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Spectral properties and asymptotic periodicity of flows in networks

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Abstract We combine functional analytical and graph theoretical methods in order to study flows in networks. We show that these flows can be described by a strongly continuous operator semigroup on a Banach space. Using Perron-Frobenius spectral theory we then prove that this semigroup behaves asymptotically periodic.

1 Introduction

Networks have been studied widely in recent years with motivations from and applications to classical natural sciences (electro-circuits, chemical processes, neural networks, population biology, etc.) as well as to social sciences or even to the World Wide Web. Much progress has been made in understanding the structure of these networks, and we refer to M.E.J. Newman [18] for a recent survey on these developments. However, on p. 224 of [18] he says: *"The next logical step after developing models of network structure, (...) is to look at the behavior of models of physical (or biological or social) processes going on on those networks. Progress on this front has been slower than progress on understanding network structure."* The main goal of the present work is to define an appropriate setting and to find the tools to investigate such processes on networks.

Several discrete or combinatorial interactions in networks have been treated in graph theory, mostly with applications to Markov processes (see e.g. [10] and

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references therein). We would like to introduce another aspect into (discrete) graph theory and are interested in so called *dynamical graphs*. Here the edges do not only link the vertices but also serve as a transmission media allowing time and space depending processes between them. Such problems have been studied by, e.g., S. Nicaise, J. von Below, F. Ali Mehmeti (see [1],[2],[3],[13]) with various diffusion processes in networks or more general structures.

In this paper, we discuss transport processes or *flows in networks*. Using spectral theory and semigroup methods we will be able to describe precisely the asymptotic behavior of such dynamical graphs. In fact, it turns out that such flows, under appropriate assumptions, converge towards a periodic flow whose period is determined by the structure of the graph (see Theorems 4.5 and 4.10 below). In our approach we make use of tools and results from various fields such as partial differential equations, theory of operator semigroups, graph theory and linear algebra.

We describe the flow in a finite network by the following equations:

$$(F)\begin{cases} \frac{\partial}{\partial t}u_j(x,t) = c_j\frac{\partial}{\partial x}u_j(x,t), \ x\in(0,1), t\geq 0, & \\ u_j(x,0) = f_j(x), \ x\in(0,1), & (IC) \\ \phi_{ij}^{-}u_j(1,t) = \omega_{ij}\sum_{k=1}^{m}\phi_{ik}^{+}u_k(0,t), \ t\geq 0, & (BC) \end{cases}$$

for $i = 1, ..., n$, and $j = 1, ..., m$.

The network is modelled by a simple, directed, topological graph G having vertices $\mathsf{v}_1, \ldots, \mathsf{v}_n$ and directed edges (or *arcs*) $\mathsf{e}_1, \ldots, \mathsf{e}_m$, normalized as $\mathsf{e}_j = [0,1]$. The arcs are parameterized contrary to the direction of the flow, i.e., every arc has its *tail* at the endpoint 1 and its *head* at the endpoint 0. We use terminology that is common to graph theory and refer to any monograph on that subject (see, e.g., [4], [7], [9] or [10]).

The distribution of material along an edge e_j at time $t \geq 0$ is described by the functions $u_j(x,t)$ for $x \in [0,1]$. The constants $c_j > 0$ are the velocities of the flow on each arc e_j. We arrange them into the diagonal matrix

$$C := \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_m \end{pmatrix}. \tag{1}$$

The boundary conditions (BC) depend on the structure of the network and are described by the following matrices. First we define the *outgoing incidence matrix* $\Phi^{-} = \left(\phi_{ij}^{-}\right)_{n\times m}$ with

$$\phi_{ij}^{-} := \begin{cases} 1, & \mathsf{v}_i = \mathsf{e}_j(1), \\ 0, & \text{otherwise.} \end{cases}$$

Accordingly, we call the edge e_j an *outgoing edge for* v_i if $\mathsf{v}_i = \mathsf{e}_j(1)$ holds. Respectively, we define the *incoming incidence matrix* $\Phi^+ = \left(\phi^+_{ij}\right)_{n\times m}$ with

$$\phi^+_{ij} := \begin{cases} 1, & \mathsf{v}_i = \mathsf{e}_j(0), \\ 0, & \text{otherwise,} \end{cases}$$

and call the edge e_j an *incoming edge for* v_i if $\mathsf{v}_i = \mathsf{e}_j(0)$ holds.

Remark 1.1. The matrix

$$\Phi = \Phi^+ - \Phi^-$$

is called the *incidence matrix* of the directed graph G.

We further define the *weighted outgoing incidence matrix* $\Phi^-_w = \left(\omega_{ij}\right)_{n\times m}$ with entries $0 \le \omega_{ij} \le 1$ satisfying

$$\omega_{ij} = \phi^-_{ij}\omega_{ij} \text{ and } \sum_{j=1}^{m} \omega_{ij} = 1 \text{ for all } i = 1, ..., n,\ j = 1, ..., m. \tag{2}$$

The entry ω_{ij} expresses the proportion of the mass leaving the vertex v_i into the edge e_j and we assume that if e_j is an outgoing edge of v_i then $\omega_{ij} \neq 0$. The boundary conditions (BC) together with (2) imply the *Kirchhoff law*

$$\sum_{j=1}^{m} \phi^-_{ij} u_j(1,t) = \sum_{j=1}^{m} \phi^+_{ij} u_j(0,t),\ i = 1, ..., n, \tag{3}$$

i.e., in each vertex the total outgoing flow is equal to the total incoming flow. This condition makes sense only if we assume that in every vertex there is at least one outgoing as well as at least one incoming edge.

The $n \times n$ matrix

$$\mathbb{A} := \Phi^+ \left(\Phi^-_w\right)^\top$$

will play an important role in our studies. Since the nonzero entries of Φ^-_w coincide with the nonzero entries of Φ^-, the matrix $\mathbb{A}$ is actually a *weighted (transposed) adjacency matrix* of the graph G. This means that its entry a_{ij} is different from zero if and only if there is an arc from the vertex v_j to the vertex v_i. Indeed,

$$(\mathbb{A})_{ij} = \begin{cases} \omega_{jk}, & \text{if } \mathsf{v}_i = \mathsf{e}_k(0) \text{ and } \mathsf{v}_j = \mathsf{e}_k(1), \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

By condition (2), $\mathbb{A}$ is column stochastic.

Analogously, the $m \times m$ matrix

$$\mathbb{B} := \left(\Phi^-_w\right)^\top \Phi^+ \tag{5}$$

is the *weighted (transposed) adjacency matrix* of the *line graph*, which is roughly the graph obtained from G by exchanging the role of the vertices and edges (maintaining the directions). Therefore, we have

$$(\mathbb{B})_{ij} = \begin{cases} \omega_{ki}, & \text{if } \mathsf{e}_j(0) = \mathsf{e}_i(1) = \mathsf{v}_k, \\ 0, & \text{otherwise.} \end{cases}$$

Again, the matrix $\mathbb{B}$ is column stochastic by (2).

To treat our problem (F) we rewrite it in the form of an abstract Cauchy problem and prove its well-posedness using semigroup methods with [8] as a standard reference. We then focus on spectral properties of the generator. This leads in Section 4 to a precise description of the asymptotic behavior of the solutions.

2 Well-posedness

Our first aim is to write the equations (F) in the form of an abstract Cauchy problem on a Banach space (see [8, Definition II.6.1]). To this purpose we introduce the *state space* of L^1-functions on the edges

$$X := \left(L^1[0,1]\right)^m,$$

on which we define the operator

$$A_w := \begin{pmatrix} c_1 \frac{d}{dx} & & 0 \\ & \ddots & \\ 0 & & c_m \frac{d}{dx} \end{pmatrix} \tag{6}$$

with (dense) domain

$$D(A_w) := \left\{ v \in \left(W^{1,1}[0,1]\right)^m \mid v(1) \in \operatorname{ran}(\Phi_w^-)^\top \right\}.$$

Before proceeding we explain the condition appearing in the definition of $D(A_w)$. The nonzero elements in the i-th row of the matrix Φ_w^- correspond to the arcs with tail v_i, and in each column of Φ_w^- there is exactly one nonzero entry. Therefore, the condition

$$v(1) = (\Phi_w^-)^\top d \text{ for some } d \in \mathbb{C}^n \tag{7}$$

implies for fixed j that

$$v_j(1) = \omega_{ij} d_i \text{ if } \omega_{ij} \neq 0.$$

Note, that the index i is uniquely defined by j and the condition $\omega_{ij} \neq 0$. If $\omega_{ik} \neq 0$ for some other k, $1 \leq k \leq m$, then (7) implies again

$$v_k(1) = \omega_{ik} d_i,$$

that is,

$$\frac{v_j(1)}{\omega_{ij}} = \frac{v_k(1)}{\omega_{ik}}.$$

This means that values of v on the arcs with the same tail are related by the corresponding weights.

The boundary conditions (BC) will now be added using two *boundary operators* L and M (see [5] where this terminology is explained and used in an abstract framework). To that purpose we call

$$\partial X := \mathbb{C}^n$$

the *boundary space*, that is the space of flow mass in the vertices, and introduce first the *outgoing boundary operator* $L : X \to \partial X$:

$$L := \Phi^- \otimes \delta_1, \quad D(L) := \left(W^{1,1}[0,1]\right)^m,$$

where δ_1 is the point evaluation at 1.

Remark 2.1. The operator L is surjective from $D(A_w)$ to ∂X.

Proof. It suffices to observe that $D(A_w)$ contains all constant functions v satisfying the boundary condition (7), i.e., $v \equiv (\Phi_w^-)^\top d$ for some $d \in \partial X$, and that $\Phi^- \left(\Phi_w^-\right)^\top = \mathbf{1}$ where $\mathbf{1}$ denotes the $n \times n$ identity matrix. □

The incoming flow will be taken into account by the *incoming boundary operator* $M : X \to \partial X$,

$$M := \Phi^+ \otimes \delta_0, \quad D(M) := \left(W^{1,1}[0,1]\right)^m, \tag{8}$$

where δ_0 is the point evaluation at 0. Observe that the equation $Lv = Mv$ expresses the Kirchhoff law (3) for each vertex.

After these preparations we are ready to introduce the operator corresponding to the problem (F).

Definition 2.2. *On the Banach space* $X = \left(L^1[0,1]\right)^m$ *we define the operator*

$$D(A) := \{v \in D(A_w) \mid Lv = Mv\}, \tag{9}$$
$$Av := A_w v.$$

A simple calculation shows that the conditions appearing in the domain of A in (9) are equivalent to (BC), hence the Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = u_0 \end{cases} \tag{10}$$

with $u_0 = \left(f_j\right)_{j=1,\dots,m}$ is an abstract version of our original problem. By standard semigroup theory (see [8, Theorem II.6.7]) this problem is well-posed if and only if A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X. In this case, the solutions of (10) have the form $u(t) = T(t)u_0$ yielding solutions also for (F). To show the generator property we will use the Phillips theorem and the following notion from the theory of positive semigroups on Banach lattices (see [16, Section C-II.1]).

Definition 2.3. *An operator A on a Banach lattice X is called* dispersive *if for every $v \in D(A)$ one has* $\mathrm{Re}\,\langle Av, \phi\rangle \leq 0$ *for some $\phi \in X'_+$ such that $\|\phi\| \leq 1$ and $\langle v, \phi\rangle = \|v^+\|$.*

In order to check this property for our operator A we use a new, equivalent lattice norm on X defined as:

$$\|f\|_c := \sum_{k=1}^{m} \int_0^1 \frac{1}{c_k}\, |f_k(s)|\, ds. \tag{11}$$

Lemma 2.4. *The operator $(A, D(A))$ is dispersive on the Banach lattice $(X, \|\cdot\|_c)$.*

Proof. Let $v \in D(A)$. Define $\phi = (\phi_k)_{k=1,\dots,m}$ by

$$\phi_k(s) := \begin{cases} \frac{1}{c_k}, & v_k(s) > 0, \\ 0, & \text{else.} \end{cases}$$

Observe that $\phi \in (L^\infty[0,1])^m$, hence it is in X', and it satisfies all the conditions in the Definition 2.3 for the new norm defined in (11). Since the operator A is real, it suffices to prove that

$$\langle Av, \phi\rangle \leq 0.$$

From the definition of A and ϕ we obtain

$$\begin{aligned}\langle Av, \phi\rangle &= \sum_{k=1}^{m} \int_0^1 c_k v_k'(s)\phi_k(s)ds = \sum_{k=1}^{m} \int_0^1 c_k v_k'(s) \frac{1}{c_k} \chi_{\{v_k>0\}} ds \\ &= \left\langle [v(1)]^+ - [v(0)]^+ , 1_{\mathbb{R}^m}\right\rangle_{\mathbb{R}^m} ,\end{aligned}$$

where $1_{\mathbb{R}^m}$ denotes the constant 1 vector in $\mathbb{R}^m$. Furthermore, for $v \in D(A)$ we have $Lv = Mv$ and $v(1) \in \mathrm{ran}\left(\Phi_w^-\right)^\top$ which implies

$$\begin{aligned}\Phi^- v(1) &= \Phi^+ v(0), \\ v(1) &= \left(\Phi_w^-\right)^\top d\end{aligned}$$

for some $d \in \partial X = \mathbb{C}^n$. Since $\Phi^- \left(\Phi_w^-\right)^\top = \mathbf{1}$ where $\mathbf{1}$ denotes the $n \times n$ identity matrix, we have

$$\Phi^- \left(\Phi_w^-\right)^\top d = d = \Phi^+ v(0)$$

and

$$v(1) = \left(\Phi_w^-\right)^\top d = \left(\Phi_w^-\right)^\top \Phi^+ v(0) = \mathbb{B} v(0).$$

Here, $\mathbb{B}$ is the positive column stochastic matrix defined in (5). Continuing the above estimate and using the positivity of $\mathbb{B}$ we obtain

$$\begin{aligned}\langle Av, \phi\rangle &= \left\langle [\mathbb{B}v(0)]^{+} - [v\,(0)]^{+}\,, 1_{\mathbb{R}^m}\right\rangle_{\mathbb{R}^m} \\ &\leq \left\langle \mathbb{B}\,[v(0)]^{+} - [v\,(0)]^{+}\,, 1_{\mathbb{R}^m}\right\rangle_{\mathbb{R}^m} = \left\langle [v(0)]^{+}\,, \mathbb{B}^{\top} 1_{\mathbb{R}^m} - 1_{\mathbb{R}^m}\right\rangle_{\mathbb{R}^m} = 0\end{aligned}$$

because of the column stochasticity of $\mathbb{B}$. □

Based on this property, we can show that the operator A generates a C_0-semigroup of *positive* operators on the Banach lattice X. We refer to [16] and [8, Section VI.1.b] for a thorough treatment of these semigroups.

Proposition 2.5. *The operator* $(A, D(A))$ *generates a positive bounded semigroup* $(T(t))_{t\geq 0}$.

Proof. Since $\left(W^{1,1}\,[0,1]\right)^m$ is dense in $\left(L^1[0,1]\right)^m$, a simple calculation shows that $D(A)$ is dense in X. It is easy to see that the operator $(A, D\,(A))$ is also closed. As we will see in Corollary 3.4, its resolvent set contains $\mathbb{R}_+$. Since it is dispersive on $(X, \|\cdot\|_c)$, the Phillips Theorem (cf. [16, Theorem C-II.1.2]) assures that it generates a positive semigroup $(T(t))_{t\geq 0}$ with

$$\|T(t)\| \leq \frac{R}{r}\,\|T(t)\|_c \leq \frac{R}{r},$$

where from (11) follows

$$\begin{aligned} r &:= \min\left\{\frac{1}{c_k} : k = 1, ..., m\right\}, \\ R &:= \max\left\{\frac{1}{c_k} : k = 1, ..., m\right\}. \end{aligned}$$

□

Observe that in the special case when the velocities on the arcs are all equal, in the above proof holds $r = R$ and we even obtain contractions.

Corollary 2.6. *If* $c_i = c$ *for all* $i = 1, ..., m$, *then the semigroup* $(T(t))_{t\geq 0}$ *is contractive.*

This gives the desired result for our original problem.

Corollary 2.7. *The problem* (F) *is well posed.*

3 Spectral properties

In order to obtain (in Section 4) qualitative properties of the solutions of (F), or of the semigroup generated by A, we now start a careful analysis of the spectrum of A. To that purpose we use a perturbation method as proposed in [17] and first introduce the operator

$$A_0 := A_w \,|_{\ker L}$$

with domain

$$D(A_0) = \{v \in D\,(A_w) : Lv = 0\}.$$

This means that we consider homogeneous boundary conditions where the right hand side of (BC) is equal to zero. In fact, the domain of A_0 is simply

$$D\,(A_0) = \left(W_0^{1,1}\,[0,1]\right)^m = \left\{v \in \left(W^{1,1}\,[0,1]\right)^m : v\,(1) = 0\right\}.$$

The corresponding Cauchy problem

$$\begin{cases} \dot{u}\,(t) = A_0 u\,(t)\,, & t \geq 0, \\ u\,(0) = u_0, \end{cases} \tag{12}$$

is well-posed since $(A_0, D(A_0))$ generates the nilpotent translation semigroup on X given by

$$(T_0(t)f)_i(s) = \begin{cases} f_i(s + c_i t), & s + c_i t \leq 1, \\ 0, & \text{otherwise}. \end{cases} \tag{13}$$

The resolvent of A_0 exists for every $\lambda \in \mathbb{C}$ and can be computed as

$$(R(\lambda, A_0)f)\,(s) = \int_s^1 \epsilon_\lambda\,(s - \tau + 1)\,C^{-1} f(\tau) d\tau,\; s \in [0,1]\,,\; f \in X, \tag{14}$$

with

$$\epsilon_\lambda(s) := \operatorname{diag}\left(\exp\left(\frac{\lambda}{c_i}\,(s-1)\right)\right)_{i=1,\ldots,m}, \quad s \in [0,1] \tag{15}$$

and C defined in (1). In order to compute the spectrum of the generator A we use operator matrix techniques as developed by A. Rhandi [20] and R. Nagel [17] and extend A to an operator on the product space

$$\mathcal{X} := X \times \partial X.$$

To that purpose we first consider the operator matrix

$$\mathcal{A}_0 := \begin{pmatrix} A_w & 0 \\ -L & 0 \end{pmatrix}, \quad D\,(\mathcal{A}_0) := D(A_w) \times \{0\}^n,$$

whose part on the closure of its domain

$$\overline{D(\mathcal{A}_0)} = \overline{D(A_w) \times \{0\}^n} = X \times \{0\}^n =: \mathcal{X}_0$$

can be identified with $(A_0, D(A_0))$.

Using ideas of Greiner [11] we are able to compute the resolvent of $\mathcal{A}_0$. To that purpose we observe that the conditions of [11, Lemma 1.2] are fulfilled (use Remark 2.1), hence $D(A_w) = D(A_0) \oplus \ker(\lambda - A_w)$ and $L\mid_{\ker(\lambda - A_w)}$ is invertible for any $\lambda \in \rho(A_0) = \mathbb{C}$. We denote its inverse by

$$D_\lambda := \left(L\mid_{\ker(\lambda - A_w)}\right)^{-1} : \partial X \to \ker(\lambda - A_w).$$

Lemma 3.1. *For every $\lambda \in \mathbb{C}$, the resolvent of $\mathcal{A}_0$ is given by*

$$R(\lambda, \mathcal{A}_0) = \begin{pmatrix} R(\lambda, A_0) & D_\lambda \\ 0 & 0 \end{pmatrix}. \tag{16}$$

Here the operator D_λ has the form

$$D_\lambda = \epsilon_\lambda (\Phi_w^-)^\top, \tag{17}$$

that is

$$(D_\lambda d)(x) = \epsilon_\lambda(x) \cdot (\Phi_w^-)^\top d \textit{ for any } d \in \partial X,\ x \in [0, 1],$$

and $\epsilon_\lambda(x)$ is defined in (15).

Proof. A simple calculation shows that the inverse of $(\lambda - \mathcal{A}_0)$ is given by the operator matrix in (16). To show (17) we set $N_\lambda := \epsilon_\lambda (\Phi_w^-)^\top$ and compute

$$LN_\lambda = \Phi^-(\Phi_w^-)^\top = \mathbf{1},$$

where $\mathbf{1}$ denotes the $n \times n$ identity matrix. We also need to show that

$$N_\lambda L\mid_{\ker(\lambda - A_w)} = I_{\ker(\lambda - A_w)}.$$

Observe, that the kernel of the operator $\lambda - A_w$ is spanned by the vectors $v(x) = \left(a_i e^{\frac{\lambda}{c_i}(x-1)}\right)_{i=1,\ldots,m}$ satisfying (7). This means that

$$v(1) = (a_i) = (\Phi_w^-)^\top d \ \text{ for some } d \in \partial X,$$

i.e., by (15),

$$v = \epsilon_\lambda (\Phi_w^-)^\top d = N_\lambda d \ \text{ for some } d \in \partial X.$$

Hence,

$$Lv = d \text{ and } N_\lambda L v = N_\lambda d = v,$$

which implies (17). ◻

In the next step we introduce a perturbing matrix

$$\mathcal{B} := \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}, \quad D(\mathcal{B}) := D(M) \times \partial X.$$

Adding $\mathcal{B}$ to $\mathcal{A}_0$ we obtain an operator on $\mathcal{X}$ given by

$$\begin{aligned} D(\mathcal{A}) &:= D(\mathcal{A}_0) = D(A_w) \times \{0\}^n, \\ \mathcal{A} &:= \mathcal{A}_0 + \mathcal{B} = \begin{pmatrix} A_w & 0 \\ M - L & 0 \end{pmatrix}. \end{aligned}$$

Remark 3.2. The part of the operator matrix $\mathcal{A}$ in $\mathcal{X}_0$ is

$$\begin{aligned} D\left(\mathcal{A}\,|_{\mathcal{X}_0}\right) &= D(A) \times \{0\}^n, \\ \mathcal{A}\,|_{\mathcal{X}_0} &= \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{18}$$

Hence it can be identified with the operator A on X.

The extension of the operator A to the operator matrix $\mathcal{A}$ helps to determine the spectrum of A using a simple perturbation argument. As a result, $\sigma(A)$ can be determined by a "characteristic equation" in $\partial X = \mathbb{C}^n$. This is based on the fact that for every $\lambda \in \mathbb{C}$, the product MD_λ is well-defined and yields an operator on ∂X – that is a $n \times n$ matrix.

Proposition 3.3. *Let A and $\mathcal{A}$ be the operators defined above on X and $\mathcal{X}$, respectively. Then the following assertions hold.*

1. *For every $\lambda \in \mathbb{C}$ we have*

$$\lambda \in \sigma(\mathcal{A}) \Longleftrightarrow \lambda \in \sigma(A) \Longleftrightarrow 1 \in \sigma(MD_\lambda).$$

2. *For $\lambda \in \sigma(A)$ and $d \in \partial X$ the following properties are equivalent.*

$$(a) \quad MD_\lambda d = d \qquad (b) \quad D_\lambda M(D_\lambda d) = D_\lambda d$$

$$(c) \quad AD_\lambda d = \lambda D_\lambda d \qquad (d) \quad \mathcal{A}\binom{D_\lambda d}{0} = \lambda\binom{D_\lambda d}{0}$$

3. *For every $\lambda \in \rho(A) = \rho(\mathcal{A})$ the resolvents of A and $\mathcal{A}$ are*

$$R(\lambda, A) = \left(I_X + D_\lambda (\mathbf{1} - MD_\lambda)^{-1} M\right) R(\lambda, A_0) \tag{19}$$

and

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} R(\lambda, A) & D_\lambda (\mathbf{1} - MD_\lambda)^{-1} \\ 0 & 0 \end{pmatrix}.$$

Proof. Since $\rho(\mathcal{A}_0) = \mathbb{C}$, for every $\lambda \in \mathbb{C}$ we can decompose

$$\lambda - \mathcal{A} = \lambda - \mathcal{A}_0 - \mathcal{B} = (\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0))(\lambda - \mathcal{A}_0). \tag{20}$$

Observe that $\lambda - \mathcal{A}$ is invertible if and only if $\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0)$ is invertible, and in this case its inverse is

$$R(\lambda, \mathcal{A}) = R(\lambda, \mathcal{A}_0)\,(\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0))^{-1}.$$

By Lemma 3.1, we have

$$\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0) = \begin{pmatrix} I_X & 0 \\ -MR(\lambda, A_0) & \mathbf{1} - MD_\lambda \end{pmatrix}. \tag{21}$$

It is easy to see that this operator matrix is invertible if and only if $\mathbf{1} - MD_\lambda$ is invertible, and in this case

$$(\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0))^{-1} = \begin{pmatrix} I_X & 0 \\ (\mathbf{1} - MD_\lambda)^{-1} MR(\lambda, A_0) & (\mathbf{1} - MD_\lambda)^{-1} \end{pmatrix}.$$

Hence, $\lambda \in \sigma(\mathcal{A})$ if and only if $1 \in \sigma(MD_\lambda)$. From these identities we also obtain the formula for the resolvent of $\mathcal{A}$. Its upper-left part is obviously the resolvent of A, since A is the part of $\mathcal{A}$ on $X \times \{0\}^n$, and from our computations follows that it can be written in the form given in (19). Thus the assertions 1. and 3. are proved.

From our setting it follows that A (and so $\mathcal{A}$) have compact resolvent – see [8, II.4.30 (4)]. Therefore they have only point spectrum, see [8, Corollary IV.1.19]. Let now $0 \neq \mathbf{u} \in D(\mathcal{A})$ be an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue λ. This is obviously true if and only if $\mathbf{u}_X \in D(A)$ is the appropriate eigenvector of A and $\mathbf{u}_{\partial X} = 0$, where we denote by $\mathbf{u}_X$ resp. $\mathbf{u}_{\partial X}$ the projection of $\mathbf{u}$ onto the space X resp. ∂X. By (20) we obtain the equivalence

$$\mathcal{A}\mathbf{u} = \lambda\mathbf{u} \Longleftrightarrow (\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0))\,((\lambda - \mathcal{A}_0)\,\mathbf{u}) = 0.$$

By (21), this is equivalent to the conditions

$$((\lambda - \mathcal{A}_0)\,\mathbf{u})_X = 0_X \text{ and} \tag{22}$$

$$((\lambda - \mathcal{A}_0)\,\mathbf{u})_{\partial X} \text{ is an eigenvector of } MD_\lambda \text{ with eigenvalue } 1. \tag{23}$$

Condition (22) is equivalent to the fact that $\mathbf{u}_X \in \ker(\lambda - A_w)$ that is

$$\mathbf{u}_X = D_\lambda d \text{ for some } d \in \partial X.$$

By (23), $L\mathbf{u}_X = LD_\lambda d = d$ is the corresponding eigenvector of MD_λ. It is easy to check that this is true if and only if $D_\lambda d$ is the appropriate eigenvector of $D_\lambda M$.

□

We can express the matrix MD_λ appearing in the characterization of $\sigma(A)$ in terms of graph matrices:

$$MD_\lambda = \left(\Phi^+ \otimes \delta_0\right)\left(\epsilon_\lambda (\Phi_w^-)^\top\right) = \Phi^+ E_\lambda (\Phi_w^-)^\top =: \mathbb{A}_\lambda,$$

where

$$E_\lambda := \epsilon_\lambda(0) = \operatorname{diag}\left(e^{-\frac{\lambda}{c_i}}\right)_{i=1,\dots,m}.$$

The matrix obtained in this way has entries

$$(\mathbb{A}_\lambda)_{ij} = \begin{cases} e^{-\frac{\lambda}{c_k}} \omega_{jk}, & \text{if } \mathsf{v}_i = \mathsf{e}_k(0) \text{ and } \mathsf{v}_j = \mathsf{e}_k(1), \\ 0, & \text{else.} \end{cases} \tag{24}$$

Clearly, $\mathbb{A}_0 = \mathbb{A}$ the column stochastic matrix defined in (4). This yields the following characteristic equation for the spectrum of A. In particular, it shows that $\rho(A) \neq \emptyset$ since $\|\mathbb{A}_\lambda\|_1 < 1$ for $\mathrm{Re}\lambda > 0$.

Corollary 3.4. *For every* $\lambda \in \mathbb{C}$ *we have*

$$\lambda \in \sigma(A) \Longleftrightarrow \det(\mathbf{1} - \mathbb{A}_\lambda) = 0. \tag{25}$$

In particular,

$$\lambda \in \rho(A) \textit{ for } Re\lambda > 0. \tag{26}$$

Corollary 3.5. *The* spectral bound

$$s(A) := \sup\{Re\lambda : \lambda \in \sigma(A)\} \tag{27}$$

of $(A, D(A))$ *satisfies*

$$s(A) = 0 \in \sigma(A).$$

Proof. By (25) and (26) we only have to prove that $1 \in \sigma(\mathbb{A}_0)$, which is true since $\mathbb{A}_0$ is column stochastic. □

If the velocities of the flow on all the arcs are equal, the matrix E_λ becomes scalar and $\mathbb{A}_\lambda = e^{-\frac{\lambda}{c}}\mathbb{A}$ a scalar multiple of the weighted (transposed) adjacency matrix $\mathbb{A}$ defined in (4).

Corollary 3.6. *If* $c_i = c$ *for* $i = 1, \ldots, m$, *then we have*

$$\lambda \in \sigma(A) \Longleftrightarrow e^{\frac{\lambda}{c}} \in \sigma(\mathbb{A}). \tag{28}$$

The following technical condition on the velocities allows us to prove more on the structure of $\sigma(A)$ and to relate it to the spectrum of the semigroup $(T(t))_{t\geq 0}$.

($\mathbf{LD}_{\mathbb{Q}}$) The set $\{c_1, \ldots, c_m\}$ is linearly dependent over $\mathbb{Q}$, i.e. $c_i/c_j \in \mathbb{Q}, 1 \leq i, j \leq m$.

Let us investigate the characteristic equation (25) in the case when condition $\left(LD_{\mathbb{Q}}\right)$ holds. This implies that there exists a real number c such that $l_i := c/c_i \in \mathbb{N}$ for every $i = 1, \ldots, m$. Therefore

$$(\mathbb{A}_\lambda)_{ij} = \begin{cases} \left(e^{-\frac{\lambda}{c}}\right)^{l_k} \omega_{jk}, & \text{if } \mathsf{v}_i = \mathsf{e}_k(0) \text{ and } \mathsf{v}_j = \mathsf{e}_k(1), \\ 0, & \text{else,} \end{cases}$$

hence the characteristic equation $\det(\mathbf{1} - \mathbb{A}_\lambda)$ becomes a polynomial in $e^{-\frac{\lambda}{c}}$, i.e.,

$$\det(\mathbf{1} - \mathbb{A}_\lambda) = q\left(e^{-\frac{\lambda}{c}}\right) \tag{29}$$

for some polynomial q. This immediately leads to the following result on the spectrum of A.

Lemma 3.7. *Suppose that the condition* $\left(LD_{\mathbb{Q}}\right)$ *is fulfilled. Then the eigenvalues of* A *lie on finitely many vertical lines.*

Proof. By (25) and (29) the zeros of $q\left(e^{-\frac{\lambda}{c}}\right)$ are exactly the eigenvalues of A, hence the statement follows. □

Now we are able to pass from the spectral properties of the generator to those of the semigroup (see [8, Chapter IV.3] for the general situation).

Proposition 3.8 (Circular Spectral Mapping Theorem)**.** *Suppose that the condition* $\left(LD_{\mathbb{Q}}\right)$ *holds. Then the semigroup* $(T(t))_{t\geq 0}$ *satisfies the so called* circular spectral mapping theorem*, that is*

$$\Gamma \cdot e^{t_0\sigma(A)} = \Gamma \cdot \sigma\left(T\left(t_0\right)\right) \setminus \{0\} \textit{ for every } t_0 \geq 0,$$

where Γ *denotes the unit circle.*

The subsequent proof is based on a result of Greiner and Schwarz [12, Corollary 1.2] and the following result on almost periodic functions (for definitions and result see [6] and [19]).

Lemma 3.9. *Let* h *be an analytic almost periodic function in the strip* $S_{(a,b)} = \{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ *and* $h(z) \neq 0$*. Then* $1/h(z)$ *is analytic and almost periodic in any strip* $S_{[a_1,b_1]} \subset S_{(a,b)}$*. Moreover, if* $h(z) = \sum_{j=1}^{\infty} a_j e^{zr_j}$, $r_j \in \mathbb{R}$*, then* $1/h(z) = \sum_{l=0}^{\infty} b_l e^{zs_l}$ *for suitable* $b_l, s_l \in \mathbb{R}$ *in any strip* $S_{[a_1,b_1]}$*.*

Proof of the circular spectral mapping theorem. The inclusion

$$\Gamma \cdot e^{t_0\sigma(A)} \subseteq \Gamma \cdot \sigma\left(T\left(t_0\right)\right) \setminus \{0\}$$

is the spectral inclusion theorem (see [8, Theorem IV.3.6]) that holds for all C_0-semigroups. Clearly, for $t_0 = 0$ the opposite inclusion also holds. If $t_0 > 0$, we have to prove that for the elements $\lambda \in \rho(A)$ for which the entire vertical line $\operatorname{Re}\lambda + i\mathbb{R}$ is contained in $\rho(A)$, we also have

$$e^{t_0(\operatorname{Re}\lambda + i\mathbb{R})} = \Gamma \cdot e^{t_0\lambda} \subseteq \rho\left(T\left(t_0\right)\right) \cup \{0\}.$$

In order to show this we use Greiner's criterion from [12, Corollary 1.2]. Take an element $e^{t_0\lambda_0} \in \Gamma \cdot e^{t_0\lambda}$. We then have to prove that $\lambda_0 + i\left(2\pi/t_0\right)\mathbb{Z} \subseteq \rho(A)$ and that the sequence

$$S_N := \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-j}^{j} R\left(\lambda_0 + i\left(2\pi/t_0\right)k, A\right),\ N \in \mathbb{N}, \tag{30}$$

is bounded in $\mathcal{L}(X)$. The first fact is obvious from the assumption. To prove the boundedness of $(S_N)_{N\in\mathbb{N}}$, we use ideas from the proof of [12, Theorem 3.1]. By (17) and (19), the resolvent of A looks like

$$\begin{aligned} R(\lambda, A) &= D_\lambda\left(\mathbf{1} - MD_\lambda\right)^{-1} MR\left(\lambda, A_0\right) + R\left(\lambda, A_0\right) \\ &= \epsilon_\lambda\left(\Phi_w^-\right)^{\top}\left(\mathbf{1} - \mathbb{A}_\lambda\right)^{-1} MR\left(\lambda, A_0\right) + R\left(\lambda, A_0\right). \end{aligned}$$

For the sake of simplicity, we write

$$\begin{aligned} R_\lambda &:= R(\lambda, A_0), \\ \lambda_k &:= \lambda_0 + i\,(2\pi/t_0)\,k. \end{aligned}$$

So, S_N has the form

$$S_N = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-j}^{j}\left(\epsilon_{\lambda_k}\left(\Phi_w^-\right)^\top\left(\mathbf{1} - \mathbb{A}_{\lambda_k}\right)^{-1} M R_{\lambda_k} + R_{\lambda_k}\right).$$

We can now estimate its L^1-norm by

$$\begin{aligned} \|S_N\|_1 \le & \left\| \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-j}^{j} \epsilon_{\lambda_k}\left(\Phi_w^-\right)^\top\left(\mathbf{1} - \mathbb{A}_{\lambda_k}\right)^{-1} M R_{\lambda_k} \right\|_1 \\ & + \left\| \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-j}^{j} R_{\lambda_k} \right\|_1 := \|U_N\|_1 + \|V_N\|_1 . \end{aligned}$$

For the estimation of the term $\|V_N\|_1$ first observe that R_λ is the resolvent of the generator of a strongly continuous nilpotent semigroup, as we have seen in (13). For any semigroup $(T(t))$ and generator A the following formula holds:

$$R(\lambda, A)\left(1 - e^{-\lambda t}T(t)\right) = \int_0^t e^{-\lambda s}T(s)ds \text{ for } \lambda \in \rho(A),\ t \ge 0.$$

Take any $t > 0$ such that for all $i : c_i t > 1$. Then from (13) follows that $T_0(t) = 0$. Using the above formula we obtain that

$$\begin{aligned} V_N &= \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-j}^{j} R(\lambda_0 + i(2\pi/t_0)k, A_0) \\ &= \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-j}^{j}\int_0^t e^{-i(2\pi/t_0)ks}e^{-\lambda_0 s}T_0(s)ds = \int_0^t \sigma_N(2\pi s/t_0)e^{-\lambda_0 s}T_0(s)ds \\ &= \frac{t_0}{2\pi}\int_0^{\frac{2\pi t}{t_0}} \sigma_N(v)e^{\frac{-\lambda_0 t_0 v}{2\pi}}T_0\left(\frac{t_0 v}{2\pi}\right)dv, \end{aligned}$$

where

$$\sigma_N(u) = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-j}^{j} e^{-iuk} \text{ for } u \in \mathbb{R}.$$

An elementary computation shows that

$$\sigma_N(u) = \frac{1}{N}\frac{1-\cos Nu}{1-\cos u}, \tag{31}$$

hence σ_N is periodic with period 2π, and

$$\sigma_N(u) \geq 0 \text{ and } \int_0^{2\pi} \sigma_N(u)\, du = 2\pi. \tag{32}$$

Choosing $t = l \cdot t_0$ for an appropriate $1 \leq l \in \mathbb{N}$, it follows that

$$\|V_N\|_1 \leq l \cdot t_0 \cdot C$$

with $C := \sup\left\{\left\|e^{-\lambda_0 s} T_0(s)\right\| : 0 \leq s \leq t\right\}$. This estimate is independent of N, hence we only have to continue with $\|U_N\|_1$.

According to Lemma 3.7, our assumption implies that the zeros of $h(\lambda) := \det(\mathbf{1} - \mathbb{A}_\lambda)$ lie on finitely many vertical lines, hence $h(\lambda) \neq 0$ on a strip $S_{(\alpha,\beta)}$ containing λ_0. In the case when $(LD_{\mathbb{Q}})$ holds, $h(\lambda)$ has the form (29), hence is a finite linear combination of exponential functions. Therefore we can apply Lemma 3.9 for $h(\lambda)$, and using the well-known formula for the entries of the inverse matrix, we have that for $\lambda \in S_{(\alpha,\beta)}$

$$(\mathbf{1} - \mathbb{A}_\lambda)^{-1} = \sum_{l=0}^{\infty} B_l e^{\lambda s_l}$$

for suitable $B_l \in M_n(\mathbb{R})$, $s_l \in \mathbb{R}$, and this series converges absolutely. Continuing the estimate of $\|U_N f\|_1$, we obtain by using (8), (14), and (15)

$$\begin{aligned}
\|U_N f\|_1 &= \sum_{p=1}^{m} \int_0^1 \left| \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=-j}^{j} \left(\epsilon_{\lambda_k}(s) \left(\Phi_w^-\right)^\top \sum_{l=0}^{\infty} B_l e^{\lambda_k s_l} \Phi^+ \right.\right. \\
&\qquad \times \left.\left. \int_0^1 \epsilon_{\lambda_k}(1-t)\, C^{-1} f(t)\, dt \right)_p \right| ds \\
&= \sum_{p=1}^{m} \int_0^1 \left| \sum_{l=0}^{\infty} \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=-j}^{j} \varepsilon_{\frac{\lambda_k}{c_p}}(s)\, e^{\lambda_k s_l} \right. \\
&\qquad \times \left. \int_0^1 \left(\left(\Phi_w^-\right)^\top B_l \Phi^+ \epsilon_{\lambda_k}(1-t)\, C^{-1} f(t) \right)_p dt \right| ds,
\end{aligned}$$

with

$$\varepsilon_{\frac{\lambda_k}{c_p}}(s) := e^{\frac{\lambda_k}{c_p}(s-1)},\ p = 1, ..., m.$$

In order to proceed we introduce the $m \times m$ matrix

$$\Psi_l := \left(\Phi_w^-\right)^\top B_l \Phi^+ = \left(\psi_{ij}^l\right)_{m \times m}$$

and obtain for the p-th coordinate

$$
\begin{aligned}
\left\| (U_N f)_p \right\|_1 &= \int_0^1 \left| \sum_{l=0}^{\infty} \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=-j}^{j} \varepsilon_{\frac{\lambda_k}{c_p}}(s)\, e^{\lambda_k s_l} \right. \\
&\quad \left. \times \int_0^1 \sum_{h=1}^{m} \psi_{p,h}^{l} \varepsilon_{\frac{\lambda_k}{c_h}}(1-t)\, \frac{1}{c_h} f_h(t)\, dt \right| ds \\
&= \int_0^1 \left| \sum_{l=0}^{\infty} \sum_{h=1}^{m} \psi_{p,h}^{l} e^{\lambda_0 s_l} \right. \\
&\quad \left. \times \int_0^1 \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=-j}^{j} e^{\frac{\lambda_k (s-1)}{c_p}} e^{i \frac{2\pi}{t_0} k s_l} e^{-\frac{\lambda_k t}{c_h}} \frac{1}{c_h} f_h(t)\, dt \right| ds \\
&= \int_0^1 \left| \sum_{l=0}^{\infty} \sum_{h=1}^{m} \psi_{p,h}^{l} e^{\lambda_0 s_l} \int_0^1 \frac{1}{c_h} e^{\frac{\lambda_0 (s-1)}{c_p} - \frac{\lambda_0 t}{c_h}} \right. \\
&\quad \left. \times \sigma_N \left(\frac{2\pi \left(\frac{1-s}{c_p} - s_l + \frac{t}{c_h} \right)}{t_0} \right) f_h(t)\, dt \right| ds \\
&\le \sum_{l=0}^{\infty} \sum_{h=1}^{m} \left| \psi_{p,h}^{l} e^{\lambda_0 s_l} \right| \int_0^1 |f_h(t)| \\
&\quad \times \int_0^1 \frac{1}{c_h} \left| e^{\frac{\lambda_0 (s-1)}{c_p} - \frac{\lambda_0 t}{c_h}} \right| \sigma_N \left(\frac{2\pi \left(\frac{1-s}{c_p} - s_l + \frac{t}{c_h} \right)}{t_0} \right) ds dt \\
&= \sum_{l=0}^{\infty} \sum_{h=1}^{m} \left| \psi_{p,h}^{l} e^{\lambda_0 s_l} \right| \int_0^1 |f_h(t)| \\
&\quad \times \int_{\frac{2\pi}{t_0}\left(-s_l + \frac{t}{c_h}\right)}^{\frac{2\pi}{t_0}\left(\frac{1}{c_p} - s_l + \frac{t}{c_h}\right)} \frac{t_0 c_p}{2\pi} \frac{1}{c_h} \left| e^{-\lambda_0 \left(\frac{t_0 v}{2\pi} + s_l - \frac{t}{c_h} \right) - \frac{\lambda_0 t}{c_h}} \right| \sigma_N(v) dv dt.
\end{aligned}
$$

Using the above properties (31) and (32) of the function $\sigma_N(u)$ we obtain that

$$
\begin{aligned}
\left\| (U_N f)_p \right\|_1 &\le C_{\lambda_0, p} \frac{t_0 c_p}{2\pi} \lceil 1/t_0 c_p \rceil 2\pi \sum_{l=0}^{\infty} \sum_{h=1}^{m} \left| \psi_{p,h}^{l} e^{\lambda_0 s_l} \right| \| f \|_1 \\
&= C_{\lambda_0, p} t_0 c_p \lceil 1/t_0 c_p \rceil \sum_{l=0}^{\infty} \sum_{h=1}^{m} \left| \psi_{p,h}^{l} e^{\lambda_0 s_l} \right| \| f \|_1
\end{aligned}
$$

with

$$
C_{\lambda_0, p} := \max_{1 \le h \le m} \left\{ \frac{1}{c_h} \sup_{s,t \in [0,1]} \left| e^{-\lambda_0 \left(\frac{1-s}{c_p} + \frac{t}{c_h} \right)} \right| \right\}
$$

and $\lceil x \rceil \in \mathbb{Z}$ meaning the upper integer part of $x \in \mathbb{R}$.

Summing up for $p = 1, ..., m$ and using the definition of $\psi^l_{p,h}$ we obtain

$$\begin{aligned}\|U_N f\|_1 &\leq C_{\lambda_0} \sum_{l=0}^{\infty}\sum_{h=1}^{m}\sum_{p=1}^{m} \left| \left(\left(\Phi_w^-\right)^\top B_l \Phi^+ \right)_{p,h} e^{\lambda_0 s_l} \right| \|f\|_1 \\ &\leq C_{\lambda_0} m \sum_{l=0}^{\infty} \left\| \left(\Phi_w^-\right)^\top B_l \Phi^+ \right\| \left| e^{\lambda_0 s_l} \right| \|f\|_1 \\ &\leq C_{\lambda_0} m \left\| \left(\Phi_w^-\right)^\top \right\| \|\Phi^+\| \left(\sum_{l=0}^{\infty} \|B_l\| \left| e^{\lambda_0 s_l} \right| \right) \|f\|_1 ,\end{aligned}$$

with

$$C_{\lambda_0} := \max_{1\leq p\leq m} C_{\lambda_0,p} t_0 c_p \lceil 1/t_0 c_p \rceil .$$

This completes the proof since the estimate is independent of $N \in \mathbb{N}$. □

4 Asymptotic behavior

The Circular Spectral Mapping Theorem and the fact that $\sigma(A)$ lies on finitely many vertical lines imply that the spectrum $\sigma(T(t))$ lies on finitely many circles. In particular, the largest of these circles is just the unit circle Γ (see Corollary 3.5). By general semigroup theory, this immediately allows a decomposition of the semigroup and a description of its asymptotic behavior.

Proposition 4.1. *Suppose that condition $\left(LD_{\mathbb{Q}}\right)$ holds. Then the following assertions are true.*

1. *The space X can be decomposed as $X = X_1 \oplus X_2$, where X_1, X_2 are closed, $(T(t))_{t\geq 0}$-invariant subspaces. Furthermore, the operators $S(t) := T(t)\,|_{X_1}$, $t \geq 0$, form a bounded C_0-group on X_1.*
2. *The semigroup $\left(T(t)\,|_{X_2}\right)_{t\geq 0}$ is uniformly exponentially stable, hence $\|T(t) - S(t)\| \leq Me^{-\varepsilon t}$ for some constants $M \geq 1,\ \varepsilon > 0$.*

Proof. Using Theorem 3.8, denote the second largest circle in $\sigma(T(t))$ by $\Gamma \cdot e^{t\cdot s}$ with $s < 0$. Take any $\varepsilon > 0$ such that $\alpha := s + \varepsilon < 0$, then the spectrum of the rescaled semigroup $(\tilde{T}(t)) := (e^{-\alpha t}T(t))$ does not intersect the unit circle. Hence we can use [8, Theorem V.1.17] for $(\tilde{T}(t))$ and obtain a decomposition that has the desired properties for the original semigroup. □

If we now take into account also the positivity of the semigroup (see Proposition 2.5), we can describe its asymptotic behavior even more precisely. The following property of positive semigroups on Banach lattices is crucial for this discussion (see [16, Definition C-III.3.1]). We state its definition only in the case of L^1-spaces.

Definition 4.2. *A positive semigroup on $L^1(\Omega, \mu)$, μ a σ-finite measure, with generator A is* irreducible *if for all $\lambda > s(A)$ – the spectral bound, defined in (27) – and $f > 0$, the resolvent satisfies $(R(\lambda, A) f)(s) > 0$ for almost all $s \in \Omega$.*

In the next step we relate the irreducibility of our semigroup on X to the strong connectedness of the underlying graph.

Definition 4.3. *A directed graph is called* strongly connected *if for every two vertices in the graph there are paths connecting them in both directions.*

According to [14, Theorem IV.3.2], a directed graph is strongly connected if and only if the corresponding adjacency matrix is irreducible. This leads to the irreducibility of our semigroup.

Lemma 4.4. *Let the graph G be strongly connected. Then the semigroup $(T(t))_{t\geq 0}$ is irreducible.*

Proof. It suffices to show that for $\lambda > 0$ and $f > 0$ the resolvent $R(\lambda, A)f$ is a. e. strictly positive. By Proposition 3.3 this means that for $0 < f \in X$

$$R(\lambda, A_0)\, f + D_\lambda \left(\mathbf{1} - MD_\lambda\right)^{-1} MR(\lambda, A_0)\, f > 0 \text{ a.e.} \tag{33}$$

Take an arbitrary $\lambda > 0$. First note that, due to (14), $R(\lambda, A_0)\, f \in X$ is strictly positive everywhere except on the largest interval $(1-\varepsilon, 1]$ for which $f\mid_{(1-\varepsilon,1]}= 0$. Applying M to it we obtain a vector $d \in \mathbb{R}^m$ of positive numbers (see (8)). Observe that under our assumptions the matrix $MD_\lambda = \mathbb{A}_\lambda$ is positive and irreducible. From the form (24) follows that $r(\mathbb{A}_\lambda) \leq \|\mathbb{A}_\lambda\| < 1$. Therefore the matrix $(\mathbf{1} - MD_\lambda)^{-1}$ is strictly positive (see [21, Proposition I.6.2]). So the vector $(\mathbf{1} - MD_\lambda)^{-1}\, d \in \mathbb{C}^m$ has only positive coordinates. Applying D_λ to it we obtain a vector of positive multiples of exponential functions which is also strictly positive. Adding it to the vector of positive functions $R(\lambda, A_0)\, f$, we finally obtain (33). □

The decomposition from Proposition 4.1 combined with the irreducibility now leads to a precise description of the asymptotic behavior of $(T(t))$. We first note that the Perron-Frobenius theory for positive irreducible semigroups implies that

$$\sigma(A) \cap i\mathbb{R} = i\alpha\mathbb{Z} \text{ for some } \alpha \geq 0, \tag{34}$$

where each $i\alpha k$ is a simple pole of the resolvent (see [8, Theorem VI.1.12] or [16, Section C-III]). From the condition $\left(LD_{\mathbb{Q}}\right)$ and from the form of the characteristic equation (29) follows that there are nonzero spectral points on the imaginary axis, hence $\alpha > 0$. Applying then a result of Nagel [15, Theorem 4.3] (combined with [16, Lemma C-IV.2.12]) we obtain that the semigroup $(T(t))_{t\geq 0}$ behaves asymptotically as a periodic group on a very concrete function space.

Theorem 4.5. *Suppose that the condition $\left(LD_{\mathbb{Q}}\right)$ holds and that the graph G is strongly connected. Then the decomposition $X = X_1 \oplus X_2$ from Theorem 4.1 has the following additional properties.*

1. *X_1 is a closed sublattice of X isomorphic to $L^1(\Gamma)$, where Γ is the unit circle.*
2. *The group $(S(t))_{t\geq 0}$ is isomorphic to the rotation group on $L^1(\Gamma)$ with period $\tau = \frac{2\pi}{\alpha}$, where α is defined in (34).*

3. The period τ equals

$$\tau = \frac{1}{c} \gcd \left\{ c \left(\frac{1}{c_{i_1}} + \cdots + \frac{1}{c_{i_k}} \right); \mathsf{e}_{i_1}, \ldots, \mathsf{e}_{i_k} \textit{ form a cycle in } G \right\}, \tag{35}$$

where c is any real number such that $c/c_i \in \mathbb{N}$ for all $i = 1, ..., m$.

Proof. We shall prove the theorem in two steps. First we assume that $c_i = c$ for $i = 1, \ldots, m$.

By Lemma 4.4, the semigroup $(T(t))_{t \geq 0}$ is irreducible and by Proposition 2.5 it is positive and bounded. Since $s(A) = 0$ (see Corollary 3.5) and because of the compactness of the resolvent, 0 is a pole of $R(\lambda, A)$. By the above remark, we also know that there are nonzero spectral points on the imaginary axis. So, all the conditions of [16, C-IV, Lemma 2.12] and [16, C-IV, Theorem 2.14] are fulfilled, and we obtain the statements 1. and 2. Observe that because of the existence of the decomposition in Proposition 4.1, we obtain uniform convergence to the rotation group $(S(t))_{t \geq 0}$.

By [16, C-IV, Lemma 2.12 (c)] the period τ equals $\frac{2\pi}{\alpha}$, where $\alpha \in \mathbb{R}$ is determined by

$$\sigma(A) \cap i\mathbb{R} = i\alpha\mathbb{Z}.$$

Due to (28), the peripheral spectrum of $\mathbb{A}$ (consisting of the spectral points of $\mathbb{A}$ with absolute value equal $r(\mathbb{A}) = 1$) consists of the points $e^{\frac{i\alpha z}{c}}$, $z \in \mathbb{Z}$, i.e., of the $c \cdot \frac{2\pi}{\alpha} = c\tau$-th roots of unity. This number is also known as index of imprimitivity of the irreducible matrix $\mathbb{A}$ – for the definition see [14, Definition III.1.1]. Using [14, Theorem IV.3.3] we obtain that $c\tau$ is equal to the index of imprimitivity of our directed graph, i.e., that is the greatest common divisor l of the lengths of all the cycles in the graph. Observe that the formula (35) holds.

In the case when the velocities c_i are not all the same, we proceed as follows. Take any $c \in \mathbb{R}$ such that $l_i = \frac{c}{c_i} \in \mathbb{N}$, $i = 1, \ldots, m$. We construct a new directed graph $\widetilde{G}$ with $\widetilde{m} := l_1 + \cdots + l_m$ arcs and $\widetilde{n} := n + \widetilde{m} - m$ vertices by adding $l_i - 1$ vertices on the arc e_i. We maintain the original direction of the arcs and we normalize the lengths. Observe, that we can consider a new problem $(\widetilde{F})$ on the new network $\widetilde{G}$ with functions $\widetilde{u_i}$ and velocities $\widetilde{c_i} := c$, $i = 1, \ldots, \widetilde{m}$. With the obvious adjustments of the initial and boundary conditions (IC) and (BC) the problem $(\widetilde{F})$ is equivalent to the original one. Note that the number of the cycles in the graphs G and $\widetilde{G}$ are the same, only the lengths have changed: if the arcs $\mathsf{e}_{i_1}, \ldots, \mathsf{e}_{i_k}$ form a cycle of length k in the graph G, then the corresponding cycle in the graph $\widetilde{G}$ has length

$$l_{i_1} + \cdots + l_{i_k} = c \left(\frac{1}{c_{i_1}} + \cdots + \frac{1}{c_{i_k}} \right). \tag{36}$$

Denote by $\widetilde{l}$ the greatest common divisor of the lengths of the cycles in $\widetilde{G}$. We may now use the first part of our proof for the graph $\widetilde{G}$, obtaining a rotation group with period $\widetilde{\tau} = \frac{\widetilde{l}}{c}$. Going back to the original graph, we have the same period, which together with (36) yields the formula (35). □

In less technical terms the above result can be expressed as follows.

Corollary 4.6. *Under the assumptions of Theorem 4.5, the semigroup* $(T(t))_{t\geq 0}$ *is* asymptotically periodic *with period*

$$\tau = \frac{1}{c}\,\mathrm{gcd}\left\{c\left(\frac{1}{c_{i_1}}+\cdots+\frac{1}{c_{i_k}}\right);\ \mathsf{e}_{i_1},\ldots,\mathsf{e}_{i_k}\ \textit{form a cycle in}\ G\right\},$$

where c *is any real number such that* $c/c_i \in \mathbb{N}$ *for all* $i = 1, ..., m$.

If the velocities on the edges coincide, the result becomes particularly simple.

Corollary 4.7. *If* $c_i = 1$ *for all* $i = 1, \ldots, m$, *then the period* τ *of the rotation group in Theorem 4.5 equals the greatest common divisor of the cycle lengths in the graph* G.

Remark 4.8. Observe that the period does not depend on the weights on the edges.

We finally extend our description of the asymptotic behavior to the case when the underlying graph is not strongly connected. To do so we consider special parts of our directed graph.

Definition 4.9. *We call a subgraph* G_p *of* G *an* invariant strongly connected component *if it is strongly connected and there are no outgoing edges of* G_p.

Theorem 4.10. *Consider a flow in an arbitrary network modelled by the directed graph* G, *and assume that* $(LD_{\mathbb{Q}})$ *holds. Then the corresponding semigroup behaves asymptotically as a direct sum of rotation groups on disjoint polygons. The period of the rotation group on each polygon is given by the modification of the formula (35) for each invariant strongly connected component of* G.

Proof. By Proposition 4.1 we have a spectral decomposition $X = X_1 \oplus X_2$ of the state space such that $(S(t))_{t\geq 0} := \left(T(t)\,|_{X_1}\right)_{t\geq 0}$ is a bounded C_0-group and $\left(T(t)\,|_{X_2}\right)_{t\geq 0}$ is uniformly exponentially stable. Since the semigroup $(T(t))_{t\geq 0}$ is bounded (see Proposition 2.5) and the resolvent of its generator is compact, it follows by [8, Corollary V.2.15] that

$$X_1 = \overline{\mathrm{lin}}\,\{x \in D(A) : \exists\gamma \in \mathbb{R} \text{ such that } Ax = i\gamma x\}.$$

Therefore, if $t \to +\infty$, the semigroup converges (exponentially) in norm to the bounded group $(S(t))_{t\geq 0}$ acting on the closed subspace generated by the eigenvectors that belong to the imaginary (that is, the boundary) spectrum of A. We want to prove that this limit is isomorphic to a direct sum of rotation groups with the proper periods.

To this purpose we first characterize the spectral values $i\gamma \in \sigma(A)$, $\gamma \in \mathbb{R}$. Taking into account the characteristic equation (25), we have to investigate in which case

$$1 \in \sigma(\mathbb{A}_{i\gamma}),\ \gamma \in \mathbb{R},$$

holds.

Observe, that the positive matrix $\mathbb{A}_0$ is similar (via a permutation P of the canonical basis) to a block-triangular matrix, i.e.,

$$P^{-1}\mathbb{A}_0 P = \begin{pmatrix} Q_0^0 & 0 & \dots & 0 \\ B_1^0 & Q_1^0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ B_q^0 & 0 & \dots & Q_q^0 \end{pmatrix}, \tag{37}$$

where the diagonal blocks $Q_1^0, ..., Q_q^0$ are irreducible and, if the $k_0 \times k_0$ matrix Q_0^0 is non-empty, then at least one B_i^0 is nonzero (see [21, Proposition I.8.8]). This form is unique up to permutations of the coordinates within each diagonal block and up to the order of $Q_1^0, ..., Q_q^0$. It is easy to see that since the zero-patterns of the matrices $\mathbb{A}_\lambda$ coincide for every λ, the same permutation matrix P yields an analogous block-form

$$P^{-1}\mathbb{A}_\lambda P = \begin{pmatrix} Q_0^\lambda & 0 & \dots & 0 \\ B_1^\lambda & Q_1^\lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ B_q^\lambda & 0 & \dots & Q_q^\lambda \end{pmatrix} \text{ for every } \lambda \in \mathbb{C}. \tag{38}$$

We now renumber the vertices of G such that the adjacency matrices $\mathbb{A}_\lambda$ have the above block-triangular form (38). Clearly, this does not change the spectral properties we need. From the block-triangular form (38) follows that

$$\sigma\left(\mathbb{A}_\lambda\right) = \bigcup_{p=0}^{q} \sigma\left(Q_p^\lambda\right) \text{ for every } \lambda \in \mathbb{C}.$$

We will show that the $k_0 \times k_0$ matrices $Q_0^{i\gamma}$ for $\gamma \in \mathbb{R}$ do not contribute to the boundary spectrum of A. This means that if $\mathbb{A}_{i\gamma}x = x$, then the first k_0 coordinates of x have to be equal to 0. Let us first investigate the case $\gamma = 0$, hence we assume $\mathbb{A}_0 x = x$. To the column stochastic matrix $\mathbb{A}_0$ we can apply [21, Corollary of I.8.4] and obtain that x is contained in the direct sum of the minimal $\mathbb{A}_0$-invariant ideals in $\mathbb{C}^n$. By the proof of [21, Proposition I.8.8] this direct sum is exactly the direct sum of the ideals spanned by the basis vectors that correspond to the blocks $Q_1^0, ..., Q_q^0$. Hence we obtain that the first k_0 coordinates of x are 0. It means that 1 is not in the spectrum of the positive matrix Q_0^0. Furthermore, because of the column stochasticity of $\mathbb{A}_0$, all the column sums of Q_0^0 are less than or equal to 1, hence $r\left(Q_0^0\right) \le 1$. The Perron-Frobenius theorem yields (see, e.g., [21, Proposition I.2.3]) that $r\left(Q_0^0\right) < 1$. From the form (24) of the entries of $\mathbb{A}_\lambda$ follows that $\left|Q_0^{i\gamma}\right| = Q_0^0$ and so, $r\left(Q_0^{i\gamma}\right) \le r\left(Q_0^0\right) < 1$. Hence $1 \notin \sigma\left(Q_0^{i\gamma}\right)$ for all $\gamma \in \mathbb{R}$ and therefore

$$1 \in \sigma\left(\mathbb{A}_{i\gamma}\right) \Longleftrightarrow 1 \in \bigcup_{p=1}^{q} \sigma\left(Q_p^{i\gamma}\right). \tag{39}$$

For each $p \in \{1, ..., q\}$, the irreducible block Q_p^λ is the weighted (transposed) adjacency matrix of a subgraph G_p of G, which is by [14, Theorem IV.3.2] strongly connected. From the form (38) of the adjacency matrix of the whole graph G follows that that there are no outgoing edges of G_p. Hence G_p is an *invariant* strongly connected component. This implies that the subspace $X^p \subset X$ of all functions having their support on the edges of G_p is invariant under the semigroup $(T(t))_{t\geq 0}$. We can apply Theorem 4.5 to the restricted positive *irreducible* semigroup $\left(T_p(t)\right)_{t\geq 0} := (T(t)\mid_{X^p})_{t\geq 0}$. Hence its generator A_p – which is the part of A in X^p – satisfies

$$\sigma(A_p) \cap i\mathbb{R} = i\alpha_p\mathbb{Z} \text{ for some } \alpha_p \in \mathbb{R}.$$

By (25) we conclude the equivalences

$$1 \in \sigma\left(Q_p^{i\gamma}\right) \Longleftrightarrow i\gamma \in \sigma(A_p) \Longleftrightarrow \gamma = \alpha_p k \text{ for some } k \in \mathbb{Z} \tag{40}$$

for each $\gamma \in \mathbb{R}$. By Theorem 4.5 the semigroup $\left(T_p(t)\right)_{t\geq 0}$ converges exponentially to a rotation group on a subspace X_1^p of X^p having the form

$$X_1^p = \overline{\operatorname{lin}}\left\{x \in D(A_p) : A_p x = i\alpha_p k x \text{ for some } k \in \mathbb{Z}\right\}. \tag{41}$$

The period of the rotation is given by the formula (35) for the cycles in G_p. Since $X_1^p \subset X_1$ for each p, we conclude that

$$Y := X_1^1 \oplus ... \oplus X_1^q \subseteq X_1.$$

We will show that equality holds, in this way proving that the semigroup converges to a direct sum of rotation groups with the appropriate periods. To this purpose it suffices to show that if for some $\gamma \in \mathbb{R}$ and $x \neq 0$ we have $Ax = i\gamma x$, then $x \in Y$. By Proposition 3.3 we know that

$$Ax = i\gamma x \Longleftrightarrow \mathbb{A}_{i\gamma}(Lx) = Lx.$$

Let $L^{(p)} := \Phi_p^- \otimes \delta_1,\ p = 0, ..., q$, where Φ_p^- denotes the matrix obtained from the rows of Φ^- belonging to the vertices that correspond to the block Q_p^λ in the adjacency matrix. Similarly, let $D_\lambda^{(p)} := \epsilon_\lambda(\Phi_{w,p}^-)^\top$ where $\Phi_{w,p}^-$ is obtained from Φ_w^- in the same way. Since $r(Q_0^{i\gamma}) < 1$, clearly

$$L^{(0)}x = 0.$$

Hence there exists $p \in \{1, ..., q\}$ such that $L^{(p)}x \neq 0$ and

$$Q_p^{i\gamma} L^{(p)}x = L^{(p)}x.$$

Again by Proposition 3.3, this is equivalent to the fact that the identity

$$A_p\left(D_{i\gamma}^{(p)} L^{(p)} x\right) = i\gamma\left(D_{i\gamma}^{(p)} L^{(p)} x\right) \tag{42}$$

holds, hence $\gamma = \alpha_p k$ for some $k \in \mathbb{Z}$. For $p = 1, \ldots, q$ we denote

$$k_p := \begin{cases} l, & \text{if } \gamma = \alpha_p l \text{ for some } l \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

A simple calculation shows that

$$x = D_{i\gamma} L x = \sum_{p=0}^{q} D_{i\gamma}^{(p)} L^{(p)} x = \sum_{p=1}^{q} D_{i\gamma}^{(p)} L^{(p)} x.$$

By (42),

$$A_p \left(D_{i\gamma}^{(p)} L^{(p)} x \right) = i \alpha_p k_p \left(D_{i\gamma}^{(p)} L^{(p)} x \right),$$

and using (41) we obtain that $x \in Y$. □

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