramification nodes

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Abstract. Combining functional analytical and graph theoretical methods, we investigate flow processes as in the papers [15] and [16], but we change the transmission process in the nodes of the network. Instead of conservation of mass, we assume that the *velocity* of the outgoing flow mass in the vertices is determined by the total incoming flow mass and by the outgoing flow in the other vertices.

1. Introduction

In this paper we investigate flows or transport processes in networks, similarly to the papers [15] and [16] – but we change the transmission process in the nodes. We assume that the *velocity* of the outgoing flow mass in the vertices is determined by the total incoming flow mass. In addition, we take into consideration a control process in each vertex, depending on the outgoing flow mass in the other vertices. However, for the sake of simplicity, we assume now that the flow velocities are constant on each edge and there is no absorption/inflow.

We show well-posedness and then describe the *asymptotic behavior* of the process using semigroup techniques combined with graph theory. As our main result we obtain, under natural positivity assumptions, that the system converges to a unique equilibrium (see Theorem 8.4.). The tools from graph theory can be found in [1], while or semigroup tools are quoted from [11]. In writing this paper we were inspired by the semigroup approach to delay equations as presented in [3] and [4].

2. Some graph theory

In this section we summarize some graph theoretical notions that we will use frequently. We use the terminology of [1], but refer also to [5], [6], [8], [12], or [13].

DEFINITION 2.1. Let $\mathsf{V} = \{\mathsf{v}_1, \dots, \mathsf{v}_n\}$ and $\mathsf{E} = \{\mathsf{e}_1, \dots, \mathsf{e}_m\}$ be two disjoint (finite) sets and G a function from E to $\mathsf{V} \times \mathsf{V}$. The triplet $(\mathsf{V}, \mathsf{E}, G)$ is called a *directed graph*. The elements of V are the *vertices* of the graph and the elements of E its (*directed*) edges (or

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arcs). The function G determines the two ordered endpoints of the edges and we say that the edge e_j *connects* the vertices v_i and v_p if $G(\mathsf{E}_j) = (\mathsf{v}_i, \mathsf{v}_p)$. For the sake of simplicity, a directed graph will only be denoted by G.

DEFINITION 2.2. If the edge e_j in a directed graph is associated to the vertex-pair $(\mathsf{v}_i, \mathsf{v}_p)$, v_i is called the *tail* of e_j and v_p is called the *head* of e_j. The edge e_j is called a *loop* if its tail coincides with its head.

From now on we assume that G satisfies the following property.

DEFINITION 2.3. A directed graph is called *simple* if it contains no loops and no multiple edges (that is, edges connecting the same vertices).

REMARK 2.4. In this paper we often use the notion *network* meaning a directed graph on which a dynamical process takes place. Hence we do not only consider the "static" structure of the graph but also some dynamics on it.

We need some more graph theoretical notions.

DEFINITION 2.5.

1. A (*directed*) path is a sequence of directed, adjoining edges in G (that is, except the last edge, the head of every edge is the tail of the following edge).
2. A directed graph is called *strongly connected* if for every pair of vertices in the graph there are directed paths connecting them in both directions.

We now introduce important matrices that can be associated to a directed graph (see [1, Chapter 3]). Let $\mathsf{V} = \{\mathsf{v}_1, \dots, \mathsf{v}_n\}$ be the set of the vertices, $\mathsf{E} = \{\mathsf{e}_1, \dots, \mathsf{e}_m\}$ the set of the edges of our simple directed graph G. We first begin with matrices describing the connection between vertices and edges in G.

DEFINITION 2.6. The *outgoing incidence matrix* $\Phi^- = (\phi^-_{ij})_{n\times m}$ of G is defined by

$$\phi^-_{ij} := \begin{cases} 1, & \text{if } \mathsf{v}_i \text{ is the tail of } \mathsf{e}_j, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Accordingly, we call the edge e_j an *outgoing edge for* v_i if $\phi^-_{ij} = 1$ holds. Respectively, we define the *incoming incidence matrix* $\Phi^+ = (\phi^+_{ij})_{n\times m}$ by

$$\phi^+_{ij} := \begin{cases} 1, & \text{if } \mathsf{v}_i \text{ is the head of } \mathsf{e}_j, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

and call the edge e_j an *incoming edge for* v_i if $\phi^+_{ij} = 1$ holds.

We also need matrices having the same zero pattern as the incidence matrices.

DEFINITION 2.7.

1. The matrix $\Phi_w^- = (\omega_{ij}^-)_{n\times m}$ with entries $\omega_{ij}^- \geq 0$ is called the *weighted outgoing incidence matrix* of the graph G if

   $$\omega_{ij}^- = 0 \Leftrightarrow \phi_{ij}^- = 0,$$

   for all $i = 1, \dots, n,\ j = 1, \dots, m$.
2. The *weighted incoming incidence matrix* $\Phi_w^+ = (\omega_{ij}^+)_{n\times m}$ is defined with entries $\omega_{ij}^+ \geq 0$ satisfying

   $$\omega_{ij}^+ = 0 \Leftrightarrow \phi_{ij}^+ = 0.$$

The name “weighted” indicates that the entries ω_{ij}^- denote the weights according to which the flow mass is distributed to the outgoing edges in the vertices. Note that contrary to [15], here we do not assume row stochasticity for Φ_w^-.

The following class of graph matrices describes the connections between the vertices.

DEFINITION 2.8. The matrix $\mathbf{A} = (a_{ip})_{n\times n}$ is called the *adjacency matrix* of G if

$$a_{ip} = \begin{cases} 1, & \text{if there exists an edge with tail } \mathsf{v}_i \text{ and head } \mathsf{v}_p, \\ 0, & \text{otherwise.} \end{cases}$$

REMARK 2.9. An easy computation shows that $\mathbf{A}$ can be obtained from the incidence matrices as

$$\mathbf{A} = \Phi^-(\Phi^+)^\top.$$

Furthermore, it is clear that the structure of the graph G is completely described once knowing either both incidence matrices or the adjacency matrix.

DEFINITION 2.10. We call a positive matrix $\mathbf{A}_w$ *weighted adjacency matrix* of G if it has the same zero pattern as the adjacency matrix $\mathbf{A}$.

Let G be a network admitting a weighted adjacency matrix $\mathbf{A}_w = (b_{ip})_{n\times n}$. Then the entry b_{ip} can be regarded as the weight of the edge connecting V_i and V_p. Hence, it is natural to associate the weighted incidence matrices of the graph with the weighted adjacency matrix.

We now quote a result (e.g., see [1, Theorem 3.2]) being important for the subsequent theory.

PROPOSITION 2.11. *A directed graph is strongly connected if and only if its adjacency matrix is irreducible.*

As usual, a positive matrix $\mathbf{D}$ is called *irreducible* if there is no permutation of the canonical basis such that in this basis the matrix has the form

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_{1,1} & 0 \\ \mathbf{D}_{2,1} & \mathbf{D}_{2,2} \end{pmatrix}.$$

Hence, this property only depends on the zero pattern of the positive matrix. Therefore in the above Proposition 2.11 we can substitute the adjacency matrix by any weighted adjacency matrix.

3. Flows with dynamic ramification nodes

We now consider a flow process on our simple graph G having vertices $\mathsf{v}_1, \dots, \mathsf{v}_n$ and directed edges (arcs) $\mathsf{e}_1, \dots, \mathsf{e}_m$. The arcs are parameterized by the interval [0, 1], in the opposite direction of the flow. Therefore we use the notation $\mathsf{e}_j(1)$ for the tail and $\mathsf{e}_j(0)$ for the head of e_j.

The distribution of the material along an edge e_j at time $t \geq 0$ is described by the function $[0, 1] \ni s \mapsto u_j(t, s)$. The positive numbers c_j are the velocities of the flow on each arc e_j. Hence, on the edges we choose the following transport equations (with adequate initial conditions):

$$\begin{cases} \frac{\partial}{\partial t} u_j(t, s) = c_j \frac{\partial}{\partial s} u_j(t, s), \ s \in (0, 1), t > 0, \\ u_j(0, s) = f_j^0(s), \ s \in (0, 1), \end{cases}$$

where $f_j^0 \in L^1(0, 1)$ for $j = 1, \dots, m$.

We now add boundary conditions in the vertices determining the distribution of the outgoing flow. Using the notation $u(t, s) = (u_1(t, s), \dots, u_m(t, s))$, we require in a first step that

$$u(t, 1) \in \operatorname{ran}(\Phi_w^-)^\top, \ t \geq 0, \tag{3}$$

where $\Phi_w^- = (\omega_{ij}^-)_{n \times m}$ is the *weighted outgoing incidence matrix* defined in Definition 2.7.1. The entries ω_{ij}^- are given by the proportions according to which the flow mass is distributed to the outgoing edges in the vertices. An easy computation shows that (3) expresses that the values of $u(t, \cdot)$ at the point 1 on the arcs with the same tail are related by the corresponding weights, see [15, (7)].

In the next step we introduce the boundary condition

$$\frac{\partial}{\partial t} \Phi^- u(t, 1) = \Phi_w^+ u(t, 0)$$

meaning that the (sum of the) outgoing flow *velocities* – and not the total outgoing flow *mass*, as in [15, (3)] – is equal to the incoming flow mass in each vertex v_i. We assume that different edges have different effects on the outgoing velocities. This is expressed by taking a weighted sum of the incoming flow mass on the right-hand side, using the *weighted incoming incidence matrix* $\Phi_w^+ = (\omega_{ij}^+)_{n\times m}$ defined in Definition 2.7.2.

In the final step we add a *boundary control* in the vertices. For this purpose we choose a control space W and a linear control operator $C : W \to \mathbb{C}^n$. Then our boundary control problem becomes

$$\frac{\partial}{\partial t}\Phi^- u(t,1) = \Phi_w^+ u(t,0) + Cw(t),\ t \geq 0.$$

Then we assume that the control function $w(\cdot)$ is given by a feedback from the values of the outgoing flow in the vertices. More precisely, we take $w(t) := D\Phi^- u(t,1)$ with the linear feedback operator $D : \mathbb{C}^n \to W$.

All these assumptions lead to the following system.

$$(DE)\begin{cases} \frac{\partial}{\partial t}u_j(t,s) = c_j \frac{\partial}{\partial s}u_j(t,s),\ s \in (0,1),\ t > 0,\\ u_j(0,s) = f_j^0(s),\ s \in (0,1),\\ u(t,1) \in \operatorname{ran}(\Phi_w^-)^\top,\ t \geq 0,\\ \frac{\partial}{\partial t}\Phi^- u(t,1) = \Phi_w^+ u(t,0) + \mathbf{B}\Phi^- u(t,1),\ t \geq 0, (BC)\\ \Phi^- u(0,1) = x^0 \in \mathbb{C}^n, \end{cases}$$

where we take $\mathbf{B} := CD \in \mathcal{M}_n(\mathbb{C})$.

We consider the boundary condition (BC) as a *delay equation* for the process in the tails of the edges, that is for $\Phi^- u(t,1)$.

This perspective allows the use of a modification of the semigroup techniques from [4] in order to solve the system (DE) and to discuss qualitative properties of the solutions.

4. Well-posedness of the problem

We convert (DE) into an abstract Cauchy problem using a space of functions on the edges, i.e.,

$$X := (L^1[0,1])^m \cong L^1([0,1],\mathbb{C}^m). \tag{4}$$

In addition, we choose the (boundary) space of the values in the vertices

$$\partial X := \mathbb{C}^n, \tag{5}$$

as in [15]. As in [15, Section 2], we introduce a "boundary operator" $L : X \to \partial X$ by

$$\begin{aligned} L &:= \Phi^- \otimes \delta_1,\\ D(L) &:= W^{1,1}([0,1],\mathbb{C}^m), \end{aligned} \tag{6}$$

and the “delay operator”

$$\begin{aligned} M &:= \Phi_w^+ \otimes \delta_0 \\ D(M) &:= W^{1,1}([0,1], \mathbb{C}^m), \end{aligned} \tag{7}$$

in analogy to [15, (8)]. Though we call M the “delay operator” as in [4], it does not act on the “history function” (depending on time), but on the spatial distribution along the edges. However, since the flow has finite velocity on every edge, the incoming flow is always delayed with respect to the outgoing flow.

If we now define the operator

$$A_w := \begin{pmatrix} c_1 \frac{d}{ds} & & 0 \\ & \ddots & \\ 0 & & c_m \frac{d}{ds} \end{pmatrix}, \tag{8}$$

$$D(A_w) := \{f \in (W^{1,1}[0,1])^m : f(1) \in \operatorname{ran}(\Phi_w^-)^\top\}, \tag{9}$$

then the problem (DE) can be written as an abstract Cauchy problem for the operator

$$\mathcal{A} := \begin{pmatrix} A_w & 0 \\ M & \mathbf{B} \end{pmatrix}, \tag{10}$$

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} f \\ x \end{pmatrix} \in D(A_w) \times \mathbb{C}^n : Lf = x \right\}$$

on the space

$$\mathcal{X} := X \times \partial X.$$

Indeed, (DE) is equivalent to

$$(ACP) \begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t), \quad t \geq 0, \\ \mathcal{U}(0) = \begin{pmatrix} f^0 \\ x^0 \end{pmatrix} \end{cases}$$

made precise by the following result proved in [4, Corollary 3.5] and [4, Proposition 3.9].

THEOREM 4.1. *The system* (DE) *admits a solution* u *with*

1. *the mapping* $[0, +\infty) \ni t \mapsto u(t,\cdot)$ *is in* $C^1([0,+\infty), X)$*, and*
2. $u(t,\cdot) \in W^{1,1}([0,1], \mathbb{C}^m)$ *for all* $t \geq 0$

if and only if (ACP) *admits a continuously differentiable solution* $\mathcal{U} : \mathbb{R}_+ \to \mathcal{X}$. *In this case*

$$\mathcal{U}(t) = \begin{pmatrix} u(t,\cdot) \\ \Phi^- u(t,1) \end{pmatrix}.$$

By standard semigroup theory (see [11, Section II.6]) it follows that (*ACP*) is well-posed if and only if $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup on $\mathcal{X}$. For the well-posedness of (*DE*) we therefore show that the above operator (10) is a generator.

In the spirit of Greiner's approach to abstract boundary value problems (see also in [7], [15], [18], [19]), we first introduce the so-called *Dirichlet operator*

$$D_\lambda := (L\ |_{\ker(\lambda - A_w)})^{-1}$$

from ∂X to $\ker(\lambda - A_w)$. To obtain its concrete form we have to define the following new weighted outgoing incidence matrix $\widetilde{\Phi}_w^-$ of the graph G (see Definition 2.7.1).

DEFINITION 4.2. Let $\Phi_w^- = (\omega_{ij}^-)_{n\times m}$ be the weighted outgoing incidence matrix of G used in (*DE*) and in the domain of A_w, see (9). Define by w_i the ith row sum of this matrix for $i = 1, \dots n$. Let $\widetilde{\Phi}_w^- = (\widetilde{\omega}_{ij}^-)_{n\times m}$ be the weighted outgoing incidence matrix with entries

$$\widetilde{\omega}_{ij}^- = \frac{\omega_{ij}^-}{w_i}, \text{ for } i = 1, \dots, n,\ j = 1, \dots, m.$$

REMARK 4.3. Clearly, $\widetilde{\Phi}_w^-$ is row stochastic, and therefore

$$\Phi^-(\widetilde{\Phi}_w^-)^\top = \mathbf{1}, \tag{11}$$

where $\mathbf{1}$ denotes the $n \times n$ identity matrix. Also

$$\operatorname{ran}(\Phi_w^-)^\top = \operatorname{ran}(\widetilde{\Phi}_w^-)^\top$$

holds, hence in (9) the condition $f(1) \in \operatorname{ran}(\Phi_w^-)^\top$ can be substituted with $f(1) \in \operatorname{ran}(\widetilde{\Phi}_w^-)^\top$.

Turning back to the Dirichlet operator D_λ, by a similar argumentation as in [15, Lemma 3.1]) we obtain that

$$D_\lambda = \epsilon_\lambda(\widetilde{\Phi}_w^-)^\top,$$

that is

$$(D_\lambda x)(s) = \epsilon_\lambda(s)\cdot((\widetilde{\Phi}_w^-)^\top x),\ x \in \partial X,\ s \in [0, 1],$$

with

$$\epsilon_\lambda(s) = \begin{pmatrix} \exp\left(\frac{\lambda}{c_1}(s-1)\right) & & 0 \\ & \ddots & \\ 0 & & \exp\left(\frac{\lambda}{c_m}(s-1)\right) \end{pmatrix}.$$

We now consider the restriction of A_w to $\ker L$, i.e.,

$$A_0 := A_w \,|_{\ker L \cap D(A_w)}, \tag{12}$$

$$D(A_0) := \{f \in D(A_w) : Lf = 0\} = \{f \in W^{1,1}([0,1], \mathbb{C}^m) : f(1) = 0\}. \tag{13}$$

This operator $(A_0, D(A_0))$ generates the nilpotent left shift semigroup $(T_0(t))_{t\geq 0}$ on X defined by

$$(T_0(t)f)_j(s) = \begin{cases} f_j(s + c_j t), & s + c_j t \leq 1, \\ 0, & \text{otherwise,} \end{cases} \tag{14}$$

see [15, (13)]. We then know that the resolvent of A_0 exists for every $\lambda \in \mathbb{C}$ and can be computed as

$$(R(\lambda, A_0) f)(s) = \int_s^1 \epsilon_\lambda (s - \tau + 1)\, C^{-1} f(\tau) \mathrm{d}\tau, \ s \in [0,1], \ f \in X, \tag{15}$$

with

$$C = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_m \end{pmatrix},$$

see [15, (14)].

In the next lemma we give a decomposition of $\lambda - \mathcal{A}$ that turns out to be very useful. In the following we denote by $\mathbf{1}$ the $n \times n$ identity matrix and by I_X the identity operator on X.

LEMMA 4.4. *For every $\lambda \in \mathbb{C}$ one has*

$$\lambda - \mathcal{A} = \begin{pmatrix} I_X & 0 \\ -MR(\lambda, A_0) & \mathbf{1} \end{pmatrix} \begin{pmatrix} \lambda - A_0 & 0 \\ 0 & \lambda - \mathbf{B} - \mathbf{A}_\lambda \end{pmatrix} \begin{pmatrix} I_X & -D_\lambda \\ 0 & \mathbf{1} \end{pmatrix} \tag{16}$$

with $\mathbf{A}_\lambda := MD_\lambda = \Phi_w^+ \epsilon_\lambda(0) (\widetilde{\Phi}_w^-)^\top$ an $n \times n$ matrix.

Proof. Let us denote the operator on the right-hand side of (16) by $\mathcal{B}$ and write

$$S := \begin{pmatrix} I_X & -D_\lambda \\ 0 & \mathbf{1} \end{pmatrix}.$$

Then the condition $\begin{pmatrix} f \\ x \end{pmatrix} \in D(\mathcal{B})$ is equivalent to the fact that $S \begin{pmatrix} f \\ x \end{pmatrix} \in D(A_0) \times \mathbb{C}^n$, which means that $f - D_\lambda x \in \ker L \cap D(A_w)$. This is again equivalent to $Lf = x$ and $f \in D(A_w)$, so to $\begin{pmatrix} f \\ x \end{pmatrix} \in D(\mathcal{A})$, hence the two domains coincide. Let $\begin{pmatrix} f \\ x \end{pmatrix} \in D(\mathcal{A})$. Then

$$\begin{aligned}
\mathcal{B}\begin{pmatrix} f \\ x \end{pmatrix} &= \begin{pmatrix} I_X & 0 \\ -MR(\lambda, A_0) & \mathbf{1} \end{pmatrix} \begin{pmatrix} \lambda - A_0 & 0 \\ 0 & \lambda - \mathbf{B} - \mathbf{A}_\lambda \end{pmatrix} \begin{pmatrix} f - D_\lambda x \\ x \end{pmatrix} \\
&= \begin{pmatrix} I_X & 0 \\ -MR(\lambda, A_0) & \mathbf{1} \end{pmatrix} \begin{pmatrix} (\lambda - A_0)(f - D_\lambda x) \\ (\lambda - \mathbf{B} - \mathbf{A}_\lambda)\, x \end{pmatrix} \\
&= \begin{pmatrix} (\lambda - A_0)(f - D_\lambda x) \\ -M(f - D_\lambda x) + (\lambda - \mathbf{B} - \mathbf{A}_\lambda) x \end{pmatrix} \\
&= \begin{pmatrix} (\lambda - A_w) f \\ -Mf + (\lambda - \mathbf{B}) x \end{pmatrix} = (\lambda - \mathcal{A}) \begin{pmatrix} f \\ x \end{pmatrix},
\end{aligned}$$

where in the last equality we used $D_\lambda x \in \ker(\lambda - A_w)$. Now the proof is complete. □

Using the above decomposition, we obtain the desired well-posedness for (*ACP*), hence for (*DE*).

THEOREM 4.5. *The operator* $(\mathcal{A}, D(\mathcal{A}))$ *defined in* (10) *generates a strongly continuous semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *on the space* $\mathcal{X} = X \times \partial X$. *Hence, the system* (*DE*) *is well-posed.*

Proof. We proceed as in [4, Theorem 3.23]. Since $\mathbf{B}$ is bounded, using the bounded perturbation theorem (see [11, Theorem III.1.3]) for the sum

$$\mathcal{A} = \begin{pmatrix} A_w & 0 \\ M & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{B} \end{pmatrix},$$

there is no loss in assuming that $\mathbf{B} = 0$. Taking $\lambda = 0$ in the decomposition (16) yields

$$\mathcal{A} = \begin{pmatrix} I_X & 0 \\ -MA_0^{-1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} A_0 & 0 \\ 0 & \mathbf{A}_0 \end{pmatrix} \begin{pmatrix} I_X & -D_0 \\ 0 & \mathbf{1} \end{pmatrix}$$

with

$$\begin{pmatrix} I_X & -D_0 \\ 0 & \mathbf{1} \end{pmatrix}$$

being an invertible operator with inverse

$$\begin{pmatrix} I_X & D_0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

By similarity, it is enough to prove that the operator

$$\begin{aligned}
\mathcal{C} &= \begin{pmatrix} I_X & -D_0 \\ 0 & \mathbf{1} \end{pmatrix} \mathcal{A} \begin{pmatrix} I_X & D_0 \\ 0 & \mathbf{1} \end{pmatrix} \\
&= \begin{pmatrix} I_X & -D_0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} I_X & 0 \\ -MA_0^{-1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} A_0 & 0 \\ 0 & \mathbf{A}_0 \end{pmatrix}
\end{aligned}$$

with domain $D(\mathcal{C}) = D(A_0) \times \partial X$ is a generator.

To proceed we compute

$$\begin{pmatrix} I_X & -D_0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} I_X & 0 \\ -MA_0^{-1} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} I_X + D_0 MA_0^{-1} & -D_0 \\ -MA_0^{-1} & \mathbf{1} \end{pmatrix}$$
$$:= \mathcal{I} + \mathcal{D},$$

where

$$\mathcal{D} := \begin{pmatrix} D_0 MA_0^{-1} & -D_0 \\ -MA_0^{-1} & 0 \end{pmatrix}.$$

Using now that $M = \Phi_w^+ \otimes \delta_0 : W^{1,1}\left([0,1], \mathbb{C}^m\right) \to \partial X$ is a bounded operator, we obtain that $MA_0^{-1} : X \to \partial X$ is bounded, hence $\mathcal{D}$ is a bounded operator on $\mathcal{X}$. So we have

$$\mathcal{C} = (\mathcal{I} + \mathcal{D}) \begin{pmatrix} A_0 & 0 \\ 0 & \mathbf{A}_0 \end{pmatrix}. \tag{17}$$

The matrix

$$\begin{pmatrix} A_0 & 0 \\ 0 & \mathbf{A}_0 \end{pmatrix}$$

with domain $D(\mathcal{C}) = D(A_0) \times \partial X$ generates the strongly continuous semigroup

$$\mathcal{S}(t) := \begin{pmatrix} T_0(t) & 0 \\ 0 & \mathrm{e}^{t\mathbf{A}_0} \end{pmatrix}, t \geq 0.$$

We now use a multiplicative version of the Desch-Schappacher Perturbation Theorem (see [11, Theorem III.3.1] and [11, Corollary III.3.4]) as stated in [4, Theorem 1.38] for the operator $\mathcal{C}$ in (17). For this purpose we take $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^p\left([0,1], \mathcal{X}\right)$ and compute

$$\begin{aligned}
&\int_0^1 \mathcal{S}(1-r)\mathcal{D}\begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} dr \\
&\quad = \int_0^1 \begin{pmatrix} T_0(1-r) & 0 \\ 0 & \mathrm{e}^{(1-r)\mathbf{A}_0} \end{pmatrix} \begin{pmatrix} D_0 MA_0^{-1} & -D_0 \\ -MA_0^{-1} & 0 \end{pmatrix} \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} dr \\
&\quad = \int_0^1 \begin{pmatrix} T_0(1-r) & 0 \\ 0 & \mathrm{e}^{(1-r)\mathbf{A}_0} \end{pmatrix} \begin{pmatrix} D_0 MA_0^{-1} f_1(r) - D_0 f_2(r) \\ -MA_0^{-1} f_1(r) \end{pmatrix} dr \\
&\quad = \begin{pmatrix} \displaystyle\int_0^1 T_0(1-r) D_0 [MA_0^{-1} f_1(r) - f_2(r)] dr \\ \displaystyle -\int_0^1 \mathrm{e}^{(1-r)\mathbf{A}_0} MA_0^{-1} f_1(r) dr \end{pmatrix}.
\end{aligned} \tag{18}$$

If we can show that the vector so obtained belongs to $D(\mathcal{C})$, we have that $\mathcal{C}$ – hence $\mathcal{A}$ – is a generator by [4, Theorem 1.38]. Using the boundedness of $MA_0^{-1} : X \to \partial X$ we have

$$g := MA_0^{-1} f_1 - f_2 \in L^p\left([0,1], \partial X\right).$$

From (14) the jth coordinate of the first component in (18) can be computed as

$$\left[\int_0^1 T_0(1-r)D_0 g(r)dr\right]_j (\cdot) = \widetilde{\omega}_{ij}^- \int_{\frac{\cdot-1}{c_j}+1}^1 \mathrm{e}^{\frac{\lambda}{c_j}(\cdot-1+c_j(1-r))} g_i(r)dr$$

with $\widetilde{\omega}_{ij}^- \neq 0$ uniquely defined by j. From this

$$\left(\left[\int_0^1 T_0(1-r)D_0 g(r)dr\right]_j\right)_{j=1,\dots,m} \in D(A_0).$$

Clearly it follows that

$$\int_0^1 \mathcal{S}(1-r)\mathcal{D}\begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix} dr \in D(A_0) \times \partial X = D(\mathcal{C}),$$

hence the proof is complete. □

5. Spectral properties

In order to describe qualitative properties of the solutions of (DE) – hence of the semigroup $(\mathcal{T}(t))_{t\geq 0}$ – we now study the spectrum of $(\mathcal{A}, D(\mathcal{A}))$ and determine its resolvent. First we state a lemma using the operator A_0 introduced in (12). For the proof see [14, Lemma 1.2].

LEMMA 5.1. *For every $\lambda \in \rho(A_0) = \mathbb{C}$ we have*

$$D(A_w) = \ker(\lambda - A_w) \oplus D(A_0). \tag{19}$$

Furthermore, the corresponding projections are $D_\lambda L|_{D(A_w)}$ onto $\ker(\lambda - A_w)$*, and* $R(\lambda, A_0)(\lambda - A_w)$ *onto* $D(A_0)$.

Using this we can characterize the spectrum of $\mathcal{A}$ by a condition in ∂X. Here the *weighted* (*transposed*) *adjacency matrix* obtained in Lemma 4.4 (see Definition 2.10),

$$\mathbf{A}_\lambda = MD_\lambda = \Phi_w^+ \epsilon_\lambda(0)(\widetilde{\Phi}_w^-)^\top,$$

with entries

$$(\mathbf{A}_\lambda)_{ip} = \begin{cases} \omega_{ij}^+ e^{-\frac{\lambda}{c_j}} \widetilde{\omega}_{pj}^-, & \text{if } \mathsf{v}_i = \mathsf{e}_j(0) \text{ and } \mathsf{v}_p = \mathsf{e}_j(1), \\ 0, & \text{otherwise} \end{cases} \tag{20}$$

plays an important role.

PROPOSITION 5.2. *For $\lambda \in \mathbb{C}$ the following characteristic equation holds:*

$$\lambda \in \sigma(\mathcal{A}) \Longleftrightarrow \lambda \in \sigma(\mathbf{B} + \mathbf{A}_\lambda).$$

Moreover, for any $\lambda \in \rho(\mathcal{A})$ the resolvent $R(\lambda, \mathcal{A})$ is given by

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} [D_\lambda R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)M + I_X]\, R(\lambda, A_0) & D_\lambda R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \\ R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) M R(\lambda, A_0) & R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \end{pmatrix}. \tag{21}$$

Proof. We follow [4, Proposition 3.19]. To compute the resolvent in λ, we have to find for $\begin{pmatrix} g \\ y \end{pmatrix} \in \mathcal{X}$ a unique $\begin{pmatrix} f \\ x \end{pmatrix} \in D(\mathcal{A})$ such that

$$(\lambda - \mathcal{A})\begin{pmatrix} f \\ x \end{pmatrix} = \begin{pmatrix} \lambda f - A_w f \\ -Mf + (\lambda - \mathbf{B})x \end{pmatrix} = \begin{pmatrix} g \\ y \end{pmatrix}. \tag{22}$$

Using Lemma 5.1 and $(\lambda - A_w)\, f = g$ we obtain that

$$f = D_\lambda L f + R(\lambda, A_0) g = D_\lambda x + R(\lambda, A_0) g, \tag{23}$$

since $Lf = x$. Plugging this into the second coordinate of (22) yields

$$(\lambda - \mathbf{B} - \mathbf{A}_\lambda)x = MR(\lambda, A_0)g + y,$$

where $\mathbf{A}_\lambda = MD_\lambda = \Phi_w^+ \epsilon_\lambda(0)(\widetilde{\Phi}_w^-)^\top$. Furthermore, if $R\,(\lambda, \mathbf{B} + \mathbf{A}_\lambda)$ exists, it follows that

$$x = R\,(\lambda, \mathbf{B} + \mathbf{A}_\lambda)\, MR(\lambda, A_0)g + R\,(\lambda, \mathbf{B} + \mathbf{A}_\lambda)\, y. \tag{24}$$

Using this and (23) we obtain

$$f = D_\lambda R\,(\lambda, \mathbf{B} + \mathbf{A}_\lambda)\, MR(\lambda, A_0)g + D_\lambda R\,(\lambda, \mathbf{B} + \mathbf{A}_\lambda)\, y + R(\lambda, A_0)g. \tag{25}$$

Equalities (22), (24) and (25) now imply (21) and

$$\lambda \in \rho\,(\mathcal{A}) \Longleftrightarrow \lambda \in \rho\,(\mathbf{B} + \mathbf{A}_\lambda)\,,$$

which is the desired characteristic equation. □

From this form of the resolvent we obtain the following property.

REMARK 5.3. The resolvent $R(\lambda, \mathcal{A})$ is compact for $\lambda \in \rho(\mathcal{A})$.

Proof. It is enough to prove that the entries of the operator matrix (21) are compact operators. In the second row this is clear since the entries have range in $\mathbb{C}^n$. In the first row, the second entry also has finite dimensional range contained in the span of finitely many exponential functions. The first entry is the sum of an operator with finite dimensional range and the resolvent of an operator having domain contained in $W^{1,1}([0,1], \mathbb{C}^m)$ – hence being compact by [11, II.4.30 (4)]. □

COROLLARY 5.4. *The operator* $(\mathcal{A}, D(\mathcal{A}))$ *has only point spectrum.*

COROLLARY 5.5. *Since* $\mathbf{B}$ *and* $\mathbf{A}_\lambda$ *are finite matrices, we can reformulate the above characteristic equation as*

$$\lambda \in P\sigma(\mathcal{A}) = \sigma(\mathcal{A}) \Longleftrightarrow \det(\lambda - \mathbf{B} - \mathbf{A}_\lambda) = 0. \tag{26}$$

6. Asymptotic behavior

In order to describe asymptotic behavior of the solutions of (*DE*), we first prove a regularity property of the solution semigroup. We show that this semigroup is eventually differentiable, that is, the orbits $t \mapsto \mathcal{T}(t)\binom{f}{x}$ are differentiable for t large enough for every $\binom{f}{x} \in \mathcal{X}$ (see [11, Definition II.4.13]). For this purpose we first show how the first coordinate of $\mathcal{T}(t)\binom{f}{x}$ can be obtained from the second one.

LEMMA 6.1. *Denoting by* π_1 *and* π_2*, resp., the projections from* $\mathcal{X}$ *to* X *and to* ∂X*, resp., we have*

$$\left[\pi_1 \mathcal{T}(t)\binom{f}{x}\right]_j (r) = \begin{cases} [(\widetilde{\Phi}_w^-)^\top \pi_2 \mathcal{T}(t - \frac{1-r}{c_j})\binom{f}{x}]_j, & \text{if } 1 - tc_j \le r \le 1, \\ f_j(r + tc_j), & \text{if } 0 \le r < 1 - tc_j, \end{cases}$$

for $j = 1, \dots, m$*, and almost all* r*.*

Proof. If $\binom{f}{x} \in D(\mathcal{A})$, then $\mathcal{T}(t)\binom{f}{x}$ defines a classical solution for (ACP), and, by Theorem 4.1, the function $\pi_1\mathcal{T}(t)\binom{f}{x}$ is a solution for (*DE*) with $L\pi_1\mathcal{T}(t)\binom{f}{x} = \pi_2\mathcal{T}(t)\binom{f}{x}$. It is easy to check that the given formula for $\pi_1\mathcal{T}(t)\binom{f}{x}$ satisfies these requirements. Heuristically, this means that $[\pi_1\mathcal{T}(t)\binom{f}{x}]_j(r)$ is the distribution of flow mass on the edges e_j at point r. If $1 - tc_j \le r \le 1$, that is $t \ge \frac{1-r}{c_j}$, this flow mass is equal to the flow mass that has been at the tail of e_j at time $t - \frac{1-r}{c_j}$. This is exactly the expression in the first part of the above formula, where we have used the condition $\pi_1\mathcal{T}(t)\binom{f}{x}(1) \in \operatorname{ran}(\widetilde{\Phi}_w^-)^\top$. If $0 \le r < 1 - tc_j$, that is $t < \frac{1-r}{c_j}$, the flow mass at point r is equal to the initial flow mass at $r + tc_j$.

If $\begin{pmatrix} f \\ x \end{pmatrix} \notin D(\mathcal{A})$, we can choose a sequence $\begin{pmatrix} f_n \\ x_n \end{pmatrix}_{n\in\mathbb{N}} \subset D(\mathcal{A})$ with $\begin{pmatrix} f_n \\ x_n \end{pmatrix} \to \begin{pmatrix} f \\ x \end{pmatrix}$ as $n \to +\infty$. From this follows for every $j = 1, \ldots, m$ that

$$\left[\pi_1 \mathcal{T}(t)\begin{pmatrix} f_n \\ x_n \end{pmatrix}\right]_j \Bigg|_{[0,1-tc_j)} = [f_n]_j \;|_{[0,1-tc_j)} \to [f]_j \;|_{[0,1-tc_j)} \text{ as } n \to +\infty.$$

We have

$$\lim_{n\to\infty} \mathcal{T}(t)\begin{pmatrix} f_n \\ x_n \end{pmatrix} = \mathcal{T}(t)\begin{pmatrix} f \\ x \end{pmatrix}, \tag{27}$$

hence, by the continuity of π_1,

$$\left[\pi_1 \mathcal{T}(t)\begin{pmatrix} f \\ x \end{pmatrix}\right]_j \Bigg|_{[0,1-tc_j)} = [f]_j \;|_{[0,1-tc_j)} .$$

Since the convergence in (27) is uniform for t in compact intervals and since π_2 is continuous, we obtain

$$\left[(\widetilde{\Phi}_w^-)^\top \pi_2 \mathcal{T}\left(t - \frac{1-r}{c_j}\right)\begin{pmatrix} f_n \\ x_n \end{pmatrix}\right]_j \to \left[(\widetilde{\Phi}_w^-)^\top \pi_2 \mathcal{T}\left(t - \frac{1-r}{c_j}\right)\begin{pmatrix} f \\ x \end{pmatrix}\right]_j$$

uniformly for $r \in [1 - tc_j, 1]$. This implies that

$$\left[\pi_1 \mathcal{T}(t)\begin{pmatrix} f_n \\ x_n \end{pmatrix}\right]_j \Bigg|_{[1-tc_j,1]} \to \left[(\widetilde{\Phi}_w^-)^\top \pi_2 \mathcal{T}\left(t - \frac{1-\cdot}{c_j}\right)\begin{pmatrix} f \\ x \end{pmatrix}\right]_j \Bigg|_{[1-tc_j,1]}$$

uniformly, hence in L^1-norm on $[1 - tc_j, 1]$. However, by (27) this limit is equal to

$$\left[\pi_1 \mathcal{T}(t)\begin{pmatrix} f \\ x \end{pmatrix}\right]_j \Bigg|_{[1-tc_j,1]},$$

and this completes the proof. $\square$

We can now prove the differentiability of the semigroup.

THEOREM 6.2. *The semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *generated by* $(\mathcal{A}, D(\mathcal{A}))$ *is differentiable for* $t > 2c$ *with* $c = \frac{1}{\min_j c_j}$.

Proof. We have to prove the differentiability of the orbits $t \mapsto \mathcal{T}(t)\begin{pmatrix} f \\ x \end{pmatrix}$ for $t > 2c$ and every $\begin{pmatrix} f \\ x \end{pmatrix} \in \mathcal{X}$. For this purpose fix a vector $\begin{pmatrix} f \\ x \end{pmatrix} \in \mathcal{X}$. We will show that both coordinates of $\mathcal{T}(t)\binom{f}{x}$ are differentiable for $t > 2c$. The formula

$$\mathcal{T}(t)\binom{f}{x} = \binom{f}{x} + \mathcal{A}\int_0^t \mathcal{T}(s)\binom{f}{x}ds \tag{28}$$

holds for any C_0-semigroup, see [11, Lemma II.1.9]. Using the form (10) of $\mathcal{A}$ and applying π_2 to both sides of (28) we obtain

$$\pi_2\mathcal{T}(t)\binom{f}{x} = x + \mathbf{B}\int_0^t \pi_2\mathcal{T}(r)\binom{f}{x}dr + M\int_0^t \pi_1\mathcal{T}(r)\binom{f}{x}dr.$$

Denoting

$$v(t) := \pi_2\mathcal{T}(t)\binom{f}{x},$$

this becomes

$$v(t) = x + \mathbf{B}\int_0^t v(r)dr + M\int_0^t \pi_1\mathcal{T}(r)\binom{f}{x}dr.$$

If $t > c$, then for every $j = 1, \dots, m$ and $s \in [0, 1]$ the relation $1 - tc_j < s \le 1$ holds. Using Lemma 6.1, we obtain

$$\begin{aligned} v(t) &= v(c) + \mathbf{B}\int_c^t v(r)dr + M\int_c^t \left(\left[(\widetilde{\Phi}_w^-)^\top v\left(r - \frac{1-\cdot}{c_j}\right)\right]_j\right)_{j=1,\dots,m} dr \\ &= v(c) + \mathbf{B}\int_c^t v(r)dr + \Phi_w^+\int_c^t \left(\left[(\widetilde{\Phi}_w^-)^\top v\left(r - \frac{1}{c_j}\right)\right]_j\right)_{j=1,\dots,m} dr. \end{aligned}$$

This formula and the continuity of $\mathbb{R}_+ \ni t \mapsto v(t) \in \partial X$ imply that the map $(c, +\infty) \ni t \mapsto v(t)$ is even continuously differentiable. Hence, the statement holds for $\pi_2\mathcal{T}(t)\binom{f}{x}$.

For the first coordinate we apply Lemma 6.1 again and obtain

$$w(t) := \pi_1\mathcal{T}(t)\binom{f}{x} = \left(\left[(\widetilde{\Phi}_w^-)^\top v\left(t - \frac{1-\cdot}{c_j}\right)\right]_j\right)_{j=1,\dots,m} \quad \text{for } t > c.$$

Observe that for every $s \in [0, 1]$, the function $(2c, +\infty) \ni t \mapsto w(t)(s)$ is continuously differentiable. We denote its derivative by

$$\dot{w}(t)(s) := \frac{d}{dt}w(t)(s).$$

We have to show that the vector-valued function $(2c, +\infty) \ni t \mapsto w(t) \in (L^1[0, 1])^m$ is differentiable. Let $t \in (2c, +\infty)$ be fixed and take a sequence $h_n \downarrow 0$. Then

$$\left| \frac{w(t + h_n)(s) - w(t)(s)}{h_n} - \dot{w}(t)(s) \right| \to 0 \text{ for every } s \in [0, 1]. \tag{29}$$

Clearly,

$$\dot{w}(t)(s) = \left(\left[(\widetilde{\Phi}_w^-)^\top \dot{v} \left(t - \frac{1 - s}{c_j} \right) \right]_j \right)_{j=1,\ldots,m}, \tag{30}$$

and this function is continuous in t (and in s) because $t - \frac{1-s}{c_j} > t - c > c$. Thus, for every $s \in [0, 1]$ there exist $0 \le \vartheta_{n,j}(s) \le h_n,\ j = 1, \ldots, m$, such that

$$\left[\frac{w(t + h_n)(s) - w(t)(s)}{h_n} \right]_j = \left[\dot{w}(t + \vartheta_{n,j}(s))(s) \right]_j .$$

Rewriting (29), we obtain

$$|[\dot{w}(t + \vartheta_{n,j}(s))(s)]_j - [\dot{w}(t)(s)]_j | \to 0 \text{ for every } s \in [0, 1],$$

$j = 1, \ldots, m$. To apply the Lebesgue dominated convergence theorem observe that by (30),

$$|[\dot{w}(t + \vartheta_{n,j}(s))(s)]_j - [\dot{w}(t)(s)]_j| \le 2 \sup_{r \in [t-c, t+1]} |[(\widetilde{\Phi}_w^-)^\top \dot{v}(r)]_j| \text{ for every } s \in [0, 1],$$

if n is large enough. We therefore obtain that

$$\left\| \left[\frac{w(t + h_n) - w(t)}{h_n} - \frac{d}{dt} w(t) \right]_j \right\|_{L^1[0,1]} \to 0,$$

$j = 1, \ldots, m$, and this is what we wanted to prove. □

We even obtain that the operators of the semigroup are compact for large t.

THEOREM 6.3. *The semigroup $(\mathcal{T}(t))_{t \ge 0}$ is eventually norm continuous and eventually compact.*

Proof. Since $R(\lambda, \mathcal{A})$ is compact by Remark 5.3 and $t \mapsto \mathcal{T}(t)$ is norm continuous for $t > 2c = 2\frac{1}{\min_j c_j}$ by the above theorem, we obtain from [11, Lemma II.4.28] that $\mathcal{T}(t)$ is compact for $t > 2c$. □

As a first consequence of the above result we observe that the Spectral Mapping Theorem from [11, Theorem IV.3.10] holds, hence the spectral bound and the growth bound of the semigroup coincide (for the definitions see [11, Definition I.1.12] and [11, Definition I.5.6]).

PROPOSITION 6.4. *For the semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *we have*

$$\sigma\,(\mathcal{T}(t)) \setminus \{0\} = \mathrm{e}^{t\sigma(\mathcal{A})},\ \ t \geq 0,$$

and

$$s(\mathcal{A}) = \omega_0(\mathcal{T}).$$

In particular, the semigroup is uniformly exponentially stable ($\omega_0(\mathcal{T}) < 0$*) if and only if the following implication holds:*

$$\lambda x - \mathbf{B}x - \mathbf{A}_\lambda x = 0 \textit{ for some } 0 \neq x \in \mathbb{C}^n \Rightarrow \mathrm{Re}\lambda < 0.$$

Proof. The first equalities follow by the eventually norm continuity of the semigroup, see [11, Theorem IV.3.10] and [11, Corollary IV.3.11]. The second statement follows from the characteristic equation (26) and the fact that the spectrum of the generator of an eventually norm continuous semigroup is bounded on halfplanes $\{\lambda : \mathrm{Re}\lambda \geq b\}$ (cf. [11, Theorem II.4.18]). □

The eventually compactness of the semigroup implies the following spectral decomposition.

PROPOSITION 6.5. *For the spectrum of* $\mathcal{A}$ *the decomposition*

$$\sigma(\mathcal{A}) = \Sigma_U \cup \Sigma_C \cup \Sigma_S$$

into closed subsets holds with

$$\begin{aligned} \Sigma_U &:= \sigma(\mathcal{A}) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}\,, \\ \Sigma_C &:= \sigma(\mathcal{A}) \cap \mathrm{i}\mathbb{R}, \\ \Sigma_S &:= \sigma(\mathcal{A}) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}\,. \end{aligned}$$

Here, Σ_U *and* Σ_C *are finite. Furthermore, all spectral points are eigenvalues with finite dimensional spectral projections.*

Proof. The statement is (i) and (ii) of [11, Corollary V.3.2]. □

Using the spectral mapping theorem from Proposition 6.4 this yields a corresponding decomposition of the spectrum of $\mathcal{T}(t)$.

COROLLARY 6.6. *For the spectrum of the semigroup operators the decomposition*

$$\sigma(\mathcal{T}(t)) = \sigma_U(t) \cup \sigma_C(t) \cup \sigma_S(t)$$

holds with

$$|\sigma_U(t)| > 1,\ \ |\sigma_S(t)| < 1,\ \ |\sigma_C(t)| = 1$$

for all $t \geq 0$*. Here,* $\sigma_U(t)$ *and* $\sigma_C(t)$ *are finite.*

Finally, this spectral decomposition implies a decomposition of the semigroup with the following asymptotic properties.

PROPOSITION 6.7. *There exist closed subspaces* $\mathcal{X}_S$, $\mathcal{X}_U$ *and* $\mathcal{X}_C$ *which are invariant under the semigroup such that* $\mathcal{X} = \mathcal{X}_S \oplus \mathcal{X}_U \oplus \mathcal{X}_C$, $\dim \mathcal{X}_C < \infty$, $\dim \mathcal{X}_U < \infty$, *and*

- *the semigroup* $\mathcal{T}_S(t) = \mathcal{T}(t)\mid_{\mathcal{X}_S}$ *is uniformly exponentially stable,*
- *the semigroup* $\mathcal{T}_U(t) = \mathcal{T}(t)\mid_{\mathcal{X}_U}$ *is invertible and the semigroup* $(\mathcal{T}_U^{-1}(t))$ *is uniformly exponentially stable,*
- *the semigroup* $\mathcal{T}_C(t) = \mathcal{T}(t)\mid_{\mathcal{X}_C}$ *is a polynomially bounded group, hence has growth bound* 0 *in both time directions.*

Proof. Using [11, Corollary V.3.2(iii)] we obtain the statement. □

7. Positivity

By the physical interpretation we expect that the semigroup describing the flow in the network is positive. Indeed, this is the case if the feedback matrix **B** satisfies a natural property. We first cite from [11, Theorem VI.1.8] the basic characterization for operators generating positive semigroups on Banach lattices (for the general theory see [17]).

PROPOSITION 7.1. *Let* $(B, D(B))$ *be the generator of a strongly continuous semigroup on a Banach lattice* Y.

(i) *The semigroup is positive if and only if the resolvent* $R(\lambda, B)$ *is a positive operator on* Y *for all* λ *large enough.*
(ii) *If* Y *is finite dimensional, the semigroup is positive if and only if the matrix* B *is real and positive off-diagonal.*

Based on the above criteria we can characterize the positivity of our semigroup.

THEOREM 7.2. *If* $\mathbf{B} \in \mathcal{M}_n(\mathbb{C})$ *is real and positive off-diagonal, then the semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *generated by* $(\mathcal{A}, D(\mathcal{A}))$ *is positive.*

Proof. By Proposition 7.1 we have to show that $R(\lambda, \mathcal{A})$ is positive for λ large enough. For this purpose we have to prove that the entries of the operator matrix (see (21))

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} [D_\lambda R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)M + I_X]\, R(\lambda, A_0) & D_\lambda R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \\ R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)MR(\lambda, A_0) & R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \end{pmatrix}$$

are all positive for large λ. As can be seen from (15), the resolvent $R(\lambda, A_0)$ is positive for real λ. Since Φ_w^+ is a positive matrix, also $MR(\lambda, A_0)$ is positive for real λ, see (7).

Under the above assumptions, $\mathbf{B}$ generates a positive (matrix)semigroup, hence $R(\lambda, \mathbf{B})$ is positive for λ large enough. Using the equality

$$\lambda - \mathbf{B} - \mathbf{A}_\lambda = (\mathbf{1} - \mathbf{A}_\lambda R(\lambda, \mathbf{B}))(\lambda - \mathbf{B})$$

and the Neumann series

$$R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) = R(\lambda, \mathbf{B}) \sum_{n=0}^{\infty} (\mathbf{A}_\lambda R(\lambda, \mathbf{B}))^n,$$

we obtain from $\mathbf{A}_\lambda \geq 0$ that $R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)$ is also positive for large λ. Combining all these facts with the positivity of $D_\lambda = \epsilon_\lambda (\widetilde{\Phi}_w^-)^\top$, we have that all the entries of (21) are positive for large λ. □

Combining the positivity and the eventually norm continuity of the semigroup $(\mathcal{T}(t))_{t\geq 0}$, we obtain that the generator $\mathcal{A}$ has a dominant eigenvalue (see [11, Theorem VI.1.10]). More precisely, the following holds.

PROPOSITION 7.3. *If* $\mathbf{B}$ *be is real and positive off-diagonal, then there exists* $\varepsilon > 0$ *such that*

$$\sigma(\mathcal{A}) = \{s(\mathcal{A})\} \cup \{\lambda \in \sigma(\mathcal{A}) : \operatorname{Re} \lambda \leq s(\mathcal{A}) - \varepsilon\}.$$

In order to determine the dominant eigenvalue $s(\mathcal{A})$ we first state an important property of the spectral bound function

$$s(\lambda) := s(\mathbf{B} + \mathbf{A}_\lambda),$$

which can be found in [4, Proposition 6.14].

LEMMA 7.4. *Let* $\mathbf{B}$ *be real and positive off-diagonal. Then the spectral bound function* $\mathbb{R} \ni \lambda \mapsto s(\lambda)$ *is decreasing and continuous.*

PROPOSITION 7.5. *Let* $\mathbf{B} \in \mathcal{M}_n(\mathbb{C})$ *be real and positive off-diagonal. Then* $s(\mathcal{A})$ *is the unique real solution of the characteristic equation*

$$\lambda = s(\lambda), \tag{31}$$

and for the spectral bound $s(\mathcal{A})$ *the following equivalences hold:*

$$s(\lambda) \lesseqqgtr \lambda \Longleftrightarrow s(\mathcal{A}) \lesseqqgtr \lambda.$$

Proof. From the assumption follows that $\mathbf{B}$, hence $(\mathcal{A}, D(\mathcal{A}))$ generate positive semigroups, see Theorem 7.2. Clearly $\sigma(\mathbf{B}) \neq \emptyset$, hence $-\infty < s(\mathbf{B}) \leq s(\lambda)$ for all $\lambda \in \mathbb{R}$ by the positivity of $\mathbf{A}_\lambda$, see [11, Corollary VI.1.11(ii)]. By the above Lemma 7.4 the equation (31) has a unique solution λ_0. Since each $\mathbf{B} + \mathbf{A}_\lambda$ generates a positive semigroup, we can again use [11, Theorem VI.1.10] and obtain $\lambda_0 = s(\lambda_0) \in \sigma(\mathbf{B} + \mathbf{A}_{\lambda_0})$, hence $\lambda_0 \in \sigma(\mathcal{A})$ by (26). However, for all $\mu > \lambda_0$, using Lemma 7.4, we have

$$\mu > \lambda_0 = s(\lambda_0) \geq s(\mu),$$

hence $\mu \notin \sigma(\mathbf{B} + \mathbf{A}_\mu)$ and so $\mu \in \rho(\mathcal{A})$ by (26). Therefore $\lambda_0 = s(\mathcal{A})$. The estimates on $s(\mathcal{A})$ follow from these considerations. □

From Proposition 6.4 and 7.5 we obtain a simple necessary and sufficient condition for the uniform exponential stability of the semigroup.

COROLLARY 7.6. *If* $\mathbf{B} \in \mathcal{M}_n(\mathbb{C})$ *is real and positive off-diagonal, then the semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *is uniformly exponentially stable if and only if the spectral bound* $s(\mathbf{B} + \mathbf{A}_0) < 0$.

Interpretation

The above characterization of uniform exponential stability depends on the spectral bound of $\mathbf{B} + \mathbf{A}_0$. Here $\mathbf{A}_0$ is a (weighted) adjacency matrix of our graph, see (20). The matrix $\mathbf{B}$ can be interpreted as a (weighted) adjacency matrix of an "imaginary" graph, whose vertices belong to the original graph but the edges do not. Its (directed) edges are those along which we control the outgoing flow velocities, depending on the outgoing flow mass in the vertices. This control happens immediately. Hence we can say that on these "imaginary edges" the information passes with infinite velocity. Observe that from Proposition 7.5 follows that

$$\begin{aligned} s(\mathcal{A}) &< 0 \iff s(\mathbf{B} + \mathbf{A}_0) < 0, \\ s(\mathcal{A}) &> 0 \iff s(\mathbf{B} + \mathbf{A}_0) > 0 \\ s(\mathcal{A}) &= 0 \iff s(\mathbf{B} + \mathbf{A}_0) = 0, \end{aligned}$$

that is only the *joint* structure of the original graph and the "imaginary graph" determines the asymptotic behavior of the system. That means, we can change "real" edges to "imaginary" edges and vice versa without changing the stability of the system. In other words:

"Stability is independent of the velocity of transportation."

8. Balanced exponential growth

In this section we always assume that $\mathbf{B} \in \mathcal{M}_n(\mathbb{C})$ is real and positive off-diagonal, hence the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is positive by Theorem 7.2. We now investigate in which sense $(\mathcal{T}(t))_{t\geq 0}$ converges to an equilibrium. For this purpose the concept of *irreducibility* is essential and we refer to [17, Chapter C–III.3] for a thorough treatment. We shall use the following characterization.

DEFINITION 8.1. A positive semigroup on $L^1(\Omega, \mu)$, μ a σ-finite measure, with generator B is *irreducible* if for all $\lambda > s(B)$ the resolvent $R(\lambda, B)$ maps positive nonzero functions to strictly positive functions.

In our case irreducibility can be characterized easily.

PROPOSITION 8.2. *If the matrix* $\mathbf{B} + \mathbf{A}_0$ *is irreducible, then the semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *is irreducible on* $\mathcal{X}$.

Proof. From Proposition 7.5 follows that $\lambda > s(\mathcal{A})$ holds if and only if $\lambda > s(\mathbf{B} + \mathbf{A}_\lambda)$. Since the zero patterns of $\mathbf{B} + \mathbf{A}_0$ and $\mathbf{B} + \mathbf{A}_\lambda$ coincide for every $\lambda \in \mathbb{C}$, the assumption implies that $\mathbf{B} + \mathbf{A}_\lambda$ is irreducible for every $\lambda \in \mathbb{R}$. Using [21, Proposition I.6.2] we obtain that for $\lambda > s(\mathcal{A})$ the matrix $R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)$ is strictly positive. Take now a vector $L^1([0,1], \mathbb{C}^m) \times \mathbb{C}^n \ni \begin{pmatrix} f \\ x \end{pmatrix} \gneqq 0$, and investigate $R(\lambda, \mathcal{A})\begin{pmatrix} f \\ x \end{pmatrix}$ using

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} [D_\lambda R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)M + I_X]\, R(\lambda, A_0) & D_\lambda R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \\ R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)MR(\lambda, A_0) & R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \end{pmatrix}$$

from (21). In the second coordinate we obtain

$$R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)MR(\lambda, A_0) f + R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)x,$$

where the second term is strictly positive by the above consideration. From the form (15) of $R(\lambda, A_0)$ follows that the function $R(\lambda, A_0)f$ is strictly positive except on the largest interval $(1-\varepsilon, 1]$ for which $f\,|_{(1-\varepsilon,1]} = 0$. Applying $M = \Phi_w^+ \otimes \delta_0$ to it we obtain a vector y of positive numbers, hence $R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)y$ yields a strictly positive vector. For the first coordinate we have

$$D_\lambda R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)Mf + R(\lambda, A_0) f + D_\lambda R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)x.$$

As before, $R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)Mf$ and $R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)x$ are strictly positive vectors of numbers. Using the strict positivity of exponential functions and the positivity of $(\widetilde{\Phi}_w^-)^\top$, we obtain that $D_\lambda = \epsilon_\lambda (\widetilde{\Phi}_w^-)^\top$ is strictly positive, hence the first and third terms are vectors of (everywhere) strictly positive functions. The second term is again positive, hence the sum yields a strictly positive vector of $L^1[0,1]$-functions. $\square$

By Proposition 2.11 the irreducibility of $\mathbf{B}+\mathbf{A}_0$ can be related to the strong connectedness of a graph. If the graph G is already strongly connected, then $\mathbf{A}_0$ is irreducible, hence for any positive off-diagonal $\mathbf{B}$, the matrix $\mathbf{B}+\mathbf{A}_0$ remains irreducible and the assumption in the above theorem is satisfied. If G is not strongly connected, we can describe the irreducibility of $\mathbf{B}+\mathbf{A}_0$ in the following way. Let us assume that $\mathbf{B}$ has positive entries $b_{ip} > 0$ for index pairs (ip) such that adding edges to G pointing from v_p to v_i we obtain a strongly connected graph. In this case $\mathbf{B}+\mathbf{A}_0$ is the (weighted) adjacency matrix of this strongly connected graph hence it becomes irreducible, and we again have the above result. The condition on the entries of $\mathbf{B}$ means that the outgoing flow is controlled along "imaginary" edges making the graph strongly connected. Hence, here again the *joint* structure of the "real" and the "imaginary" graph determines irreducibility.

COROLLARY 8.3. *Assume that after adding edges to G from v_p to v_i, where the corresponding entry of $\mathbf{B} = (b_{ip})_{n\times n}$ is different from* 0*, the graph G becomes strongly connected. Then the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is irreducible on $\mathcal{X}$.*

The following result on the asymptotics of the semigroup now follows from the general theory of positive semigroups (see [17, Chapter C–IV] and [11, Section V.3]) as can be found in [2, Section 3.5].

THEOREM 8.4. *Under the conditions of Corollary* 8.3 *there exists a one-dimensional projection $\mathcal{P} = \mu \otimes \mathsf{y}$ for strictly positive $\mathsf{y} \in \mathcal{X}$ and $\mu \in \mathcal{X}'$ such that*

$$\lim_{t\to+\infty} \|\mathrm{e}^{-s(\mathcal{A})t}\mathcal{T}(t) - \mathcal{P}\| = 0.$$

If $s(\mathcal{A}) = 0$ (e.g. $\mathbf{B} = -s(\mathbf{A}_0)\cdot\mathbf{1}$), then $(\mathcal{T}(t))_{t\geq 0}$ converges to the projection $\mathcal{P}$.

Proof. By Proposition 7.3 we know that the spectral bound $s(\mathcal{A})$ is a dominant eigenvalue of $\mathcal{A}$. By the irreducibility and [17, Proposition C-III.3.5], $s(\mathcal{A})$ is a first-order pole of the resolvent and the corresponding residue has the form $\mathcal{P} = \mu \otimes \mathsf{y}$, where μ and y are strictly positive eigenvectors of $\mathcal{A}'$ and $\mathcal{A}$, respectively. By [11, Corollary V.3.3] we now have the desired result. □

The property above is called *balanced exponential growth* (or *asynchronous exponential growth*) and plays an important role in applications, e.g., to population equations (see [10] and [20]).

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