Appl Math Optim 55:219–240 (2007)
DOI: 10.1007/s00245-006-0887-9

Applied Mathematics & Optimization

Variational and Semigroup Methods for Waves and Diffusion in Networks

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Abstract. We study diffusion and wave equations in networks. Combining semigroup and variational methods we obtain well-posedness and many nice properties of the solutions in general L^p-context. Following earlier articles of other authors, we discuss how the spectrum of the generator can be connected to the structure of the network. We conclude by describing asymptotic behavior of solutions to the diffusion problem.

Key Words. Evolution equations on networks, Closed sesquilinear forms, Analytic semigroups, Graph spectral theory.

AMS Classification. 34B45, 47D06, 47D09.

1. Introduction

In this paper we continue the study of dynamical processes in networks using semigroup methods. While [KS4] and [MS] studied flow and transport processes, the aim of the present paper is to combine variational and semigroup methods in order to obtain the well-posedness of initial value problems associated with diffusion and wave equations. We thus consider first- and second-order problems

$$\dot{u}_j(t,x) = (c_j u_j')'(t,x) \quad \text{and} \quad \ddot{u}_j(t,x) = (c_j u_j')'(t,x), \qquad t \geq 0, \quad x \in (0,1),$$

where $c_j(\cdot)$ and $u_j(t,\cdot)$ are functions on parameterized edges e_j of a finite network. The node conditions in (2.1) and (3.1) below impose continuity and Kirchhoff laws in the ramification vertices. Problems of this kind have already been treated by many authors both from the mathematical and physical communities—among others, we mention the earlier articles of Lumer [L], Ali Mehmeti [Al1], Roth [R], von Below [B1], Nicaise [Ni1], Exner [E], Cattaneo [Ca], Kottos and Smilansky [KS3], Kostrykin and Schrader [KS1], and Kuchment [Ku], as well as the monographs [Ni4], [Al3], and [LLS], and the proceedings [ABN].

Since the pioneering work of Beurling and Deny in the 1950s, variational methods have been greatly developed. In combination with the theory of strongly continuous semigroups of operators, they provide a powerful tool to discuss properties of solutions to many parabolic and hyperbolic problems; see [D], [Ar], and [O]. While L^2-techniques like the lemma of Lax–Milgram have been used in most of the above-mentioned papers, our paper seems to be the first applying variational methods to obtain positivity, ultracontractivity, and stability for network equations in a general L^p-context. This is the main aim of Sections 2 and 3. We remark that positivity of the semigroup governing the diffusion problem with much more general nodal conditions has been characterized, by algebraic methods, in [KS2].

We then proceed to study the qualitative behavior of the solutions. To that purpose we obtain in Section 4 a characteristic equation for the spectrum of the generator and describe the appropriate eigensolutions. We reprove some results from [B1], [Ni1], [Ni3], and [B3] in our setting with slight generalizations. We see that the spectrum is determined by the structure of the network and corresponds to the spectrum of the Laplacian matrix known from graph theory; see [Mo1]. We give an explicit connection between the two spectra and show the impact of this to our problem. This relates to the well-known question: "Can one hear the shape of a drum?," first addressed by Kac in [Ka]. Concerning differential operators on graphs, the analogous question "Can one hear the shape of a network?" has been formulated and answered in the negative by von Below in his contribution to [ABN]. Quite surprisingly, the same question was raised at the same time in the almost homonymous paper [GS] by Gutkin and Smilansky. They answered it *in the positive*, by studying the Schrödinger operator on a finite, simple graph *with rationally independent arc lengths* and imposing some further assumptions on matching conditions at the vertices. In graph theory, however, it is well known that there are many graphs sharing the same spectrum; see [DH]. Also in our case, the spectrum itself does not determine the network.

In Section 5 we study the asymptotic behavior of solutions to the diffusion problem. To our knowledge this topic has not yet been properly treated by other authors. We show that the solutions always converge toward an equilibrium with rate of convergence depending on the structure of the network. This is discussed for special classes of networks. In similar contexts, convergence to equilibria has already been discussed, e.g., in [BN].

2. The Wave Equation on a Network

We consider a finite connected network, represented by a finite graph G with m edges $\mathsf{e}_1, \dots, \mathsf{e}_m$ and n vertices $\mathsf{v}_1, \dots, \mathsf{v}_n$. We assume that all the vertices have degree at least 2, i.e., that each vertex is incident to at least two edges. Furthermore, we assume that G is *simple*, that is, it has no multiple edges or loops. We normalize and parameterize the edges on the interval $[0, 1]$. The structure of the network is given by the $n \times m$ matrices $\Phi^+ := (\varphi_{ij}^+)$ and $\Phi^- := (\varphi_{ij}^-)$ defined by

$$\varphi_{ij}^+ : \begin{cases} 1, & \text{if} \quad \mathsf{e}_j(0) = \mathsf{v}_i, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \varphi_{ij}^- : \begin{cases} 1, & \text{if} \quad \mathsf{e}_j(1) = \mathsf{v}_i, \\ 0, & \text{otherwise}. \end{cases}$$

We refer to [KS4] for terminology. The $n \times m$ matrix $\Phi := (\varphi_{ij})$ defined by

$$\Phi := \Phi^+ - \Phi^-$$

is known in graph theory as the *incidence matrix* of the graph G. Further, let $\Gamma(\mathsf{v}_i)$ be the set of all the indices of the edges having an endpoint at v_i, i.e.,

$$\Gamma(\mathsf{v}_i) := \left\{ j \in \{1, \dots, m\} \colon \mathsf{e}_j(0) = \mathsf{v}_i \text{ or } \mathsf{e}_j(1) = \mathsf{v}_i \right\}.$$

For the sake of simplicity, we denote the value of the functions $c_j(\cdot)$ and $u_j(t, \cdot)$ at 0 or 1 by $c_j(\mathsf{v}_i)$ and $u_j(t, \mathsf{v}_i)$, if $\mathsf{e}_j(0) = \mathsf{v}_i$ or $\mathsf{e}_j(1) = \mathsf{v}_i$, respectively. With an abuse of notation, we also set $u_j'(t, \mathsf{v}_i) = c_j(\mathsf{v}_i) := 0$ whenever $j \notin \Gamma(\mathsf{v}_i)$. When convenient, we shall also write the functions u_j in vector form, i.e., $u = (u_1, \dots, u_m)^\top$.

We start with the second-order problem

$$\ddot{u}_j(t, x) = (c_j u_j')'(t, x), \qquad t \in \mathbb{R}, \quad x \in (0, 1), \quad j = 1, \dots, m, \tag{2.1a}$$

$$u_j(t, \mathsf{v}_i) = u_\ell(t, \mathsf{v}_i), \qquad t \in \mathbb{R}, \quad j, \ell \in \Gamma(\mathsf{v}_i), \quad i = 1, \dots, n, \tag{2.1b}$$

$$\sum_{j=1}^m \varphi_{ij} \mu_j c_j(\mathsf{v}_i) u_j'(t, \mathsf{v}_i) = 0, \qquad t \in \mathbb{R}, \quad i = 1, \dots, n, \tag{2.1c}$$

$$u_j(0, x) = \mathsf{f}_j(x), \qquad x \in (0, 1), \quad j = 1, \dots, m, \tag{2.1d}$$

$$\dot{u}_j(0, x) = \mathsf{g}_j(x), \qquad x \in (0, 1), \quad j = 1, \dots, m, \tag{2.1e}$$

on the network. Note that $c_j(\cdot)$ and $u_j(t, \cdot)$ are functions on the edge e_j of the network, so that the right-hand side of (2.1a) reads in fact as

$$(c_j u_j')'(t, \cdot) = \frac{\partial}{\partial x} \left(c_j \frac{\partial}{\partial x} u_j \right)(t, \cdot), \qquad t \in \mathbb{R}, \quad j = 1, \dots, m.$$

The functions $c_1, \dots, c_m$ are the *weights* of the edges, and throughout this section we assume that $0 < c_j \in H^1(0, 1)$, $j = 1, \dots, m$. They represent the different speeds of propagation along each edge of the network G. Equation (2.1b) represents the continuity of the values attained by the system at the vertices. The coefficients μ_j, $j = 1, \dots, m$, are strictly positive constants that influence the distribution of impulse happening in the ramification nodes according to the Kirchhoff-type law (2.1c).

We now introduce *weighted incidence matrices* $\Phi_w^+ := (\omega_{ij}^+)$ and $\Phi_w^- := (\omega_{ij}^-)$ with entries

$$\omega_{ij}^+ := \begin{cases} \mu_j c_j(\mathsf{v}_i), & \text{if } \quad \mathsf{e}_j(0) = \mathsf{v}_i, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\omega_{ij}^- := \begin{cases} \mu_j c_j(\mathsf{v}_i), & \text{if } \quad \mathsf{e}_j(1) = \mathsf{v}_i, \\ 0, & \text{otherwise.} \end{cases}$$

With these notations, (2.1b) can be rewritten as

$$\exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^\top d = u(t,0) \quad \text{and} \quad (\Phi^-)^\top d = u(t,1), \qquad t \in \mathbb{R}, \tag{2.2}$$

while the Kirchhoff law (2.1c) becomes

$$\Phi_w^+ u'(t,0) = \Phi_w^- u'(t,1), \qquad t \in \mathbb{R}.$$

We are now in the position to rewrite our system in form of a second-order abstract Cauchy problem. First we consider the (complex) Hilbert space

$$X_2 := \prod_{j=1}^m L^2(0,1;\mu_j\, dx)$$

endowed with the natural inner product

$$(f,g)_{X_2} := \sum_{j=1}^m \int_0^1 f_j(x)\overline{g_j(x)}\mu_j\, dx, \qquad f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix} \in X_2.$$

Observe that X_2 is isomorphic to $(L^2(0,1))^m$ with equivalence of norms. Moreover, X_2 is in fact a Hilbert lattice whose positive cone consists of m copies of the positive cone of $L^2(0,1;\mu_j dx) \approx L^2(0,1)$. On X_2 we define an operator

$$A := \begin{pmatrix} \frac{d}{dx}\left(c_1 \frac{d}{dx}\right) & & 0 \\ & \ddots & \\ 0 & & \frac{d}{dx}\left(c_m \frac{d}{dx}\right) \end{pmatrix} \tag{2.3}$$

with domain

$$\begin{aligned} D(A) := \{ f \in \left(H^2(0,1)\right)^m : \Phi_w^+ f'(0) = \Phi_w^- f'(1) \text{ and} \\ \exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^\top d = f(0) \text{ and } (\Phi^-)^\top d = f(1)\}. \end{aligned} \tag{2.4}$$

With this notation, we can finally rewrite (2.1) in form of a second-order abstract Cauchy problem

$$\begin{cases} \ddot{u}(t) = Au(t), & t \in \mathbb{R}, \\ u(0) = \mathsf{f}, \\ \dot{u}(0) = \mathsf{g}, \end{cases} \tag{2.5}$$

on X_2. By means of variational techniques, we are going to show that A enjoys several nice properties. We follow the techniques of [D] and [ABHN, Section 7.1].

Lemma 2.1. *Consider the sesquilinear form*

$$\mathfrak{a}(f, g) := \sum_{j=1}^{m} \int_0^1 \mu_j c_j(x) f_j'(x) \overline{g_j'(x)}\, dx$$

on the Hilbert space X_2 with domain

$$D(\mathfrak{a}) = V := \{f \in (H^1(0,1))^m \colon \exists d \in \mathbb{C}^n \ s.t.\ (\Phi^+)^\top d = f(0) \ \text{and}\ (\Phi^-)^\top d = f(1)\}.$$

Then $\mathfrak{a}$ is densely defined and has the following properties:

- (symmetry) $\mathfrak{a}(f, g) = \overline{\mathfrak{a}(g, f)}$ *for all* $f, g \in D(\mathfrak{a})$,
- (positivity) $\mathfrak{a}(f, f) \geq 0$ *for all* $f \in D(\mathfrak{a})$,
- (closedness) V *is complete for the form norm* $\|f\|_{\mathfrak{a}} := \sqrt{\mathfrak{a}(f, f) + \|f\|^2_{X_2}}$,
- (continuity) $|\mathfrak{a}(f, g)| \leq M\|f\|_{\mathfrak{a}}\|g\|_{\mathfrak{a}}$ *for some* $M > 0$ *and all* $f, g \in D(\mathfrak{a})$.

Proof. It is apparent that V is a linear subspace of X_2. Observe that $\left(C_c^\infty(0,1)\right)^m \subset V$. It follows that V is dense in X_2, as by definition $L^2(0,1)$ is the closure of $C_c^\infty(0,1)$ in the L^2-norm. By assumption, the weights c_j are strictly positive, so that in particular $\mathfrak{a}$ is symmetric and also positive, since

$$\mathfrak{a}(f, f) = \sum_{j=1}^{m} \int_0^1 \mu_j c_j(x) |f_j'(x)|^2\, dx \geq 0 \qquad \text{for all} \quad f \in V.$$

Furthermore, V becomes a Hilbert space whenever equipped with the inner product

$$(f, g)_V := \sum_{j=1}^{m} \int_0^1 \left(f_j'(x)\overline{g_j'(x)} + f_j(x)\overline{g_j(x)} \right) \mu_j\, dx, \qquad f, g \in V,$$

since V is a closed subspace of $\left(H^1(0,1)\right)^m$. Set

$$c := \min_{1 \leq j \leq m} \min_{x \in [0,1]} c_j(x), \qquad C := \max_{1 \leq j \leq m} \max_{x \in [0,1]} c_j(x).$$

Then one has

$$(c \wedge 1)\|f\|_V^2 \leq \|f\|_{\mathfrak{a}}^2 \leq (C \vee 1)\|f\|_V^2, \qquad f \in V,$$

so that the form norm $\|\cdot\|_{\mathfrak{a}}$ is equivalent to the norm $\|\cdot\|_V$. Since V is complete with respect to $\|\cdot\|_V$, the closedness of $\mathfrak{a}$ follows at once.

Finally, $\mathfrak{a}$ is continuous. To see this, take $f, g \in V$ and observe that

$$\begin{aligned}
|\mathfrak{a}(f,g)| &\le C \sum_{j=1}^{m} \left| \int_0^1 \mu_j f_j'(x) g_j'(x)\, dx \right| \\
&\le C \sum_{j=1}^{m} \|f_j'\|_{L^2(0,1;\mu_j dx)} \|g_j'\|_{L^2(0,1;\mu_j dx)} \\
&\le \frac{C}{2} \left(\sum_{j=1}^{m} \|f_j'\|^2_{L^2(0,1;\mu_j dx)} \right)^{1/2} \left(\sum_{j=1}^{m} \|g_j'\|^2_{L^2(0,1;\mu_j dx)} \right)^{1/2} \\
&\le \frac{C}{2 \cdot (c \wedge 1)} \|f\|_{\mathfrak{a}} \|g\|_{\mathfrak{a}},
\end{aligned}$$

by the Cauchy–Schwartz inequality. ◻

Definition 2.2. From the form $\mathfrak{a}$ we can obtain a unique operator $(B, D(B))$ in the following way:

$$\begin{aligned}
D(B) &:= \{ f \in V \colon \exists g \in X_2 \text{ s.t. } \mathfrak{a}(f,h) = (g,h)_{X_2},\ \forall h \in V \}, \\
Bf &:= -g.
\end{aligned}$$

We say that the operator $(B, D(B))$ is *associated with the form* $\mathfrak{a}$.

Lemma 2.3. *The operator associated with the form* $\mathfrak{a}$ *is* $(A, D(A))$ *defined in* (2.3)–(2.4).

Proof. Denote by $(B, D(B))$ the operator associated with $\mathfrak{a}$. We first show that $A \subset B$. Take $f \in D(A)$. Then for all $h \in V$

$$\begin{aligned}
\mathfrak{a}(f,h) &= \sum_{j=1}^{m} \int_0^1 \mu_j c_j(x) f_j'(x) \overline{h_j'(x)}\, dx \\
&= \sum_{j=1}^{m} \left[\mu_j c_j f_j' \overline{h_j} \right]_0^1 - \sum_{j=1}^{m} \int_0^1 \mu_j (c_j f_j')'(x) \overline{h_j(x)}\, dx. \qquad (2.6)
\end{aligned}$$

Using the incidence matrix $\Phi = \Phi^+ - \Phi^-$, the first term above can be written as

$$\sum_{j=1}^{m} \left[\mu_j c_j f_j' \overline{h_j} \right]_0^1 = \sum_{j=1}^{m} \sum_{i=1}^{n} \mu_j c_j(\mathsf{v}_i) (\varphi_{ij}^- - \varphi_{ij}^+) f_j'(\mathsf{v}_i) \overline{h_j(\mathsf{v}_i)}.$$

Observe now that the condition

$$\exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^\top d = h(0) \quad \text{and} \quad (\Phi^-)^\top d = h(1)$$

in the definition of V implies that h is continuous in the vertices, i.e., there exist $d_1, \dots, d_n \in \mathbb{C}$ such that $h_j(\mathsf{v}_i) = d_i$ for all $j \in \Gamma(\mathsf{v}_i)$, $i = 1, \dots, n$. Summing up and using the other condition $\Phi_w^+ f'(0) = \Phi_w^- f'(1)$ in $D(A)$ we obtain that

$$\begin{aligned}\mathfrak{a}(f,h) &= \sum_{i=1}^{n} \overline{d_i} \underbrace{\sum_{j=1}^{m} (\omega_{ij}^- - \omega_{ij}^+) f_j'(\mathsf{v}_i)}_{=0} - \sum_{j=1}^{m} \int_0^1 (c_j f_j')'(x) \overline{h_j(x)} \mu_j \, dx \\ &= -(Af, h)_{X_2},\end{aligned}$$

which makes sense because $Af \in X_2$. The proof of the inclusion $A \subset B$ is completed.

To check the converse inclusion $B \subset A$ take $f \in D(B)$. By definition, there exists $g \in X_2$ such that

$$\begin{aligned}\sum_{j=1}^{m} \int_0^1 \mu_j c_j(x) f_j'(x) \overline{h_j'(x)} \, dx = \mathfrak{a}(f,h) &= (g,h)_{X_2} \\ &= \sum_{j=1}^{m} \int_0^1 g_j(x) \overline{h_j(x)} \mu_j \, dx \qquad (2.7)\end{aligned}$$

for all $h \in V$, hence in particular for all $h^j \in V$ of the form

$$h^j = \begin{pmatrix} 0 \\ \vdots \\ h_j \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{th row}, \qquad h_j \in H_0^1(0,1).$$

From this it follows that (2.7) in fact implies

$$\begin{aligned}&\int_0^1 \mu_j c_j(x) f_j'(x) \overline{h_j'(x)} \, dx \\ &\quad = \int_0^1 g_j(x) \overline{h_j(x)} \mu_j \, dx \qquad \text{for all} \quad j = 1, \ldots, m, \quad h_j \in H_0^1(0,1).\end{aligned}$$

By definition of weak derivative this means that $c_j \cdot f_j' \in H^1(0,1)$ for all $j = 1, \ldots, m$. Since $0 < c_j \in H^1(0,1)$, it follows that in fact $f_j' \in H^1(0,1)$ for all $j = 1, \ldots, m$. We conclude that $f \in \left(H^2(0,1)\right)^m$. Moreover, integrating by parts as in (2.6) we see that if (2.7) holds for some $h \in V$, then

$$\sum_{i=1}^{n} d_i \sum_{j=1}^{m} (\omega_{ij}^- - \omega_{ij}^+) f_j'(\mathsf{v}_i) = 0,$$

where d_i is the joint value attained at the vertex v_i by all h_j, $j \in \Gamma(\mathsf{v}_i)$. Since $h \in V$ is arbitrary, this means that

$$\sum_{j=1}^{m} (\omega_{ij}^- - \omega_{ij}^+) f_j'(\mathsf{v}_i) = 0 \qquad \text{for all} \quad i = 1, \ldots, n,$$

that is, $\Phi_w^+ f'(0) = \Phi_w^- f'(1)$. Therefore $f \in D(A)$ and

$$-\sum_{j=1}^{m} \int_0^1 \mu_j (c_j f_j')'(x)\overline{h_j(x)}\,dx = \sum_{j=1}^{m} \int_0^1 g_j(x)\overline{h_j(x)}\mu_j\,dx$$

holds for all $h \in V$. This implies that $Af = -g$, and the proof is complete. □

We are now able to use some well-known results on sesquilinear forms (see [ABHN], [D], and [O]) in order to obtain nice properties of our operator A.

Proposition 2.4. *The operator* $(A, D(A))$ *defined in* (2.3)–(2.4) *is self-adjoint and dissipative. Thus, it generates a cosine operator function with associated phase space* $V \times X_2$.

Proof. By Lemmas 2.1 and 2.3 we are in the situation described in Section 7.1 of [ABHN] for $H = X_2$, $V = D(\mathfrak{a})$, $(\cdot \mid \cdot)_V = \mathfrak{a}(\cdot,\cdot)$, $\omega = 1$, and $A = A_H$. Thus the claim follows by Proposition 7.1.1 of [ABHN], Example 7.1.2 of [ABHN], and the fact that self-adjoint operators are unitarily equivalent to multiplication operators. (See also the remark at p. 413 in [ABHN].) □

We can now state the main result of this section. This generalizes the well-posedness and regularity results in [Al1], [Al2], [B2], and [CF], where only the case of constant or smooth coefficients $c_1, \dots, c_m$ was considered.

Theorem 2.5. *The problem* (2.1) *is well-posed, i.e., for all* $\mathsf{f} \in V$ *and* $\mathsf{g} \in X_2$ *it admits a unique mild solution that continuously depends on the initial data. If further* $c_j \in C^\infty[0,1]$, $j = 1, \dots, m$, *and the initial conditions* $\mathsf{f}, \mathsf{g} \in \left(C_c^\infty[0,1]\right)^m$, *then the solution is of class* $(C^\infty[0,1])^m$.

Proof. It is well known (see e.g. Corollary 3.14.12 of [ABHN]) that

$$u(t) := C(t, A)\mathsf{f} + S(t, A)\mathsf{g}, \qquad t \in \mathbb{R},$$

yields the unique mild solution to (2.5) for all initial data (f, g) in the phase space, where we denote by $(C(t,A))_{t\in\mathbb{R}}$ and $(S(t,A))_{t\in\mathbb{R}}$ the cosine and sine operator functions generated by A, respectively. The assertion about regularity of solutions follows directly from basic properties of cosine and sine operator functions. □

3. The Heat Equation on a Network

We now consider again the same network G and, under the same assumptions and with the same notations of Section 2, we turn our attention to the first-order problem

$$\begin{cases} \dot u_j(t,x) = (c_j u_j')'(t,x), & t \ge 0, \quad x \in (0,1), \quad j = 1, \dots, m, \\ u_j(t, \mathsf{v}_i) = u_\ell(t, \mathsf{v}_i), & t \ge 0, \quad j, \ell \in \Gamma(\mathsf{v}_i), \quad i = 1, \dots, n, \\ \sum_{j=1}^m \mu_j \varphi_{ij} c_j(\mathsf{v}_i) u_j'(t, \mathsf{v}_i) = 0, & t \ge 0, \quad i = 1, \dots, n, \\ u_j(0,x) = \mathsf{f}_j(x), & x \in (0,1), \quad j = 1, \dots, m. \end{cases} \tag{3.1}$$

This equation describes a diffusion process that takes place in a network and $c_1, \dots, c_m \in C^1[0,1]$ are (variable) diffusion coefficients or conductances. Again, we are imposing continuity and Kirchhoff-type conditions in the ramification nodes (controlled by some constants $\mu_1, \dots, \mu_m$).

It is already known that such a problem is well posed in an L^2-context; see [B2]. Moreover, at least for the case of constant weights $c_1, \dots, c_m$ and $\mu_1 = \cdots = \mu_m = 1$ the heat kernel has been computed in [Ni2], thus yielding well-posedness in other L^p-spaces. We show by variational methods that the semigroup governing (3.1) is L^∞-contractive, and hence we can extend the well-posedness result to an L^p-context by interpolation in the general case of variable diffusion coefficients. In particular, the analyticity of the L^p-semigroups seems to be a new result. Also observe that, by the bounded perturbation theorem, this also yields well-posedness for the Cauchy problem associated to the analogous cable equation; see [Ni3].

Let

$$X_p := \prod_{j=1}^{m} L^p(0,1;\mu_j dx), \qquad p \in [1,\infty].$$

We have already seen in Proposition 2.4 that A is a self-adjoint and dissipative operator on X_2. By the spectral theorem, this shows that A generates a contractive, analytic semigroup of angle $\pi/2$, and in particular the first-order abstract Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = \mathsf{f}, \end{cases}$$

is well posed in X_2. In fact, much more can be said.

Lemma 3.1. *The semigroup $(T_2(t))_{t\geq 0}$ on X_2, associated with $\mathfrak{a}$, is* sub-Markovian, *i.e., it is real, positive, and contractive on X_∞.*

Proof. By Proposition 2.5, Theorem 2.7, and Corollary 2.17 of [O], we need to check that the following criteria are verified for the domain V of $\mathfrak{a}$:

- $f \in V \Rightarrow \overline{f} \in V$ and $\mathfrak{a}(\mathrm{Re} f, \mathrm{Im} f) \in \mathbb{R}$,
- $f \in V$, f real-valued $\Rightarrow |f| \in V$ and $\mathfrak{a}(|f|,|f|) \leq \mathfrak{a}(f,f)$,
- $0 \leq f \in V \Rightarrow 1 \wedge f \in V$ and $\mathfrak{a}(1 \wedge f, (f-1)^+) \geq 0$.

It is clear that $\overline{k} \in H^1(0,1)$ if $k \in H^1(0,1)$. Further, if k is real-valued, then $|k| \in H^1(0,1)$ and $|k|' = \mathrm{sign} k \cdot k'$, and if $0 \leq k$, then $1 \wedge k \in H^1(0,1)$ with $(1 \wedge k)' = k' \mathbf{1}_{\{k<1\}}$ and $((k-1)^+)' = k' \mathbf{1}_{\{k>1\}}$.

By definition, the subspace V contains exactly those functions on the network that are continuous in the vertices (see (2.2)). Take any $f \in V$. By definition we have $\overline{f_j} = (\overline{f})_j$, $1 \leq j \leq m$. It follows from the above arguments that $\overline{f} \in (H^1(0,1))^m$, and one can see that the continuity of the values attained by f in the vertices is preserved after taking the complex conjugate $\overline{f}$. Hence, $\overline{f} \in V$. Moreover, $\mathfrak{a}(\mathrm{Re}\, f, \mathrm{Im}\, f)$ is the sum of m integrals. Recall that the weights are real-valued, positive functions. Since all the integrated functions are real-valued, it follows that $\mathfrak{a}(\mathrm{Re} f, \mathrm{Im} g) \in \mathbb{R}$. Thus, the first criterion has been checked.

Moreover, if f is a real-valued function in V, then $|f_j| = |f|_j$, $1 \leq j \leq m$, and one sees as above that $|f| \in V$. In particular, $||f|'|^2 = |f'|^2$, and it holds that

$$\mathfrak{a}(|f|, |f|) = \sum_{j=1}^{m} \int_0^1 \mu_j c_j(x) |f_j'(x)|^2 \, dx = \mathfrak{a}(f, f).$$

This shows that the second criterion applies.

Finally, take $0 \leq f \in V$. Then

$$1 \wedge f = 1 \wedge \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} 1 \wedge f_1 \\ \vdots \\ 1 \wedge f_m \end{pmatrix},$$

with all the functions $1 \wedge f_j \in H^1(0, 1)$, hence $1 \wedge f \in \left(H^1(0, 1)\right)^m$. Again, the continuity of f in the vertices imposes the same property to the function $1 \wedge f$, i.e., $1 \wedge f \in V$. Further, it holds that

$$\begin{aligned} \mathfrak{a}(1 \wedge f, (f-1)^+) &= \sum_{j=1}^{m} \int_0^1 \mu_j c_j (1 \wedge f_j)'(x)((f_j - 1)^+)'(x) \, dx \\ &= \sum_{j=1}^{m} \int_0^1 \mu_j c_j f_j'(x) \mathbf{1}_{\{f_j < 1\}}(x) f_j'(x) \mathbf{1}_{\{f_j > 1\}}(x) \, dx = 0. \end{aligned}$$

We have also checked the third criterion, thus the claim follows. □

Lemma 3.2. *The semigroup $(T_2(t))_{t \geq 0}$ on X_2 associated with $\mathfrak{a}$ is ultracontractive. In particular, it satisfies the estimate*

$$\|T_2(t) f\|_{X_\infty} \leq M t^{-1/4} \|f\|_{X_2} \qquad \text{for all} \quad t \in (0, 1], \quad f \in X_2, \tag{3.2}$$

for some constant M.

Proof. The form norm $\|\cdot\|_{\mathfrak{a}}$ on V is equivalent to the norm $\|\cdot\|_V$; see the proof of Lemma 2.1. Thus, by Theorem 6.3 and the following remark in [O] it suffices to show that it holds that

$$\|f\|_{X_2} \leq M \|f\|_V^{1/3} \cdot \|f\|_{X_1}^{2/3} \qquad \text{for all} \quad f \in V,$$

for some constant M. Recall the Nash inequality

$$\begin{aligned} \|k\|_{L^2(0,1)} &\leq M_1 (\|k'\|_{L^2(0,1)} + \|k\|_{L^1(0,1)})^{1/3} \cdot \|k\|_{L^1(0,1)}^{2/3} \\ &\leq M_1 \|k\|_{H^1(0,1)}^{1/3} \cdot \|k\|_{L^1(0,1)}^{2/3}, \end{aligned} \tag{3.3}$$

which is valid for all $k \in H^1(0, 1)$ and some constant M_1; see Theorem 1.4.8.1 of [Ma].

Take finally $f \in V$ and observe that by (3.3)

$$\begin{aligned}\|f\|^2_{X_2} &= \sum_{j=1}^m \|f_j\|^2_{L^2(0,1;\mu_j dx)} \le M_1^2 \sum_{j=1}^m \|f_j\|^{2/3}_{H^1(0,1;\mu_j dx)} \cdot \|f_j\|^{4/3}_{L_1(0,1;\mu_j dx)} \\ &\le M_2 \left(\sum_{j=1}^m \|f_j\|_{H^1(0,1;\mu_j dx)}\right)^{2/3} \cdot \left(\sum_{j=1}^m \|f\|_{L^1(0,1;\mu_j dx)}\right)^{4/3} \\ &\le M_2 \|f\|^{2/3}_V \cdot \|f\|^{4/3}_{X_1},\end{aligned}$$

using the Hölder inequality. Thus, the claim follows. □

The following now holds by Theorems 1.4.1, 1.6.4, and 2.1.5 of [D] and Theorem 3.13 of [O].

Corollary 3.3. *The semigroup $(T_2(t))_{t\ge0}$ extends to a family of compact, contractive, positive one-parameter semigroups $(T_p(t))_{t\ge0}$ on X_p, $1 \le p \le \infty$. Such semigroups are strongly continuous if $p \in [1,\infty)$, and analytic of angle $\pi/2 - \arctan(|p-2|/2\sqrt{p-1})$ for $p \in (1,\infty)$.*

Moreover, the spectrum of A_p is independent of p, where A_p denotes the generator of $(T_p(t))_{t\ge0}$, $1 \le p \le \infty$.

The estimate on the analyticity angle of $(T_p(t))_{t\ge0}$ is not sharp; see [Mu] for details.

Remark 3.4. Consider the part $\tilde{A}$ of A in $(C[0,1])^m$, whose domain is given by

$$\begin{aligned}D(\tilde{A}) = \{f \in (C^2(0,1))^m\colon\ &\Phi_w^+ f'(0) = \Phi_w^- f'(1) \text{ and} \\ &\exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^\top d = f(0) \text{ and } (\Phi^-)^\top d = f(1)\}.\end{aligned}$$

Define

$$C(G) := \{f \in (C[0,1])^m : \exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^\top d = f(0) \text{ and } (\Phi^-)^\top d = f(1)\},$$

which can be looked at as the space of all continuous functions on the graph G. It is easy to see that $\overline{D(\tilde{A})} = C(G)$. By Corollary 3.3 $\tilde{A}$ has positive resolvent, and it follows by Theorem 3.11.9 of [ABHN] that its part in $C(G)$ generates a positive strongly continuous semigroup.

In the next lemma we show that the generators of the semigroups in the spaces X_p, $1 \le p \le \infty$ (see Corollary 3.3), have in fact the same form as in X_2, with appropriate domain.

Lemma 3.5. *For all $p \in [1,\infty]$ the generator A_p of the semigroup $(T_p(t))_{t\ge0}$ is given by the operator defined in* (2.3) *with domain*

$$D(A_p) = \left\{ f \in \prod_{j=1}^m W^{2,p}(0,1;\mu_j dx)\colon\ \Phi_w^+ f'(0) = \Phi_w^- f'(1) \textit{ and} \right. \\ \left. \exists d \in \mathbb{C}^n \textit{ s.t. } (\Phi^+)^\top d = f(0) \textit{ and } (\Phi^-)^\top d = f(1) \right\}.$$

Proof. We prove the claim for $p > 2$. We have already remarked that $X_p \hookrightarrow X_q$ for all $1 \leq q \leq p \leq \infty$. Moreover, it follows by the ultracontractivity of $(T_2(t))_{t\geq 0}$ (see Lemma 3.2) that X_p is invariant under $(T_2(t))_{t\geq 0}$ for all $p > 2$ because if $f \in X_p$ then $f \in X_2$, and by (3.2),

$$\|T_2(t)f\|_{X_p} \leq C \cdot \|T_2(t)f\|_{X_\infty} \leq C \cdot Mt^{-1/4}\|f\|_{X_2} \leq C' \cdot Mt^{-1/4}\|f\|_{X_p}.$$

Thus, by Proposition II.2.3 of [EN] the generator of $(T_p(t))_{t\geq 0}$ is the part of A in X_p. A direct computation yields the claim.

For $1 \leq p < 2$ the claim can be proven by duality, mimicking the proof of Lemma 4.11 of [Mu]. □

Theorem 3.6. *The first-order problem* (3.1) *is well posed on* X_p, $p \in [1, \infty)$, *as well as on* $C(G)$, *i.e., for all initial data* $\mathsf{f} \in X_p$ *or* $\mathsf{f} \in C(G)$ *problem* (3.1) *admits a unique mild solution that continuously depends on the initial data.*

Such a solution is bounded in the time as well as (if $p > 1$*) in the space variables. If further* $\mathsf{f} \in X_p$, $1 < p < \infty$, *and* $c_j \in C^\infty[0, 1]$, $j = 1, \ldots, m$, *then the solution* $u(t, \cdot)$ *is of class* $(C^\infty[0, 1])^m$ *for all* $t > 0$, *and in particular the problem is solved pointwise for* $t > 0$.

Proof. The well-posedness and boundedness results follow from the fact that the operators A_p generate ultracontractive analytic semigroups. If $c_j \in C^\infty[0, 1]$, $j = 1, \ldots, m$, then we can show as in the proof of Theorem 2.5 that $D(A_p^\infty) \subset (C^\infty[0, 1])^m$ for all $p \in [1, \infty]$. Since the semigroup $(T_p(t))_{t\geq 0}$ is analytic, $1 < p < \infty$, it maps X_p into $D(A_p^\infty)$, and the claim follows. □

4. A Characteristic Equation

Having proved that the Cauchy problem (3.1) is well posed, we want to study the qualitative behavior of its solutions. To this end we investigate the spectrum of the generator $(A, D(A))$. Since by Corollary 3.3 the spectra of all A_p on X_p, $1 \leq p \leq \infty$, coincide, it suffices to study the operator $A = A_2$ on X_2. Hence, we are interested in the spectrum of the operator

$$A := \begin{pmatrix} c_1 \dfrac{d^2}{dx^2} & & 0 \\ & \ddots & \\ 0 & & c_m \dfrac{d^2}{dx^2} \end{pmatrix} \tag{4.1}$$

with domain

$$D(A) := \Big\{ f \in \prod_{j=1}^m \left(H^2(0, 1); \mu_j dx\right) : \ \Phi_w^+ f'(0) = \Phi_w^- f'(1) \text{ and } \\ \exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^\top d = f(0) \text{ and } (\Phi^-)^\top d = f(1) \Big\}. \tag{4.2}$$

Recalling properties of $(A, D(A))$ already yields some information on its spectrum.

Lemma 4.1. *The spectrum of $(A, D(A))$ lies on the negative real line and consists of eigenvalues only. Moreover, $s(A) = 0 \in \sigma(A)$.*

Proof. First note that $1 \in D(A)$ and $A1 = 0$, thus A is not invertible and $0 \in \sigma(A)$. Since A generates a contractive semigroup (cf. Proposition 2.4), $s(A) = 0$. It follows from Exercise II.4.30(4) of [EN] that, since $D(A)$ is contained in $\left(H^2(0,1); \mu_j dx\right)^m$, the resolvent of A is compact. Therefore the operator A only has point spectrum. Recall that by Proposition 2.4 the operator A is self-adjoint, hence all its eigenvalues are real. □

From now on we assume that all the weights c_i, $i = 1, \ldots, m$, are constant. Our aim is to find a "characteristic equation" for the spectrum of A. In particular, we will be able to connect the eigenvalues of the operator A to the eigenvalues of the *Laplacian* or *admittance matrix* of the corresponding graph. This is the $n \times n$ matrix

$$\mathcal{L} := D - \mathbb{A}, \tag{4.3}$$

where $\mathbb{A}$ is the standard 0–1 adjacency matrix of the graph and D the diagonal matrix of vertex degrees. It is well known that its spectrum reveals many properties of the graph, hence it is used in many applications (see e.g. [Ch], [Me], and [Mo2]).

We further define the *generalized weighted adjacency matrix* of the graph G in the case $0 < \lambda \neq c_j l^2 \pi^2$, $j = 1, \ldots, m$, $l \in \mathbb{Z}$, as

$$(\mathbb{A}_C(\lambda))_{ik} := \begin{cases} 0, & \text{if} \quad \Gamma(\mathsf{v}_i) \cap \Gamma(\mathsf{v}_k) = \emptyset, \\ \dfrac{\mu_j}{\sqrt{c_j}} \sin^{-1} \sqrt{\dfrac{\lambda}{c_j}}, & \text{if} \quad j \in \Gamma(\mathsf{v}_i) \cap \Gamma(\mathsf{v}_k). \end{cases}$$

By $D_C(\lambda)$ we denote the $n \times n$ diagonal matrix (again for $0 < \lambda \neq c_j l^2 \pi^2$, $j = 1, \ldots, m$, $l \in \mathbb{Z}$) defined as

$$D_C(\lambda) := \operatorname{diag}\left(\sum_{j \in \Gamma(\mathsf{v}_i)} \frac{\mu_j}{\sqrt{c_j}} \cot \sqrt{\frac{\lambda}{c_j}} \right)_{i=1,\ldots,n}.$$

Finally, we define the *generalized weighted Laplacian matrix* as

$$\mathcal{L}_C(\lambda) := D_C(\lambda) - \mathbb{A}_C(\lambda).$$

We now express the above matrices using the weighted incidence matrices. For this purpose we define diagonal matrices

$$\operatorname{Sin} x := \operatorname{diag}\left(\sin \frac{x}{\sqrt{c_1}}, \ldots, \sin \frac{x}{\sqrt{c_m}} \right),$$

$$\operatorname{Cos} x := \operatorname{diag}\left(\cos \frac{x}{\sqrt{c_1}}, \ldots, \cos \frac{x}{\sqrt{c_m}} \right),$$

$$\operatorname{Cot} x := \operatorname{Sin}^{-1} x \cdot \operatorname{Cos} x, \quad \text{and}$$

$$C := \operatorname{diag}(1/\sqrt{c_1}, \ldots, 1/\sqrt{c_m}).$$

Lemma 4.2. *For* $0 < \lambda \neq c_r l^2\pi^2$, $r = 1, \ldots, m$, $l \in \mathbb{Z}$ *we have*

$$\mathbb{A}_C(\lambda) = \Phi_w^+ \cdot C \cdot \operatorname{Sin}^{-1}\sqrt{\lambda} \cdot (\Phi^-)^\top + \Phi_w^- \cdot C \cdot \operatorname{Sin}^{-1}\sqrt{\lambda} \cdot (\Phi^+)^\top \quad \text{and}$$
$$D_C(\lambda) = \Phi_w^+ \cdot C \cdot \operatorname{Cot}\sqrt{\lambda} \cdot \left(\Phi^+\right)^\top + \Phi_w^- \cdot C \cdot \operatorname{Cot}\sqrt{\lambda} \cdot (\Phi^-)^\top.$$

We are now able to describe the spectrum of our operator A in terms of spectral values of $\mathcal{L}_C(\lambda)$. Similar results have already been obtained in much the same way as in [B1], [Ni1]–[Ni3], [B3], and [Ca] for the cases $\mu_j = 1$ and/or $c_j = 1$.

Theorem 4.3. *For the spectrum of the operator* $(A, D(A))$, *defined in* (4.1)–(4.2), *we obtain*

$$\sigma(A) = \{0\} \cup \sigma_C \cup \sigma_{\mathcal{L}},$$

where

$$\sigma_C \subseteq \{-c_i k^2\pi^2 \colon k \in \mathbb{Z}\backslash\{0\}, i = 1, \ldots, m\} \quad \textit{and}$$
$$\sigma_{\mathcal{L}} = \{-\lambda \in \mathbb{R}_- \colon \lambda \neq c_i k^2 \pi^2,\ \det \mathcal{L}_C(\lambda) = 0\}.$$

Furthermore:

(1) $\lambda = 0 \in \sigma(A)$ *is always an eigenvalue of* (*geometric and algebraic*) *multiplicity* 1 *with an eigenvector* $f(x) \equiv \mathbf{1}$, *the constant* 1 *function.*

(2) $-\lambda \in \sigma_{\mathcal{L}}$ *is an eigenvalue of A with corresponding eigenvector*

$$f(x) = \operatorname{Cos}\sqrt{\lambda}x \cdot (\Phi^+)^\top d + \operatorname{Sin}^{-1}\sqrt{\lambda} \cdot \operatorname{Sin}\sqrt{\lambda}x$$
$$\cdot((\Phi^-)^\top - \operatorname{Cos}\sqrt{\lambda} \cdot (\Phi^+)^\top)d,$$

where $d \in \ker \mathcal{L}_C(\lambda)$, *and so the multiplicity* $m(-\lambda)$ *of this eigenvalue is equal to* $\dim\ker \mathcal{L}_C(\lambda)$;

(3) $-c_i k^2\pi^2 \in \sigma_C$ *is an eigenvalue of A if and only if there exist* $b \in \mathbb{C}^m$ *and* $d \in \mathbb{C}^n$ *such that whenever* $j \in \Gamma(\mathsf{v}_r) \cap \Gamma(\mathsf{v}_s)$, $j \in \{1, \ldots, m\}$, *we have*

$$\begin{cases} d_r = (-1)^{\sqrt{(c_i/c_j)}k} d_s, & \textit{if} \quad \sqrt{\dfrac{c_i}{c_j}}k \in \mathbb{Z}, \\ b_j = \sin^{-1}\sqrt{\dfrac{c_i}{c_j}}k\pi \cdot d_r - \cot^{-1}\sqrt{\dfrac{c_i}{c_j}}k\pi \cdot d_s, & \textit{otherwise}. \end{cases} \tag{4.4}$$

These vectors further satisfy the equation

$$\Phi_w^- \cdot C \cdot \operatorname{Sin}\sqrt{c_i}k\pi \cdot (\Phi^+)^\top d = (\Phi_w^- \cdot C \cdot \operatorname{Cos}\sqrt{c_i}k\pi - \Phi_w^+ \cdot C) \cdot b. \tag{4.5}$$

If the eigenvector exists, *then it has the form*

$$f(x) = \operatorname{Cos}\sqrt{c_i}k\pi x \cdot (\Phi^+)^\top d + \operatorname{Sin}\sqrt{c_i}k\pi x \cdot b.$$

Proof. By Lemma 4.1, we need to solve the equation

$$Af = -\lambda f \qquad \text{for} \quad f \in D(A) \quad \text{and} \quad \lambda \geq 0.$$

We distinguish three cases.

Case 1: *Assume that* $\lambda \neq c_i k^2 \pi^2$ *for all* $k \in \mathbb{Z}$, $i = 1, \ldots, m$. In this case the eigenfunctions of A are of the form

$$f(x) = \mathrm{Cos}\sqrt{\lambda}x \cdot a + \mathrm{Sin}\sqrt{\lambda}x \cdot b \qquad \text{for some} \quad a, b \in \mathbb{C}^m.$$

From the continuity assumption in the domain of A (see (4.2))

$$\exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^\top d = f(0) \quad \text{and} \quad (\Phi^-)^\top d = f(1)$$

we obtain

$$f(x) = \mathrm{Cos}\sqrt{\lambda}x \cdot (\Phi^+)^\top d + \mathrm{Sin}^{-1}\sqrt{\lambda} \cdot \mathrm{Sin}\sqrt{\lambda}x \cdot ((\Phi^-)^\top - \mathrm{Cos}\sqrt{\lambda} \cdot (\Phi^+)^\top)d$$

for some $d \in \mathbb{C}^n$. The other condition $\Phi_w^+ f'(0) = \Phi_w^- f'(1)$ in the domain $D(A)$ (i.e., the Kirchhoff law) yields that $f \in \ker(\lambda - A)$ if and only if the vector $d \in \mathbb{C}^n$ satisfies

$$\begin{aligned}
&\Phi_w^+ \cdot C \cdot \mathrm{Sin}^{-1}\sqrt{\lambda} \cdot ((\Phi^-)^\top - \mathrm{Cos}\sqrt{\lambda} \cdot (\Phi^+)^\top)d \\
&\quad = \Phi_w^- \cdot C \cdot (\mathrm{Cot}\sqrt{\lambda} \cdot ((\Phi^-)^\top - \mathrm{Cos}\sqrt{\lambda} \cdot (\Phi^+)^\top) - \mathrm{Sin}\sqrt{\lambda} \cdot (\Phi^+)^\top)d.
\end{aligned}$$

Observe now that, by Lemma 4.2, the following two terms are the previously defined diagonal matrix

$$\Phi_w^+ \cdot C \cdot \mathrm{Cot}\sqrt{\lambda} \cdot \left(\Phi^+\right)^\top + \Phi_w^- \cdot C \cdot \mathrm{Cot}\sqrt{\lambda} \cdot (\Phi^-)^\top = D_C(\lambda),$$

while rearranging the remaining terms yields the weighted adjacency matrix

$$\begin{aligned}
&\Phi_w^+ \cdot C \cdot \mathrm{Sin}^{-1}\sqrt{\lambda} \cdot (\Phi^-)^\top + \Phi_w^- \cdot C \cdot (\mathrm{Sin}^{-1}\sqrt{\lambda} \cdot \mathrm{Cos}^2\sqrt{\lambda} + \mathrm{Sin}\sqrt{\lambda}) \cdot (\Phi^+)^\top \\
&\quad = \Phi_w^+ \cdot C \cdot \mathrm{Sin}^{-1}\sqrt{\lambda} \cdot (\Phi^-)^\top + \Phi_w^- \cdot C \cdot \mathrm{Sin}^{-1}\sqrt{\lambda} \cdot (\Phi^+)^\top = \mathbb{A}_C(\lambda).
\end{aligned}$$

Summing up, the condition for $d \in \mathbb{C}^n$ becomes

$$(\mathbb{A}_C(\lambda) - D_C(\lambda))\, d = 0, \qquad \text{that is} \quad d \in \ker \mathcal{L}_C(\lambda).$$

Case 2: $\lambda = 0$. The eigenfunctions of A, corresponding to $\lambda = 0$, are of the form

$$f(x) = x \cdot a + b \qquad \text{for some} \quad a, b \in \mathbb{C}^m.$$

We repeat the above procedure and the conditions in the domain (2.4) yield

$$f(x) = (\Phi^+)^\top d - x \cdot \Phi^\top d \qquad \text{for} \quad d \in \ker \Phi_w \Phi^\top \quad \text{with} \quad \Phi_w = \Phi_w^+ - \Phi_w^-.$$

Since our graph is connected, the multiplicity of 0 in $\sigma(\Phi_w \Phi^\top)$ is 1 (see Lemma 1.7(iv) of [Ch] or Proposition 2.3 of [Mo2]). It is easy to see that the corresponding eigenvector equals $d = \mathbf{1} := (1, \ldots, 1)^\top$. Now compute

$$f(x) = (\Phi^+)^\top \mathbf{1} - x \cdot \Phi^\top \mathbf{1} \equiv \mathbf{1} \qquad \text{for all } x.$$

Case 3: $\lambda = c_i k^2 \pi^2$ *for some* $k \in \mathbb{Z} \backslash \{0\}$ *and some* $i \in \{1, \ldots, m\}$. We proceed as before while some care need to be taken with zero entries that arise. Before applying the inverse of $\operatorname{Sin}\sqrt{\lambda}$, the continuity condition in (4.2) implies

$$f(x) = \operatorname{Cos}\sqrt{c_i} k\pi x \cdot (\Phi^+)^\top d + \operatorname{Sin}\sqrt{c_i} k\pi x \cdot b,$$

where b satisfies the equation

$$\operatorname{Sin}\sqrt{c_i} k\pi \cdot b = \left((\Phi^-)^\top - \operatorname{Cos}\sqrt{c_i} k\pi \cdot (\Phi^+)^\top\right) d. \tag{4.6}$$

Since the ith entry on the left-hand side equals 0, the vector d should satisfy the condition

$$d_p = (-1)^k d_q \qquad \text{for} \quad i \in \Gamma(\mathsf{v}_p) \cap \Gamma(\mathsf{v}_q).$$

Moreover, if for any other $j \in \{1, \ldots, m\}$ we have $\sqrt{(c_i/c_j)}k \in \mathbb{Z}$, then also

$$d_r = (-1)^{\sqrt{(c_i/c_j)}k} d_s \quad \text{for} \quad j \in \Gamma(\mathsf{v}_r) \cap \Gamma(\mathsf{v}_s).$$

For each of these cases we have no conditions on b_j. If on the other hand $\sqrt{(c_i/c_j)}k \notin \mathbb{Z}$, (4.6) yields

$$b_j = \operatorname{Sin}^{-1}\sqrt{\frac{c_i}{c_j}}k\pi \cdot d_r - \operatorname{Cot}^{-1}\sqrt{\frac{c_i}{c_j}}k\pi \cdot d_s, \qquad j \in \Gamma(\mathsf{v}_r) \cap \Gamma(\mathsf{v}_s).$$

Furthermore, the condition $\Phi_w^+ f'(0) = \Phi_w^- f'(1)$ in the domain $D(A)$ (i.e., the Kirchhoff law) implies that above vectors d and b have to satisfy (4.5). □

We emphasize that the condition (3) in the above theorem is not always satisfied, therefore the spectrum of our operator A strongly relies on the underlying graph and on the weights c_j.

From now on we assume $c_j = 1$, $j = 1, \ldots, m$. In this case we are able to connect the spectrum of the operator A to the spectrum of yet another matrix known in graph theory. The *transition matrix* is defined as

$$\mathbb{P} := D^{-1}\mathbb{A}$$

and is studied in connection with random walks on graphs (see e.g. Section 5.2 of [Mo2] or Section 1.5 of [Ch]). The matrix $\mathbb{P}$ is always a positive, symmetric, row stochastic matrix with eigenvalues

$$\sigma(\mathbb{P}) = \{\alpha_1, \ldots, \alpha_n\} \qquad \text{where} \quad -1 \le \alpha_n \le \ldots \le \alpha_2 < \alpha_1 = 1.$$

By Claim 5.3 of [Mo2], 1 is a simple eigenvalue whenever G is connected (what we assumed at the beginning) and -1 is an eigenvalue if and only if G is bipartite (see also Lemma 1 in Section 5 of [B1]). It turns out that an important subset of the spectrum depends on the fact of whether or not the graph G is *bipartite*. This property means that the set of vertices can be divided into two disjoint subsets V_1 and V_2 such that any edge of G has one endpoint in one and the other endpoint in the other subset. Note that G is bipartite if and only if it does not have any odd cycle.

The following characteristic equation has already been proved by von Below [B1]. We state and sketch the proof in our context for the convenience of the reader.

Theorem 4.4. *Let $(A, D(A))$ be the operator defined in* (4.1)–(4.2), *with $c_j = 1$, $j = 1, \dots, m$. Then for the spectrum of A we have*

$$\sigma(A) = \{0\} \cup \sigma_p \cup \sigma_k,$$

where

$$\sigma_p = \{-(2l\pi \pm \arccos\alpha)^2\colon\ \alpha \in \sigma(\mathbb{P})\backslash\{-1, 1\} \text{ and } l \in \mathbb{Z}\}$$

and

$$\sigma_k = \{-k^2\pi^2\colon\ k \in \mathbb{Z}\backslash\{0\}\}.$$

For the multiplicities of the eigenvalues we have:

(1) $m(0) = 1$;
(2) $m(-\lambda) = \dim\ker(\cos\sqrt{\lambda} \cdot I - \mathbb{P})$ *for* $-\lambda \in \sigma_p$;
(3) $m(-k^2\pi^2) = m - n + 2$, *if G is bipartite*;
(4) $m(-4l^2\pi^2) = m - n + 2$ *and* $m(-(2l+1)^2\pi^2) = m - n$, *if G is not bipartite*.

Proof. We use Theorem 4.3 for $C = I$. For the spectral point $\lambda = 0$ the statement follows directly from Theorem 4.3. Assume first that $\lambda \neq k^2\pi^2$ for any $k \in \mathbb{Z}$. Then

$$\mathcal{L}_I(\lambda) = \sin^{-1}\sqrt{\lambda}(\cos\sqrt{\lambda} \cdot D - \mathbb{A}),$$

and the characteristic equation becomes

$$-\lambda \in \sigma(A) \quad \Longleftrightarrow \quad \det(\cos\sqrt{\lambda} \cdot D - \mathbb{A}) = 0 \quad \Longleftrightarrow \quad \det(\cos\sqrt{\lambda} \cdot I - \mathbb{P}) = 0$$

for the transition matrix $D^{-1}\mathbb{A} = \mathbb{P}$. The last equivalence says that

$$-\lambda \in \sigma\,(A) \quad \Longleftrightarrow \quad \cos\sqrt{\lambda} \in \sigma\,(\mathbb{P}) \quad \Longleftrightarrow \quad \lambda = (2l\pi \pm \arccos\alpha)^2$$

for some $\alpha \in \sigma\,(\mathbb{P})$, $-1 < \alpha < 1$, and $l \in \mathbb{Z}$. Since $\dim\ker(\cos\sqrt{\lambda} \cdot D - \mathbb{A}) = \dim\ker(\cos\sqrt{\lambda} \cdot I - \mathbb{P})$, statement (2) also follows by Theorem 4.3(2).

Now, let $\lambda = k^2\pi^2$ for some $k \in \mathbb{Z}\backslash\{0\}$. Observe that condition (4.4) in Theorem 4.3(3) becomes

$$d_r = (-1)^k d_s \qquad \text{whenever} \quad \Gamma(\mathsf{v}_r) \cap \Gamma(\mathsf{v}_s) \neq \emptyset.$$

If k is even this condition is always fulfilled for $d = c \cdot \mathbf{1}$ for any $c \in \mathbb{R}$. Because the network is assumed to be connected, there is no other solution. For odd k we can always choose $d = 0$. However, we can find a nonzero d only in the case G does not have any odd cycles, that is, when G is bipartite—hence, when its set of vertices can be divided into two disjoint subsets V_1 and V_2 such that any edge of G has one endpoint in one and the other endpoint in the other subset. If this holds, the coordinates of d can be chosen in such a way that at the places of vertex indices belonging to V_1 we set c and at the places of vertex indices belonging to V_2 we set $-c$, $c \in \mathbb{R}$. By connectivity these are again all possible solutions.

Since $\sin k\pi = 0$ and $\cos k\pi = (-1)^k$, the other condition (4.5) in Theorem 4.3(3) becomes

$$(\Phi_w^+ - (-1)^k \Phi_w^-)b = 0.$$

Using the proof of Theorem 5(17) of [B1] we obtain (3) and (4). □

5. Stability Results for the Diffusion Problem

In the last section we were interested in the asymptotic behavior of solutions to the problem (3.1). By Corollary 3.3, the corresponding semigroup $T_p(t)_{t\geq 0}$ on $X_p = (L^p(0,1);\mu_j dx)^m$, $1 \leq p < \infty$, has many nice properties: it is contractive, compact, positive. These properties already yield norm convergence of the solutions to an equilibrium (see Corollary V.2.15 of [EN]).

From the connectedness of our graph, used in the proof of Theorem 4.3(1), we obtain another useful property of the semigroup.

Proposition 5.1. *The semigroup $(T_p(t))_{t\geq 0}$ on X_p, $p \in [1,\infty)$, is irreducible.*

Proof. It is enough to prove the statement for $p = 2$, because the irreducibility is inherited for the extrapolation semigroups in X_p, $1 \leq p < \infty$, using Corollary 3.3 and [Ar, Theorem 7.2.2]. Since X_2 is reflexive and $(T_2(t))_{t\geq 0}$ is bounded, by Example V.4.7 of [EN] we have that the semigroup is *mean ergodic* (see Definition V.4.3 of [EN]). By Theorem 4.3(1), we obtain that the corresponding *mean ergodic projection* P defined by

$$Px := \lim_{r\to\infty} \frac{1}{r}\int_0^r T(s)x\, ds$$

is the projection $\mathbf{1} \otimes \mathbf{1}$ onto the subspace $\langle \mathbf{1} \rangle$. Let $\{0\} \neq J \subset X_2$ be a closed invariant ideal for $(T(t))$, that is, $T(t)J \subset J$, $t \geq 0$. Then also $PJ \subset J$ holds. By definition, $PJ \subset \langle \mathbf{1} \rangle$ and so J contains the closed ideal generated by a constant function—hence the whole space X_2. From this it follows that the semigroup is irreducible. □

Knowing that our semigroup is irreducible we can now show its norm convergence toward a projection of rank one.

Corollary 5.2. *For the semigroup $(T_p(t))_{t\geq 0}$ on X_p, $p \in [1, \infty)$, the following hold*:

1. *The limit $Pf := \lim_{t\to\infty} T_p(t)f$ exists for every $f \in X_p$.*
2. *P is a strictly positive projection onto $\langle \mathbf{1} \rangle = \mathrm{Ker}A$, the one-dimensional subspace of X_p spanned by the constant function $\mathbf{1}$.*
3. *For every $\varepsilon > 0$ there exists $M > 0$ such that*

$$\|T_p(t) - P\| \leq Me^{(\varepsilon+\lambda_2)t} \qquad \text{for all} \quad t \geq 0, \tag{5.1}$$

where λ_2 is the largest nonzero eigenvalue of the generator A.

Proof. The first assertion follows from Corollary C-IV.2.10 of [Na] and the second from C-III.3.5(d) of [Na] and Theorem 4.3. Since P is the residue corresponding to the spectral value $\lambda_1 = 0$ and 0 is a first-order pole of the resolvent, estimate (3.2) in Corollary V.3.2 of [EN] yields the third assertion. □

Note that the property of converging to an equilibrium does not depend on the structure of the network. However, the speed of the convergence toward a projection is determined by the second largest eigenvalue λ_2 of A and thus relies on the network.

Combining graph theory and results about the spectrum of A, obtained in Section 4, we can now draw some further estimates containing graph parameters for the speed of convergence in (5.1). Let us demonstrate this for the case when all $c_j = 1$ and the graph is *regular*, that is, every vertex has the same degree. This means that

$$|\Gamma(\mathsf{v}_i)| = \gamma \qquad \text{for all} \quad i = 1, \ldots, n, \quad \text{and} \quad D = \gamma \cdot I.$$

Two generic examples of regular graphs are the *n-cycle C_n* and the *complete graph K_n* on n vertices (in the latter, every two vertices are connected by an edge).

In this case the characteristic equation becomes

$$-\lambda \in \sigma(A) \quad \iff \quad \det(\cos\sqrt{\lambda} \cdot \gamma \cdot I - \mathbb{A}) = 0 \quad \iff \quad \cos\sqrt{\lambda} \cdot \gamma \in \sigma(\mathbb{A});$$

see the proof of Theorem 4.4 or Section 6 of [B1]. Using the Laplacian of the graph, defined in (4.3), we obtain for its spectrum

$$\nu \in \sigma(\mathcal{L}) \quad \iff \quad \det(\nu \cdot I - (\gamma \cdot I - \mathbb{A})) = 0 \quad \iff \quad -\nu + \gamma \in \sigma(\mathbb{A}).$$

Hence, investigating the spectrum of the generator A, we are looking for λs such that

$$\lambda = -\left(2l\pi \pm \arccos\left(1 - \frac{\nu}{\gamma}\right)\right)^2, \qquad l \in \mathbb{Z}, \tag{5.2}$$

where $\nu \in \sigma(\mathcal{L})$. The spectrum of $\mathcal{L}$ is sometimes also called the *spectrum of the graph* and is well investigated in graph theory. In the following we always refer to the survey paper [Mo1].

Example 5.3. In the case when $G = C_n$, the eigenvalues of $\mathcal{L}$ are precisely the numbers

$$\nu_k = 2 - 2\cos\left(\frac{2(k-1)\pi}{n}\right), \quad k = 1, \ldots, n,$$

while for $G = K_n$ we have

$$\nu_1 = 0 \quad \text{and} \quad \nu_k = n \qquad \text{for} \quad 2 \le k \le n.$$

As explained above we are interested in the *second-smallest* eigenvalue $\nu_2 \in \sigma(\mathcal{L})$, which is also called the *algebraic connectivity* of the graph; see [Mo1]. It is related to the classical connectivity parameters of the graph (see below). If we look at (5.2) we can easily conclude that

$$\lambda_2 = -\left(\operatorname{arc\,cos}\left(1 - \frac{\nu_2}{\gamma}\right)\right)^2, \tag{5.3}$$

since the function arc cos is strictly monotone decreasing and assumes its values between 0 and π.

Example 5.4. By increasing the number of vertices n, the convergence gets slower on the cycle C_n and on the complete graph K_n. In fact, for C_n we have $\gamma = 2$, hence by Example 5.3 we obtain that

$$\lambda_2 = -\left(\operatorname{arc\,cos}\left(1 - (1 - \cos\frac{2\pi}{n})\right)\right)^2 = -\frac{4\pi^2}{n^2}.$$

For K_n we have $\gamma = n - 1$, hence by Example 5.3,

$$\lambda_2 = -\left(\operatorname{arc\,cos}\left(1 - \frac{n}{n-1}\right)\right)^2 = -\left(\operatorname{arc\,cos}\left(-\frac{1}{n-1}\right)\right)^2.$$

If we have an estimate from *below* for ν_2, using (5.1) and formula (5.3), we obtain an *upper* estimate for the convergence speed of the semigroup to the one-dimensional projection. In [Mo1] we find many estimates for ν_2 that use several graph parameters. As an example we mention:

Definition 5.5. The *edge connectivity parameter* $\eta = \eta(G)$ of the graph G is defined as the minimum number of edges whose deletion from G disconnects it.

Example 5.6. By Theorem 6.2(b) of [Mo1],

$$\nu_2 \ge 2\eta\left(1 - \cos\frac{\pi}{n}\right)$$

holds, where η is the edge connectivity parameter of the graph. From this we obtain for

(5.1) that for every $\varepsilon > 0$ there exists an $M > 0$ such that

$$\|T_p(t) - P\| \le Me^{(\varepsilon-(\arccos(1-(2\eta/\gamma)(1-\cos(\pi/n))))^2)t} \qquad \text{for all} \quad t \ge 0.$$

Another estimate can be obtained using the *diameter* of G.

Definition 5.7. The *diameter* of G denoted by $\mathrm{diam}(G)$ is the largest number of vertices that must be traversed in order to travel from one vertex to another when paths that backtrack, detour, or loop are excluded from consideration.

Example 5.8. By (6.10) of [Mo1], we have

$$\nu_2 \ge \frac{4}{n \cdot \mathrm{diam}(G)}.$$

Hence, again we can conclude that for every $\varepsilon > 0$ there exists an $M > 0$ such that

$$\|T_p(t) - P\| \le Me^{(\varepsilon-(\arccos(1-4/(\gamma\cdot n\cdot\mathrm{diam}(G))))^2)t} \qquad \text{for all} \quad t \ge 0.$$

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Accepted 9 *July* 2006. *Online publication* 6 *March* 2007.