3. Integrálszámítás

49. Alapintegrálokra visszavezethető feladatok

(a) $\int \frac{\sqrt[4]{x\cdot\sqrt[5]{x}}}{\sqrt[6]{x}} = \frac{15}{17}x^{\frac{17}{15}}$

(b) $\int (5\cdot 2^x + 4\cdot\sin x - 3\cos x) = \frac{5\cdot 2^x}{\ln 2} - 4\cos x - 3\sin x$

(c) $\int \tan^2 x = \tan x - x$

(d) $\int \frac{\cos^2 x - 5}{1+\cos(2x)} = \frac{1}{2}x - \frac{5}{2}\tan x$

(e) $\int \coth^2 x = -\coth x + x$

(f) $\int \frac{1}{\cosh x + \sinh x} = \sinh x - \cosh x$

(g) $\int \frac{1}{4\sqrt{5-5x^2}} = \frac{1}{4\sqrt{5}}\arcsin x$

(h) $\int \frac{x^2}{x^2+1} = x - \arctan x$

50. Integrálás helyettesítéssel

(a) $f(ax+b)$ alakú integrandus:
$F'(x) = f(x) \Rightarrow \int f(ax+b) = \frac{1}{a}\cdot(F(ax+b)) + c$

i. $\int \sqrt[4]{7x-16} = \frac{4}{35}(7x-16)^{\frac{5}{4}}$

ii. $\int \mathrm{e}^{5x+4} = \frac{1}{5}\mathrm{e}^{5x+4}$

iii. $\int \left(5 - \tanh^2(1-x)\right) = 4x - \tanh(1-x)$

iv. $\int \frac{7}{4x^2-4x+2} = \frac{7}{2}\arctan(2x-1)$

v. $\int \frac{1}{1-x} = -\ln|1-x|$

vi. $\int \frac{1}{(1-x)^2} = \frac{1}{1-x}$

vii. $\int \frac{1}{\sqrt{4x^2+4x}} = \frac{1}{2}\mathrm{arcosh}(2x+1)$

(b) $\int f^\alpha(x)\cdot f'(x) = \frac{f^{\alpha+1}(x)}{\alpha+1} + c, \alpha \neq -1$

i. $\int x^2\cdot(2x^3+4)^{99} = \frac{1}{6}\frac{(2x^3+4)^{100}}{100}$

ii. $\int \sin^4 x \sin(2x) = \frac{1}{3}\sin^6 x$

iii. $\int \frac{\sqrt{\ln^3 x}}{x} = \frac{2}{5}\ln^{\frac{5}{2}} x$

iv. $\int \frac{x}{\sqrt{x^2+1}} = \sqrt{x^2+1}$

v. $\int (x^2+1)\cdot\sqrt[3]{x^3+3x+1} = \frac{1}{4}(x^3+3x+1)^{\frac{4}{3}}$

vi. $\int 2^{x+1}\cdot\sqrt{2^x-1} = \frac{4}{3\ln 2}(2^x-1)^{\frac{3}{2}}$

vii. $\int \frac{1}{(1+x^2)\cdot\sqrt{\arctan x}} = 2\sqrt{\arctan x}$

viii. $\int \frac{\sin x\cdot\sqrt[3]{\tan^2 x - 1}}{\cos^3 x} = \frac{3}{8}(\tan^2 x - 1)^{\frac{4}{3}}$

ix. $\int \frac{\sin 2x}{(5-\sin^2 x)^7} = \frac{1}{6(5-\sin^2 x)^6}$

x. $\int \frac{\sin^5 x}{\cos^7 x} = \frac{1}{6}\tan^6 x$

xi. $\int 2\cdot\mathrm{e}^{2\cdot\sin x}\cdot\cos x = \mathrm{e}^{2\cdot\sin x}$

(c) $\int \frac{f'(x)}{f(x)} = \ln(|f(x)|) + c$

i. $\int \frac{x^3+2x}{x^4+4x^2+1} = \frac{1}{4}\ln(x^4+4x^2+1)$

ii. $\int \frac{\sin(2x)}{5+\cos^2 x} = -\ln(5+\cos^2 x)$

iii. $\int \frac{1}{\cosh^2 x\cdot\tanh x} = \ln|\tanh x|$

iv. $\int \tan x = -\ln|\cos x|$

v. $\int \frac{1}{\sqrt{1-x^2}\cdot\arcsin x} = \ln|\arcsin x|$

vi. $\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}+3} = \frac{1}{2}\ln(\mathrm{e}^{2x}+3)$

vii. $\int \frac{1}{x\cdot\ln x} = \ln|\ln x|$

(d) További feladatok helyettesítéssel

i. $\int \frac{1}{36+16x^2} = \frac{1}{24}\arctan\frac{2}{3}x$

ii. $\int \frac{1}{\sqrt{25x^2-16}} = \frac{1}{5}\mathrm{arcosh}\frac{5}{4}x$

iii. $\int \frac{\mathrm{e}^{2x}}{1+\mathrm{e}^x} = \mathrm{e}^x - \ln(1+\mathrm{e}^x)$

iv. $\int \sqrt{1-x^2} = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right)$

v. $\int \sqrt{x^2-1} = \frac{1}{2}\left(x\sqrt{x^2-1} - \mathrm{arcosh}x\right)$

51. Parciális integrálás

(a) Polinomfüggvénnyel szorzott exp, trigon. és hiperbolikus függvények

i. $\int (2x+3)\cdot \sin(6x) = -\frac{1}{6}(2x+3)\cos 6x + \frac{1}{18}\sin 6x$

ii. $\int x\cdot \mathrm{e}^{\pi x} = \frac{1}{\pi}x\mathrm{e}^{\pi x} - \frac{1}{\pi^2}\mathrm{e}^{\pi x}$

iii. $\int (1+2x^2)\cdot \cosh 3x = \frac{1}{27}(18x^2+13)\sinh 3x - \frac{4}{9}x\cosh 3x$

(b) Logaritmus, arcus és area függvények integrálása

i. $\int \ln x = x\ln x - x$

ii. $\int \arcsin x = x\arcsin x + \sqrt{1-x^2}$

iii. $\int \arctan x = x\arctan x - \frac{1}{2}\ln(1+x^2)$

iv. $\int \operatorname{arsinh} x = x\cdot \operatorname{arsinh} x - \sqrt{1+x^2}$

v. $\int \operatorname{arcosh} x = x\cdot \operatorname{arcosh} x - \sqrt{x^2-1}$

vi. $\int \operatorname{arcoth} x = x\cdot \operatorname{arcoth} x + \frac{1}{2}\ln|1-x^2|$

(c) Exp függvénnyel szorzott trigonometrikus és hiperbolikus függvények

i. $\int \mathrm{e}^{3x}\cdot \sin(2x) = \frac{1}{13}\mathrm{e}^{3x}(3\sin 2x - 2\cos 2x)$

ii. $\int 2^x\cdot \cos(3x-1) = \frac{1}{\ln^2 2+9}2^x(\ln 2\cos(3x-1) + 3\sin(3x-1))$

iii. $\int 3^{2x+1}\cdot \sinh(4x-1) = \frac{1}{2(\ln^2 3+4)}3^{2x+1}(\ln 3\sinh(4x-1) - 2\cosh(4x-1))$

(d) További feladatok (parciális integrálás)

i. $\int \ln^3 x = x\ln^3 x - 3x\ln^2 x + 6(x\ln x - x)$

ii. $\int \arcsin^2 x = x\arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2x$

iii. $\int \mathrm{e}^{\arcsin x} = \frac{1}{2}\mathrm{e}^{\arcsin x}\left(x+\sqrt{1-x^2}\right)$

iv. $\int \arctan\sqrt{x} = (x+1)\arctan\sqrt{x} - \sqrt{x}$

v. $\int \sin\sqrt{x} = 2\left(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}\right)$

vi. $\int \frac{\ln^2 x}{\sqrt[3]{x}} = x^{\frac{2}{3}}\left(\frac{3}{2}\ln^2 x - \frac{9}{2}\ln x + \frac{27}{4}\right)$

52. Racionális törtfüggvények integrálása

(a) Nevező $(ax+b)^n$, számláló elsőfokú vagy konstans

i. $\int \frac{14}{(6-4x)^7} = \frac{7}{12}\frac{1}{(6-4x)^6}$

ii. $\int \frac{5}{(2x+3)^4} = -\frac{5}{6}\frac{1}{(2x+3)^3}$

iii. $\int \frac{4x-3}{(3x-5)^3} = -\frac{4}{9}\frac{1}{3x-5} - \frac{11}{18}\frac{1}{(3x-5)^2}$

iv. $\int \frac{3x+1}{(1-2x)^{101}} = -\frac{1}{132}\frac{1}{(1-2x)^{99}} + \frac{1}{80}\frac{1}{(1-2x)^{100}}$

(b) Nevező másodfokú, számláló konstans

i. $\int \frac{2}{3x^2+6x+15} = \frac{1}{6}\arctan\frac{x+1}{2}$

ii. $\int \frac{1}{2x^2-3x+20} = \frac{2}{\sqrt{151}}\arctan\left(\frac{2}{\sqrt{151}}(x-\frac{3}{4})\right)$

iii. $\int \frac{1}{x^2+6x+9} = -\frac{1}{x+3}$

iv. $\int \frac{1}{x^2+8x+12} = \frac{1}{4}\ln\left|\frac{x+2}{x+6}\right|$

(c) Nevező másodfokú, számláló elsőfokú

i. $\int \frac{2x-3}{x^2+4x-5} = -\frac{1}{6}\ln|x-1| + \frac{13}{6}\ln|x+5|$

ii. $\int \frac{5x-6}{x^2-2x+10} = \frac{5}{2}\ln(x^2-2x+10) - \frac{1}{3}\arctan\frac{x-1}{3}$

iii. $\int \frac{x+2}{x^2-x+2} = \frac{1}{2}\ln(x^2-x+2) + \frac{5}{\sqrt{7}}\arctan\left(\frac{2}{\sqrt{7}}x - \frac{1}{\sqrt{7}}\right)$

(d) Parciális törtekre bontás

i. $\int \frac{14}{(x-3)\cdot(x+2)\cdot(x-4)} = \frac{7}{15}\ln\frac{|x+2|}{(x-3)^2} + \frac{7}{3}\ln|x-4|$

ii. $\int \frac{x^3-4}{5x^3-x} = \frac{1}{5}x + 2\ln\frac{x^2}{|5x^2-1|} - \frac{1}{10\sqrt{5}}\ln\left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right|$

iii. $\int \frac{x^4}{(x-1)(x+2)} = \frac{x^3}{3} - \frac{x^2}{2} + 3x + \frac{1}{3}\ln\frac{|x-1|}{(x+2)^{16}}$

iv. $\int \frac{5}{x(x^2+4)} = \frac{5}{4}\ln|x| - \frac{5}{8}\ln(x^2+4)$

v. $\int \frac{2x-4}{(x+1)^2(x-1)^2} = \ln\left|\frac{x-1}{x+1}\right| + \frac{3}{2}\frac{1}{x+1} + \frac{1}{2}\frac{1}{x-1}$

vi. $\int \frac{2x^2}{x^4-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + \arctan x$

53. Trigonometrikus függvények racionális kifejezéseinek integrálása

(a) $\int \sin^7 x = \frac{1}{7}\cos^7 x - \frac{3}{5}\cos^5 x + \cos^3 x - \cos x$

(b) $\int \cos^4 x = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$

(c) $\int \sin^3 x \cdot \cos^4 x = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7}$

(d) $\int \sin^4 x \cdot \cos^6 x = \frac{3}{256}x - \frac{1}{256}\sin 4x + \frac{1}{2048}\sin 8x + \frac{1}{320}\sin^5 2x$

(e) $\int \sin^2 2x \cdot \cos^3 x = \frac{4}{3}\sin^3 x - \frac{8}{5}\sin^5 x + \frac{4}{7}\sin^7 x$

(f) $\int \cos^2 2x \cdot \cos 3x = -\frac{16}{7}\sin^7 x + 4\sin^5 x - \frac{8}{3}\sin^3 x + \sin x$

54. $t := \tan\frac{x}{2}$ helyettesítés: $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

(a) $\int \frac{1}{\sin x} = \ln\left|\tan\frac{x}{2}\right|$

(b) $\int \frac{1}{\cos x} = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right|$

(c) $\int \frac{1+\sin x}{1-\cos x} = 2\ln\left|\tan\frac{x}{2}\right| - \ln\left(1+\tan^2\frac{x}{2}\right) - \cot\frac{x}{2}$

(d) $\int \frac{1}{1+\cos x} = \tan\frac{x}{2}$

Gyakorló feladatok

55. $\int \frac{x^4-4x^3+2\sqrt[3]{x}}{\sqrt[5]{x^4}} = \frac{5}{21}x^{\frac{21}{5}} - \frac{5}{4}x^{\frac{16}{5}} + \frac{15}{4}x^{\frac{8}{15}}$

56. $\int \frac{5\cdot\cos 2x}{\sin x+\cos x} = 5\sin x + 5\cos x$

57. $\int \left(\frac{3}{\cos^2 x} - \frac{7}{5\cdot\sin^2 x}\right) = 3\tan x - \frac{7}{5}\cot x$

58. $\int \left(x^{-2}+x^{-1}-2\cdot 3^{x-1}\right) = \frac{-1}{x} + \ln|x| - 2\frac{3^{x-1}}{\ln 3}$

59. $\int \frac{1}{\sqrt{6+6x^2}} = \frac{1}{\sqrt{6}}\text{arsinh}x$

60. $\int \frac{5}{4-4x^2} = \frac{5}{4}\ln\left|\frac{x+1}{x-1}\right|$

61. $\int \sin(4x+5) = -\frac{1}{4}\cos(4x+5)$

62. $\int \sinh(2-7x) = -\frac{1}{7}\cosh(2-7x)$

63. $\int \frac{5}{\cos^2(-6x+4)} = -\frac{5}{6}\tan(-6x+4)$

64. $\int \tan^2(2-3x) = -\frac{1}{3}\tan(2-3x) - x$

65. $\int \frac{1}{\sinh^2(1-x)} = \coth(1-x)$

66. $\int (3x^2-\sin x)(x^3+\cos x) = \frac{1}{2}\left(x^3+\cos x\right)^2$

67. $\int \frac{\ln x}{x} = \frac{1}{2}\ln^2 x$

68. $\int \mathrm{e}^x\cdot\sqrt{(\mathrm{e}^x+2005)^{27}} = \frac{2}{29}\left(\mathrm{e}^x+2005\right)^{\frac{29}{2}}$

69. $\int \frac{x}{1+x^2} = \frac{1}{2}\ln(1+x^2)$

70. $\int \frac{100}{(x+1)\cdot\ln(x+1)} = 100\ln|\ln(x+1)|$

71. $\int \frac{1}{(x^2+1)\cdot\arctan x} = \ln|\arctan x|$

72. $\int \cos\frac{x}{2}\cos\frac{x}{3} = 3\sin\frac{x}{6} + \frac{3}{5}\sin\frac{5x}{6}$

73. $\int \frac{1}{\sin^2 x\cos^2 x} = \tan x - \cot x$

74. $\int \frac{1}{\sin x\cos^3 x} = \frac{1}{2}\tan^2 x + \ln|\tan x|$

75. $\int \frac{1}{\cos^4 x} = \frac{1}{3}\tan^3 x + \tan x$

76. $\int \frac{x}{\sqrt[3]{1-3x}} = \frac{1}{15}(1-3x)^{\frac{5}{3}} - \frac{1}{6}(1-3x)^{\frac{2}{3}}$

77. $\int \frac{\arctan\sqrt{x}}{\sqrt{x}}\frac{1}{1+x} = \arctan^2\sqrt{x}$

78. $\int \frac{x}{4+x^4} = \frac{1}{4}\arctan\frac{x^2}{2}$

79. $\int \frac{x^3}{x^8+3} = \frac{1}{4\sqrt{3}}\arctan\frac{x^4}{\sqrt{3}}$

80. $\int x\frac{\mathrm{e}^{\arctan x}}{(1+x^2)^{\frac{3}{2}}} = \frac{x-1}{2\sqrt{x^2+1}}\mathrm{e}^{\arctan x}$

81. $\int (\arcsin x)^2 = x\arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - x$

82. $\int (\sin x)\ln(\tan x) = \ln\left|\tan\frac{x}{2}\right| - \cos x\cdot\ln(\tan x)$

83. $\int \left(\frac{\ln x}{x}\right)^2 = -\frac{\ln^2 x}{x} - 2\frac{\ln x}{x} - \frac{2}{x}$

84. $\int \frac{x}{\cos^2 x} = x\tan x + \ln|\cos x|$

85. $\int \frac{x^3}{(x-1)^{100}} = -\frac{1}{96}(x-1)^{-96} - \frac{3}{97}(x-1)^{-97} - \frac{3}{98}(x-1)^{-98} - \frac{1}{99}(x-1)^{-99}$

86. $\int \ln\left(x+\sqrt{1+x^2}\right) = x\text{arsinh}x - \sqrt{1+x^2}$

87. $\int \frac{\arcsin x}{x^2}\frac{1+x^2}{\sqrt{1-x^2}} = -\arcsin x\cdot\frac{\sqrt{1-x^2}}{x} + \frac{\arcsin^2 x}{2} + \ln|x|$

88. $*\int \frac{x\mathrm{e}^x}{(x+1)^2} = \frac{\mathrm{e}^x}{x+1}$

89. $*\int \frac{x^{2n-1}}{x^n+1} = \frac{1}{n}x^n - \frac{1}{n}\ln|x^n+1|$