Functorial relations in the p-adic Langlands program

Habilitation thesis

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Contents

1 Introduction

1.1 From the Riemann ζ to L-functions

The Langlands programme is one of the central topics in modern Number Theory. To explain the starting point consider the Riemann ζ-function $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ (valid for complex s with $\mathrm{Re}(s) > 1$). This function carries a lot of arithmetic information, partly because it can be written as an Euler-product

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}} .$$

Therefore it is no surprise that certain analytic properties of $\zeta(s)$ (such as e.g. meromorphic continuation to the whole complex plane; zero-free regions in the critial strip; functional equation relating $\zeta(1-s)$ to $\zeta(s)$) imply deep theorems on the distribution of primes. It is less well-known that the special values of ζ (at odd negative integers) have connections to algebraic number theory, too—the values $\zeta(-1), \zeta(-3), \dots, \zeta(2-p)$ carry information on the arithmetic of the cyclotomic field $\mathbb{Q}(\mu_p)$ adjoining the pth roots of unity to $\mathbb{Q}$, especially whether or not the ring of integers in $\mathbb{Q}(\mu_p)$ is a unique factorization domain. The primary goal of the Langlands programme is to establish similar methods relating more general questions in arithmetic to certain properties of more general functions similar to ζ, the so-called L-functions. The central question on the analytic side is analytic continuation to the whole complex plane and a possible functional equation. The practically only known methods to show these properties for a large class of such L-functions is via *modular forms*, or via similar, but more general objects called *automorphic forms*. Modular forms are certain functions f on the complex upper half plane $\mathcal{H}$ with a lot of symmetries: the group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}$ via fractional linear maps and the modular forms transform in a well-described way under a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{R})$. For instance, the invariance under the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ means that the function f is periodic by 1, ie. we have $f(z) = f(z+1)$. This allows us (using further properties, too) to expand $f(z)$ as a q-series $f(z) = \sum_{n=1}^{\infty} a_n q^n$ where we put $q := e^{2\pi i z}$ (strictly speaking, for $a_0 = 0$ we need the vanishing of f at the cusp at infinity). In particular, we can attach to f an L-series $L_f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$. On the other hand, the transformation under the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (ie. under $z \mapsto -1/z$) leads to the functional equation relating $L_f(s)$ and $L_f(1-s)$, similar to the functional equation of the ζ-function. It is a theorem of Hecke that the L-function L_f of a modular form f indeed has the required analytic continuation to the whole complex plane (and satisfies a functional equation). Moreover, Weil's converse theorem (and its generalizations) explains why this is essentially the only way of proving these analytic properties: whenever an L-series $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ can be continued analytically to $\mathbb{C}$, satisfies certain growth condition, and a functional equation, then it indeed comes from a modular form (or, in the generalizations assuming different growth conditions, implies that $L(s)$ comes from an automorphic form). Therefore the main task of the Langlands programme is to prove that a certain wide class of arithmetic objects are *modular*, ie. the L-function attached to them comes from a modular form, or more generally from an automorphic form (and therefore satisfies a functional equation and can be continued analytically to $\mathbb{C}$).

1.2 Arithmetic examples

1.2.1 The L-function of a Galois representation

In order to describe these "arithmetic objects"—that we hope to attach an L-function to—let us have a closer look at the above formula for $\zeta(s)$. The denominators in each term in the Euler product are polynomials $P_p(T)$ in $T = p^{-s}$—in this case these polynomials have degree one and they are equal to $P_p(T) = 1 - T$ for each prime p. The natural objects, to which one can attach such a bunch of polynomials, are *Galois representations.* A Galois representation (of $\mathbb{Q}$) is a finite dimensional continuous representation of the absolute Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (see section A.1) over a (for now arbitrary, topological) field K, ie. a continuous group homomorphism $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(K)$ for some natural number n. Putting V for the n-dimensional K-vectorspace on which $G_{\mathbb{Q}}$ acts via ρ the polynomial $P_p(T)$ is the—slightly modified—characteristic polynomial

$$P_p(T) := \det(\mathrm{id}_{V^{I_p}} - \rho(\mathrm{Frob}_p)_{|V^{I_p}} T) \in K[T]$$

of the matrix of $\rho(\mathrm{Frob}_p)$ acting on the subspace $V^{I_p} := \{v \in V \mid \rho(g)v = v\ \forall g \in I_p\}$. Here Frob_p is a lift of the Frobenius element to characteristic 0 and I_p denotes the inertia subgroup at p (for the definition and background on these see appendix A.1). Note that this is independent of all our choices therein. Now suppose K is (abstractly isomorphic to) a subfield of $\mathbb{C}$ (in particular, $\mathrm{char}(K) = 0$). Then the L-function of the Galois representation ρ is defined as

$$L(\rho, s) := \prod_{p \text{ prime}} \frac{1}{P_p(p^{-s})}$$

for $s \in \mathbb{C}$ with sufficiently big real part.

1.2.2 The special case of Dirichlet L-functions

In this language the Riemann ζ-function is the L-function of the trivial Galois representation (sending each group element to the identity matrix in dimension 1). Further, if $\chi\colon (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$ is a Dirichlet character (assumed to be primitive for simplicity) then identifying $\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$ with $(\mathbb{Z}/n\mathbb{Z})^\times$ we also obtain a Galois representation as a composite map

$$\rho_\chi\colon G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times .$$

The Dirichlet L-function $L(\chi, s) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}$ (putting $\chi(m) := 0$ if $(m, n) \neq 1$) is the L-function of the Galois representation obtained this way. Indeed, the primitivity of χ implies that for each prime p dividing n the representation ρ_χ *ramifies* at p, ie. the image $\rho_\chi(I_p)$ of the inertia is not just $\{1\}$. Therefore $V^{I_p} = \{0\} \leq V \cong \mathbb{C}$ whence the Euler factor at p is just 1. On the other hand, if $p \nmid n$ then ρ_χ is unramified and the p-Frobenius element in $\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$ raises an nth root of unity to its pth power therefore the image of Frob_p is $p \mod n$ under the identification with $(\mathbb{Z}/n\mathbb{Z})^\times$. In other words $\rho_\chi(\mathrm{Frob}_p) = \chi(p)$ and the L-function $L(\rho_\chi, s)$ is defined as the Euler product

$$L(\rho_\chi, s) = \prod_{p \nmid n} \frac{1}{1 - \chi(p)p^{-s}} = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} .$$

Both these above examples are *Artin representations*: $K = \mathbb{C}$ with its usual topology. However, it is not hard to see that Artin representations factor through a finite quotient of $G_{\mathbb{Q}}$ (due to the fact that a sufficiently small neighbourhood of the identity in $\mathrm{GL}_n(\mathbb{C})$ contains no nontrivial subgroups, however its pre-image is a finite index subgroup in $G_{\mathbb{Q}}$ as it is open in the compact space $G_{\mathbb{Q}}$).

1.2.3 Galois representations attached to elliptic curves

Arithmetic geometry provides us with more complicated examples. The easiest case is that of elliptic curves. Let therefore E be an elliptic curve defined over $\mathbb{Q}$, ie. a smooth projective curve of genus 1 defined by equations with rational coefficients together with a distinguished rational point $O \in E(\mathbb{Q})$. Now if K is any field of characteristic 0 then the set $E(K)$ of K-rational points on E forms an abelian group with a geometrically defined operation usually denoted by $+$, and O as zero element. In case $K = \mathbb{C}$ the group of complex points $E(\mathbb{C})$ is a torus $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \cong S^1 \times S^1$ for some non-real complex number τ. In particular, the subgroup $E(\mathbb{C})[n]$ of points P of order dividing n (for some positive integer n) is isomorphic to $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ since on each copy of S^1 the points of order dividing n are the nth roots of unity. Now it follows from the formula for addition of points that in fact the coordinates of these n^2 points in $E(\mathbb{C})[n]$ are algebraic numbers and therefore we obtain an action of $G_{\mathbb{Q}}$ on the abelian group $E(\mathbb{C})[n]$, ie. a group homomorphism

$$\rho_{E,n}\colon G_{\mathbb{Q}} \to \mathrm{Aut}(E(\mathbb{C})[n]) \cong \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}) \ .$$

These homomorphisms are compatible in the sense that whenever $n \mid m$ then $\rho_{E,m}(g)$ is congruent to $\rho_{E,n}(g)$ modulo n for all $g \in G_{\mathbb{Q}}$. So choosing $n = \ell^r$ for some prime ℓ and letting the integer r go to ∞ we obtain a representation

$$\rho_{E,\ell^\infty}\colon G_{\mathbb{Q}} \to \varprojlim_r \mathrm{GL}_2(\mathbb{Z}/\ell^r\mathbb{Z}) = \mathrm{GL}_2(\mathbb{Z}_\ell) \leq \mathrm{GL}_2(\mathbb{Q}_\ell)$$

where $\mathbb{Z}_\ell$ is the ring of ℓ-adic integers, ie. the closed unit ball in the field $\mathbb{Q}_\ell$ of ℓ-adic numbers. Now the Euler factor at a prime $p \neq \ell$ is defined using this Galois representation which is, for fixed p, independent of ℓ. Moreover, if (and only if) E has *good reduction at* p (still with $p \neq \ell$), ie. the curve E is smooth when reduced modulo p, or equivalently, p does not divide the discriminant of E, then, by the criterion of Néron–Ogg–Shafarevich, the representation ρ_{E,ℓ^∞} is unramified at p, ie. $\rho_{E,\ell^\infty}(I_p) = \{1\}$. In this case $P_{E,p}(T)$ has degree 2 and equals $1-a_pT+pT^2$ where $N_p := 1-a_p+p$ is the number of $\mathbb{F}_p$-rational points on the reduced curve $\widetilde{E}_p$ modulo p. In particular, this polynomial has integer coefficients whence the complex L-function, as an Euler-product $L(E,s) := \prod_{p \text{ prime}} \frac{1}{P_{E,p}(p^{-s})}$, makes sense. The Taniyama–Shimura conjecture (now proven by the work of Wiles [41], Taylor–Wiles [37], and Breuil–Conrad–Diamond–Taylor [4]) states that all elliptic curves over $\mathbb{Q}$ are modular, ie. their L-function comes from a modular form.

1.3 Geometric Galois representations

The above example of an ℓ-adic Galois representation attached to an elliptic curve can be generalized to other smooth projective varieties over $\mathbb{Q}$ the following way. The fundamental idea (that goes back to Weil's formulation of his conjectures around 1940 and was systematically developed by Grothendieck and his students since the 1960s) is to construct an algebraic cohomology theory for smooth projective varieties (or more generally, for schemes) that have nice properties similar to singular cohomology for topological spaces and admit an action of the absolute Galois group of the ground field (mainly because of being "algebraic"). Certainly the most important such cohomology theory (at least in characteristic 0, but for some applications also in characteristic p) is *étale cohomology* (see appendix A.2). In case of elliptic curves the ℓ-adic Tate module $T_\ell(E) := \varprojlim_r E[\ell^r]$ in the above example is isomorphic to the first étale cohomology group $H^1_{et}(E, \mathbb{Z}_\ell)$. The *geometric* ℓ-adic Galois representations are the étale cohomology groups of smooth projective varieties defined over $\mathbb{Q}$ with coefficients in $\mathbb{Q}_\ell$ (see appendix A.2). For technical reasons one has to allow *Tate twists* of these representations, ie. tensor products with integral powers of the cyclotomic character $\chi_{cyc}\colon G_{\mathbb{Q}} \to \mathbb{Z}_\ell^\times$ sending $g \in G_{\mathbb{Q}}$ to the number modulo ℓ^r to which power g sends an ℓ^rth root of unity. The open question of fundamental importance in the theory is to characterize those ℓ-adic Galois representations that arise in geometry. The following is a conjecture of Fontaine and Mazur:

Conjecture 1.1 (Fontaine–Mazur)**.** *An ℓ-adic Galois representation $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ comes from geometry if and only if the following two conditions hold:*

(i) ρ is unramified (ie. $\rho(I_p) = \{1\}$) at all but finitely many primes p.

(ii) ρ is de Rahm at the prime $p = \ell$.

For the definition what "de Rahm" means here see appendix A.3. The "only if" part of the above conjecture is known by now: Grothendieck proved that a geometric Galois representation is always unramified at all but finitely many primes (note that in the case of elliptic curves those primes ramify which divide the discriminant, so these are only finitely many). Assertion (*ii*) was deeper and was first proven by Faltings [22], and later, using different methods, by many others. For instance, there is a beautiful, more geometric proof recently by Beilinson [3] resembling the comparison isomorphism between de Rahm and singular cohomology in the complex case. The latter proof is surveyed in the present author's joint paper [36] with T. Szamuely (see the appendix A.3 for more on this).

The "if" part of the Fontaine–Mazur conjecture is extremely difficult in general. The case of dimension $n = 1$ is essentially the theorem of Kronecker and Weber and is therefore known. The case $n = 2$ has been proven (upto certain non-generic cases) recently by Kisin [28] (see also Emerton's modularity result [20] that we briefly recall in section 1.5) using crucially the p-adic Langlands correspondence in dimension 2 constructed by Colmez [17] and its local–global compatibility by Breuil and Emerton [7]. It is generally expected that a strong enough version of the p-adic Langlands correspondence in higher dimensions will have nontrivial consequences towards the Fontaine–Mazur conjecture.

1.4 The local Langlands theorems

One key input into the global Langlands programme is the local Langlands correspondence. On the Galois side we have seen that we can restrict any n-dimensional representation ρ of $G_{\mathbb{Q}}$ to a representation ρ_p of $G_{\mathbb{Q}_p}$ for any prime p. On the other hand, as recalled briefly in the appendix B.1, one can break up each global automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$

into a restricted tensor product $\bigotimes'_p \pi_p$ of representations π_p of $\mathrm{GL}_n(\mathbb{Q}_p)$ for the primes p (and a term π_∞ corresponding to the archimedean field $\mathbb{R}$). Informally, the goal of the local Langlands programme is to match ρ_p with π_p such that their local L- and ε-factors—the formal definition of the latter would take us too much aside—are the same. For the group GL_n such a bijection is a theorem first proven by Harris and Taylor [25] and then, using simpler methods, by Henniart [26]. For other reductive groups it is still a conjecture. Note that irreducible admissible smooth representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ correspond to n-dimensional (not necessarily irreducible) representations of $G_{\mathbb{Q}_p}$ as one can already expect this from the structure of L-functions: On one hand, the local L-function on the automorphic side of an irreducible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ is the reciprocal of a degree n polynomial in p^{-s} for all but finitely many primes p (see appendix B.2). On the other hand, the Euler factor of a Galois representation on a vectorspace V at unramified (therefore all but finitely many) primes p is the reciprocal of a degree $\dim V$ polynomial (namely, the characteristic polynomial of a lift of Frobenius) in p^{-s}, so we need to have $\dim V = n$ in order for these to match.

We do not attempt to state precisely the local Langlands theorems here as there are better introductory texts on them anyway (see for instance Wedhorn's note [39]). However, in order to formulate and motivate the p-adic Langlands programme, we will need the idea of the proof which is as follows. One considers certain "highly symmetric" algebraic varieties X, called Shimura varieties on which both the groups $\mathrm{GL}_n(\mathbb{Q}_p)$ and $G_{\mathbb{Q}_p}$ act such that these two actions commute with each other. The Shimura varieties are in some sense higher dimensional analogues of the modular curves $X_1(N) := \Gamma_1(N)\backslash\mathcal{H}^*$ introduced in the appendix B.1. The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ comes from the adélic setting: instead of $\Gamma_1(N)\backslash\mathcal{H}^*$ one works with the double coset space $\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_0(N)$ (see appendix B.1). Since $K_0(N)$ is not a normal subgroup in $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, we do not quite have an action of $\mathrm{GL}_2(\mathbb{Q}_p) \leq \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ on this space yet. The idea is to replace the subgroup $K_0(N)$ by an arbitrary compact open subgroup $K \leq \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ and vary K. The Shimura variety in this case is the inverse system

$$(X_K)_K := (\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K)_K$$

of curves for all compact open subgroups $K \leq \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ (they indeed form an inverse system as we have a projection map $X_{K_1} \to X_{K_2}$ for compact open subgroups $K_1 \leq K_2$). This way we obtain an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on the inverse system for all primes p. On the other hand, the action of the Galois group $G_\mathbb{Q}$ (or, in fact, of a finite index subgroup) is constructed the following way. Each Shimura variety can be defined, as an algebraic variety, over a canonical number field E called the *reflex field*. Therefore one has an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on the set of $\overline{\mathbb{Q}}$-points. In our case we have $E = \mathbb{Q}$, so we truly have an action of $G_\mathbb{Q}$ and its subgroups $G_{\mathbb{Q}_p}$ for all primes p.

Now for some fixed prime $\ell \neq p$ we obtain an action of the direct product $\mathrm{GL}_n(\mathbb{Q}_p) \times G_{\mathbb{Q}_p}$ on the ℓ-adic cohomology groups $H^i_{et}(X, \mathbb{Q}_\ell)$. It is a general principle that irreducible representations of the direct product of two groups can be split as the tensor product of irreducible representations of the two terms. So, informally, the local Langlands correspondence matches π_p with ρ_p if the tensor product $\pi_p \otimes \rho_p$ appears as a subrepresentation of $\mathrm{GL}_n(\mathbb{Q}_p) \times G_{\mathbb{Q}_p}$ in $H^i_{et}(X, \mathbb{Q}_\ell)$.

For the expert reader we remark here that even though on the local Galois side one has to work in the Langlands programme with so-called Weil–Deligne representations (certain enhanced representations of a dense subgroup $W_{\mathbb{Q}_p}$ of $G_{\mathbb{Q}_p}$), we chose not to confuse the reader with the notion of these as the category of ℓ-adic representations has a fully faithful functor into the category of Weil–Deligne representations—and working with the former will suffice for our purposes.

1.5 The p-adic (local) Langlands programme

The p-adic Langlands programme is the elaboration of the question what happens in the above theories for $\ell = p$ (for a more detailed survey see [8]). Inspired by the result of Vignéras [38] recalled in the appendix B.2, the aim is to match certain n-dimensional representations ρ of $G_{\mathbb{Q}_p}$ over finite extensions of $\mathbb{Q}_p$ to certain linear continuous Banach space representations π of $\mathrm{GL}_n(\mathbb{Q}_p)$ on p-adic Banach spaces over finite extensions of $\mathbb{Q}_p$ (see appendix B.3 for the definition). In this picture there are no L- and ε-factors any more, so the expected properties of this bijection are expressed in a different way. The p-adic local Langlands correspondence should be compatible with cohomology, with reduction modulo p, and with p-adic families. The current state of the art of the p-adic Langlands correspondence is that it is very well understood in case $n = 2$ (for the field $\mathbb{Q}_p$) through the work of Berger [2], Breuil [5, 6, 7], Colmez [15], [17], Emerton [19], Kisin [29], and Paškūnas [31], but very little is known for $n > 2$ and for finite extensions $F/\mathbb{Q}_p$ (apart from the case $n = 1$ which is local class field theory and has been known for a century).

1.5.1 Compatibility with reduction mod p

By the compatibility with reduction modulo p we mean the following: On the Galois side if V is an n-dimensional continuous representation of $G_{\mathbb{Q}_p}$ then it can be shown using a compactness argument that V admits a $\mathbb{Z}_p$-lattice $T \leq V$ (ie. a free module T of rank n over $\mathbb{Z}_p$) stable under the action of $G_{\mathbb{Q}_p}$. Therefore T/pT is a representation of $G_{\mathbb{Q}_p}$ over $\mathbb{F}_p$ whose semisimplification $\overline{V}$ does not depend on the choice of the lattice T. On the other hand, if Π is a p-adic unitary Banach space representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ then as recalled in the appendix B.3, the closed unit ball $\Pi^{\leq 1}$ with respect to some invariant norm $\|\cdot\|$ on Π is invariant under the action of $\mathrm{GL}_n(\mathbb{Q}_p)$ and the quotient $\Pi^{\leq 1}/\Pi^{<1}$ is a smooth modulo p representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ whose semisimplification $\overline{\Pi}$ is independent of the choice of the invariant norm on Π. In case the representation V_1 (resp. V_2) of $G_{\mathbb{Q}_p}$ corresponds to Π_1 (resp. to Π_2) on the automorphic side via a p-adic Langlands correspondence then we should have that $\overline{V_1}$ is isomorphic to $\overline{V_2}$ if and only if so is $\overline{\Pi_1}$ to $\overline{\Pi_2}$.

1.5.2 Compatibility with families

The objects on the two sides of the p-adic Langlands programme often come in *families*, parametrized by certain algebraic (or p-adic analytic) varieties. For instance, for each modulo p representation $\overline{V}$ the representations V lifting $\overline{V}$ to characteristic 0 amount to points in the spectrum of a "universal deformation ring". In the p-adic Langlands programme there should exist a correspondence on the level of families specializing to the original correspondence for single representations on both sides at each point of the family.

1.5.3 Colmez's correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$

Before saying what we mean by compatibility with cohomology we recall briefly Colmez's correspondence for the group $\mathrm{GL}_2(\mathbb{Q}_p)$ here. By Fontaine's theory (see appendix A.4), it suffices to match (φ, Γ)-modules with p-adic Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. The so-called "Montréal-functor" associates to a smooth p^h-torsion representation of the Borel subgroup $B_2(\mathbb{Q}_p)$ of upper triangular matrices in $\mathrm{GL}_2(\mathbb{Q}_p)$ a p^h-torsion (φ, Γ)-module over Fontaine's ring $\mathcal{O}_{\mathcal{E}}$. If we are given a unitary Banach space representation Π over $\mathbb{Q}_p$ of the group $\mathrm{GL}_2(\mathbb{Q}_p)$ with closed unit ball $\Pi^{\leq 1}$, then $\Pi^{\leq 1}/p^h$ is a smooth p^h-torsion representation that we restrict now to $B_2(\mathbb{Q}_p)$. The (φ, Γ)-module associated to Π is the projective limit (as $h \to \infty$) of the (φ, Γ)-modules associated to $\Pi^{\leq 1}/p^h$ via the Montreal functor. (Note that this automatically gives the desired compatibility with reduction mod p explained in section 1.5.1.) For more on this, including generalizations, see section 3.

The reverse direction, how one adjoins a unitary continuous p-adic representation to a 2-dimensional (φ, Γ)-module D over Fontaine's ring, is even more subtle. One first constructs a unitary p-adic Banach space representation $\Pi(D)$ to each 2-dimensional *trianguline* (see Appendix A.4) (φ, Γ)-module D over $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[p^{-1}]$ using some kind of parabolic induction. This Banach space is well described as a quotient of the space of p-adic functions satisfying certain properties by a certain $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant subspace (see [16] for details), however, a priori it is not clear whether or not it is nontrivial. On the other hand, there is a general construction of a (somewhat bigger) $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $D \boxtimes_\delta \mathbb{P}^1$ which is in fact the space of global sections of a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant sheaf $U \mapsto D \boxtimes_\delta U$ ($U \subseteq \mathbb{P}^1$ open) on the projective space $\mathbb{P}^1(\mathbb{Q}_p) \cong \mathrm{GL}_2(\mathbb{Q}_p)/B_2(\mathbb{Q}_p)$ for any (not necessarily 2-dimensional) (φ, Γ)-module D and any unitary character $\delta \colon \mathbb{Q}_p^\times \to \mathbb{Z}_p^\times$. This sheaf has the following properties: (i) the centre of $\mathrm{GL}_2(\mathbb{Q}_p)$ acts via δ on $D \boxtimes_\delta \mathbb{P}^1$; (ii) we have $D \boxtimes_\delta \mathbb{Z}_p \cong D$ as a module over the monoid $\begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ (where we regard $\mathbb{Z}_p$ as an open subspace in $\mathbb{P}^1 = \mathbb{Q}_p \cup \{\infty\}$). (See section 2 recalling the results in [35] on a generalization of this construction to general reductive groups.) Then Colmez shows that in case D is 2-dimensional and trianguline, then there exists a unitary character δ (namely $\delta = \chi^{-1} \det D$ where χ is the cyclotomic character and $\det D$ is the character associated to the 1-dimensional (φ, Γ)-module $\bigwedge^2 D$ via Fontaine's equivalence composed with class field theory) such that a certain subspace $D^\natural \boxtimes_\delta \mathbb{P}^1$ (for the definition see section 2 and [17]) of $D \boxtimes_\delta \mathbb{P}^1$ is isomorphic to the dual of the Banach space representation $\Pi(\check{D})$ associated earlier to the dual (φ, Γ)-module $\check{D}$—therefore showing in particular that the previous construction is nonzero. This subspace makes sense also in case D is not trianguline (nor of rank 2), but a priori only known to be $B_2(\mathbb{Q}_p)$-invariant. Moreover, whenever D is indecomposable and 2-dimensional, then the above δ is unique [31], and whenenever D is absolutely irreducible and ≥ 3-dimensional, then there does not exist [31] such a character δ (so that the subspace $D^\natural \boxtimes_\delta \mathbb{P}^1$ is $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant). Since the construction of $D \mapsto D^\natural \boxtimes_\delta \mathbb{P}^1$ behaves well in families (see chapter II in [15]) and the trianguline Galois-representations are Zariski-dense in the deformation space of 2-dimensional (φ, Γ)-modules with given reduction mod p [28], Colmez [15] shows that this subspace $D^\natural \boxtimes_\delta \mathbb{P}^1$ is not only $B_2(\mathbb{Q}_p)$, but also $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant for general 2-dimensional (φ, Γ)-modules under the choice $\delta = \chi^{-1} \det D$. In this case we omit the subscript δ from the notation and we have a short exact sequence

$$0 \to D^\natural \boxtimes \mathbb{P}^1 \to D \boxtimes \mathbb{P}^1 \to \Pi(D) \to 0$$

of $\mathrm{GL}_2(\mathbb{Q}_p)$-representations where $\Pi(D)$ is the unitary Banach-space representation associated to D via the p-adic Langlands correspondence.

1.5.4 Compatibility with cohomology and the global Langlands

Now we describe the compatibility with cohomology (and with the global Langlands programme) here in the special case of $n = 2$ as this is a theorem of Emerton [20], all the other cases being widely open. We follow Breuil's survey [8] here. Consider the following complex curve (cf. appendix B.1)

$$Y(K_f)(\mathbb{C}) := \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})/K_f\mathbb{R}^\times \mathrm{SO}_2(\mathbb{R})$$

for any compact open subgroup $K_f \leq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},fin})$. For varying K_f, these varieties form a projective system $(Y(K_f)(\mathbb{C}))_{K_f}$ on which $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},fin})$ acts on the right ($g \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},fin})$ maps $Y(K_f)(\mathbb{C})$ to $Y(g^{-1}K_fg)(\mathbb{C})$). Further, for each fixed compact open subgroup $K_f^p \leq \mathrm{GL}_2(\mathbb{A}^p_{\mathbb{Q},fin})$ (where $\mathbb{A}^p_{\mathbb{Q},fin}$ stands for the ring of finite adèles outside p), and varying compact open subgroup $K_{f,p} \leq \mathrm{GL}_2(\mathbb{Q}_p)$ the group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on the projective system $(Y(K_f^pK_{f,p})(\mathbb{C}))_{K_{f,p}}$. One considers the following "completed cohomology spaces"

$$\begin{aligned}\widehat{H}^1(K_f^p) &:= \left(\varprojlim_r \varinjlim_{K_{f,p}} H^1(Y(K_f^pK_{f,p})(\mathbb{C}), \mathbb{Z}/p^r\mathbb{Z})\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ \widehat{H}^1 &:= \varinjlim_{K_f^p} \widehat{H}^1(K_f^p) \ ,\end{aligned}$$

where H^1 stands for usual singular cohomology of the topological space (or, equivalently, for étale cohomology of the algebraic variety). The group $\mathrm{GL}_2(\mathbb{Q}_p)$ (resp. $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},fin})$) acts on the $\mathbb{Q}_p$-vectorspace $\widehat{H}^1(K_f^p)$ (resp. on $\widehat{H}^1$) and Emerton [18] showed that in fact $\widehat{H}^1(K_f^p)$ is a unitary Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ in the sense of the appendix B.3. Moreover, the curve $Y(K_f^pK_{f,p})$ can be defined over $\mathbb{Q}$, so the above cohomology groups admit an action of the Galois group $G_{\mathbb{Q}}$. Now let ρ be an arbitrary continuous representation of $G_{\mathbb{Q}}$ on a 2-dimensional vectorspace over $\mathbb{Q}_p$. To the restriction ρ_ℓ of ρ to $G_{\mathbb{Q}_\ell}$ for a prime $\ell \neq p$ corresponds, via the classical local Langlands correspondence, a smooth representation π'_ℓ of $\mathrm{GL}_2(\mathbb{Q}_\ell)$. We modify these as follows: If π'_ℓ is infinite dimensional then we put $\pi_\ell := \pi'_\ell \otimes |\det|^{-\frac{1}{2}}$. If π'_ℓ is finite dimensional (hence it is a character) then we let π_ℓ be the unique principal series representation with irreducible quotient $\pi'_\ell \otimes |\det|^{-\frac{1}{2}}$. On the other hand if $\ell = p$ then we have the p-adic Banach space representation $\Pi(\rho_p)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ corresponding to ρ_p via Colmez's p-adic Langlands correspondence [17]. Now assume that p is odd and ρ is unramified at all but finitely many primes ℓ and the determinant of the complex conjugation in $G_{\mathbb{Q}}$ is -1 under the map ρ. Suppose further that the restriction of the semisimplification $\overline{\rho}$ of the modulo p reduction of ρ to the subgroup $G_{\mathbb{Q}(\mu_p)} \leq G_{\mathbb{Q}}$ is irreducible, and the restriction $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ to $G_{\mathbb{Q}_p}$ is not an extension of two copies of the trivial character, nor an extension of the modulo p cyclotomic character by the trivial character. Then we can express [20] the $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},fin})$-representation $\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, \widehat{H}^1)$ as a restricted tensor product

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, \widehat{H}^1) \cong \Pi(\rho_p) \otimes_{\mathbb{Q}_p} \left(\sideset{}{'}\bigotimes_{\ell \neq p} \pi_\ell\right) \ .$$

Note that, in particular, $\mathrm{Hom}_{G_\mathbb{Q}}(\rho, \widehat{H}^1)$ is nonzero, ie. ρ is modular: this result of Emerton is both a compatibility and a modularity result! One should expect something similar, but more complicated in higher dimensions, too (see, for instance, section 5).

1.6 Plan of the thesis

We could summarize the results surveyed in this thesis as the various attempts to generalize several parts of Colmez's work on the p-adic Langlands programme from the case of $\mathrm{GL}_2(\mathbb{Q}_p)$ to other groups of higher rank.

In section 2 we construct a G-equivariant sheaf on the p-adic flag variety G/B for groups like $G = \mathrm{GL}_n(\mathbb{Q}_p)$ starting from a (φ, Γ)-module, or equivalently, from a representation of $G_{\mathbb{Q}_p}$ (see appendix A.4). Note that the global sections of such a sheaf is a representation of G. This part of the thesis is based on a joint work of the author with Peter Schneider and Marie-France Vignéras [35] and is one of the key steps in Colmez's p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ in the special case, due to Colmez, of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Our approach is conceptual and more algebraic whereas Colmez used a lot of p-adic analysis instead. The idea of the equivalence of categories in Thm. 2.1 originates in the paper [42] and is also proven in a slightly more general setting in [43].

In section 3 we recall previous approaches (by Schneider and Vignéras [34], and by Breuil [9]) to generalize Colmez's Montréal functor (sometimes also called "Magical functor"). Moreover, we show that Breuil's functor can be obtained from that of Schneider and Vigneras by a process of taking "étale hull" (the notion introduced by the present author and his student Márton Erdélyi), localizing, and completing. This is based on the joint work [21] with Erdélyi.

Section 4 is devoted to combining all the above approaches in a uniform way. The difficulty is the following. Breuil's functor depends (crucially) on the choice of a Whittaker-type functional $\ell\colon N \to \mathbb{Q}_p$ (where N is the group of strictly upper triangular matrices) which is chosen in [9] to be "generic". This choice is needed for some of the very nice and useful properties of the functor. However, in order to use the results in section 2 to pass back to the automorphic side, the choice of a generic ℓ is not suitable. The reason for this is that the torus T acts on the set of functionals ℓ, and the stabilizer of a generic ℓ is rather small, so we lose the action of a big part of the group T which cannot be recovered. The idea is that instead of fixing such a functional ℓ, consider all the possible choices at once, ie. take the map $N \twoheadrightarrow N/[N,N] = \bigoplus_{\alpha\in\Delta} \mathbb{Q}_p$ instead. This way we construct a functor $D^\vee_\Delta$ from smooth modulo p^n representations of G to $|\Delta|$-variable (φ, Γ)-modules. These objects correspond to representations of the $|\Delta|$th power $G_{\mathbb{Q}_p,\Delta}$ of the local Galois group $G_{\mathbb{Q}_p}$ on finitely generated p^n-torsion abelian groups as shown by the present author in [45] (see section 4.5). The functor $D^\vee_\Delta$ has the same very nice properties as Breuil's functor (compatibility with products and parabolic induction, finiteness and exactness on principal series) and, in addition, one can indeed recover the representation π from $D^\vee_\Delta$ in certain special cases.

In section 5 we make an optimistic conjecture on the value of $D^\vee_\Delta$ at a smooth modulo p representaion π of G coming from the ρ-isotypic components of certain cohomology groups of Shimura varieties for a fixed global Galois representation ρ. This is inspired by the $\mathrm{GL}_2(\mathbb{Q}_p)$-situation and by a conjecture of Breuil, Herzig, and Schraen [11] on the value of Breuil's functor at π.

1.7 Notations

We fix a finite extension $K/\mathbb{Q}_p$ of ring of integers o, prime element ϖ, residue field $\kappa = o/(\varpi)$, and an algebraic closure $\overline{\mathbb{Q}}_p$ of K. This will serve as a coefficient field for representations on both the Galois and automorphic sides. We denote by $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ the absolute Galois group of $\mathbb{Q}_p$, by $\Lambda(\mathbb{Z}_p) = o[[\mathbb{Z}_p]]$ the Iwasawa o-algebra of maximal ideal $\mathcal{M}(\mathbb{Z}_p)$, and by $\mathcal{O}_\mathcal{E}$ the Fontaine ring which is the p-adic completion of the localisation of $\Lambda(\mathbb{Z}_p)$ with respect to the elements not in $p\Lambda(\mathbb{Z}_p)$. We put on $\mathcal{O}_\mathcal{E}$ the weak topology inducing the $\mathcal{M}(\mathbb{Z}_p)$-adic topology on $\Lambda(\mathbb{Z}_p)$, a fundamental system of neighbourhoods of 0 being $(p^n\mathcal{O}_\mathcal{E} + \mathcal{M}(\mathbb{Z}_p)^n)_{n\in\mathbb{N}}$. The action of $\mathbb{Z}_p - \{0\}$ by multiplication on $\mathbb{Z}_p$ extends to an action on $\mathcal{O}_\mathcal{E}$.

We fix an arbitrary split reductive connected $\mathbb{Q}_p$-group G and a Borel $\mathbb{Q}_p$-subgroup $B = TN$ with maximal $\mathbb{Q}_p$-subtorus T and unipotent radical N. We denote by w_0 the longest element of the Weyl group of T in G, by Φ^+ the set of roots of T in N, by $\Delta \subseteq \Phi^+$ the set of simple roots, and by $u_\alpha : \mathbb{G}_a \to N_\alpha$, for $\alpha \in \Phi^+$, a $\mathbb{Q}_p$-homomorphism onto the root subgroup N_α of N such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for $x \in \mathbb{Q}_p$ and $t \in T$, and $N_0 = \prod_{\alpha\in\Phi^+} u_\alpha(\mathbb{Z}_p)$ is a subgroup of N. We denote by T_+ the monoid of dominant elements t in T such that $\mathrm{val}_p(\alpha(t)) \geq 0$ for all $\alpha \in \Phi^+$, by $T_0 \subset T_+$ the maximal subgroup, by T_{++} the subset of strictly dominant elements, i.e. $\mathrm{val}_p(\alpha(t)) > 0$ for all $\alpha \in \Phi^+$, and we put $B_+ = N_0T_+, B_0 = N_0T_0$. The natural action of T_+ on N_0 extends to an action on the Iwasawa o-algebra $\Lambda(N_0) = o[[N_0]]$. The compact set G/B contains the open dense subset $\mathcal{C} = Nw_0B/B$ homeomorphic to N and the compact open subset $\mathcal{C}_0 = N_0w_0B/B$ homeomorphic to N_0. We put $\overline{B} = w_0Bw_0^{-1}$.

Each simple root $\alpha \in \Delta$ gives a $\mathbb{Q}_p$-homomorphism $x_\alpha : N \to \mathbb{G}_a$ with section u_α. We denote by $\ell_\alpha : N_0 \to \mathbb{Z}_p$, resp. $\iota_\alpha : \mathbb{Z}_p \to N_0$, the restriction of x_α, resp. u_α, to N_0, resp. $\mathbb{Z}_p$.

Since the centre of G is assumed to be connected, there exists a cocharacter $\xi\colon \mathbb{Q}_p^\times \to T$ such that $\alpha \circ \xi$ is the identity on $\mathbb{Q}_p^\times$ for each $\alpha \in \Delta$. We put $\Gamma := \xi(\mathbb{Z}_p^\times) \leq T$.

By a smooth o-torsion representation π of G (resp. of B) we mean a torsion o-module π together with a smooth (ie. stabilizers are open) and linear action of the group G (resp. of B).

For example, $G = \mathrm{GL}_n$, B is the subgroup of upper triangular matrices, N consists of the strictly upper triangular matrices (1 on the diagonal), T is the diagonal subgroup, $N_0 = N(\mathbb{Z}_p)$, the simple roots are $\alpha_1, \ldots, \alpha_{n-1}$ where $\alpha_i(\mathrm{diag}(t_1, \ldots, t_n)) = t_it_{i+1}^{-1}$, x_{α_i} sends a matrix to its $(i, i+1)$-coefficient, $u_{\alpha_i}(.)$ is the strictly upper triangular matrix, with $(i, i+1)$-coefficient . and 0 everywhere else.

We denote by $C^\infty(X, o)$ the o-module of locally constant functions on a locally profinite space X with values in o, and by $C_c^\infty(X, o)$ the subspace of compactly supported functions.

For a finite index subgroup $\mathcal{G}_2$ in a group $\mathcal{G}_1$ we denote by $J(\mathcal{G}_1/\mathcal{G}_2) \subset \mathcal{G}_1$ a (fixed) set of representatives of the left cosets in $\mathcal{G}_1/\mathcal{G}_2$.

2 G-equivariant sheaves on G/B attached to (φ, Γ)-modules

This section is devoted to recalling the results in the joint paper [35] of the author with Peter Schneider a Marie-France Vignéras. The aim is to generalize Colmez's $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant sheaf on $\mathbb{P}^1(\mathbb{Q}_p)$ to other p-adic groups $G \neq \mathrm{GL}_2(\mathbb{Q}_p)$ in a conceptual way.

2.1 The rings $\Lambda_{\ell_\alpha}(N_0)$ and $\mathcal{O}_{\mathcal{E}_\alpha}$

For a fixed surjective group homomorphism $\ell\colon N_0 \twoheadrightarrow \mathbb{Z}_p$ with kernel $H_0 := \mathrm{Ker}(\ell)$ we define the topological local ring $\Lambda_\ell(N_0)$, generalizing Fontaine's ring $\mathcal{O}_\mathcal{E}$ as follows. We denote by $\mathcal{M}(N_{\ell_\alpha})$ the maximal ideal of the Iwasawa o-algebra $\Lambda(N_{\ell_\alpha}) = o[[N_{\ell_\alpha}]]$ of the kernel N_{ℓ_α} of ℓ_α. The ring $\Lambda_{\ell_\alpha}(N_0)$ is the $\mathcal{M}(N_{\ell_\alpha})$-adic completion of the localisation of $\Lambda(N_0)$ with respect to the Ore subset of elements which are not in $\mathcal{M}(N_{\ell_\alpha})\Lambda(N_0)$. This is a noetherian local ring with maximal ideal $\mathcal{M}_{\ell_\alpha}(N_0)$ generated by $\mathcal{M}(N_{\ell_\alpha})$. We put on $\Lambda_{\ell_\alpha}(N_0)$ the weak topology with fundamental system of neighbourhoods of 0 equal to $(\mathcal{M}_{\ell_\alpha}(N_0)^n + \mathcal{M}(N_0)^n)_{n\in\mathbb{N}}$. The action of T_+ on N_0 extends to an action on $\Lambda_{\ell_\alpha}(N_0)$. The ring $\Lambda(N_0)$ can be viewed as the ring $\Lambda(H_0)[[X]]$ of skew Taylor series over $\Lambda(H_0)$ in the variable $X = [u] - 1$ where $u \in N_0$ and $\ell(u)$ is a topological generator of $\ell(N_0) = \mathbb{Z}_p$. Then $\Lambda_\ell(N_0)$ is viewed as the ring of infinite skew Laurent series $\sum_{n\in\mathbb{Z}} a_n X^n$ over $\Lambda(H_0)$ in the variable X with $\lim_{n\to-\infty} a_n = 0$ for the compact topology of $\Lambda(H_0)$. For a different characterization of this ring in terms of a projective limit $\Lambda_\ell(N_0) \cong \varprojlim_{n,k} \Lambda(N_0/H_k)[1/X]/\varpi^n$ for $H_k \lhd N_0$ normal subgroups contained and open in H_0 satisfying $\bigcap_{k\geq 0} H_k = \{1\}$ see also [43].

Fixing a simple root $\alpha \in \Delta$, in section 2 we make the choice $\ell := \ell_\alpha\colon N_0 \to \mathbb{Z}_p$.

We denote by $\mathcal{O}_{\mathcal{E}_\alpha}$ the ring $\mathcal{O}_\mathcal{E}$ with the action of T_+ induced by $(t, x) \mapsto \alpha(t)x : T_+ \times \mathbb{Z}_p \to \mathbb{Z}_p$. The homomorphism ℓ_α and its section ι_α induce T_+-equivariant ring homomorphisms

$$\ell_\alpha : \Lambda_{\ell_\alpha}(N_0) \to \mathcal{O}_{\mathcal{E}_\alpha} \ , \ \iota_\alpha : \mathcal{O}_{\mathcal{E}_\alpha} \to \Lambda_{\ell_\alpha}(N_0) \ , \text{ such that } \ell_\alpha \circ \iota_\alpha = \mathrm{id}\,.$$

2.2 Equivalence of categories

An étale T_+-module over $\Lambda_{\ell_\alpha}(N_0)$ is a finitely generated $\Lambda_{\ell_\alpha}(N_0)$-module M with a semilinear action $T_+ \times M \to M$ of T_+ which is étale, i.e. the action φ_t on M of each $t \in T_+$ is injective and

$$M = \oplus_{u\in J(N_0/tN_0t^{-1})} u\varphi_t(M) \ ,$$

if $J(N_0/tN_0t^{-1}) \subset N_0$ is a system of representatives of the cosets N_0/tN_0t^{-1}; in particular, the action of each element of the maximal subgroup T_0 of T_+ is invertible. We denote by ψ_t the left inverse of φ_t vanishing on $u\varphi_t(M)$ for $u \in N_0$ not in tN_0t^{-1}. These modules form an abelian category $\mathcal{M}^{et}_{\Lambda_{\ell_\alpha}(N_0)}(T_+)$.

We define analogously the abelian category $\mathcal{M}^{et}_{\mathcal{O}_{\mathcal{E}_\alpha}}(T_+)$ of finitely generated $\mathcal{O}_{\mathcal{E}_\alpha}$-modules with an étale semilinear action of T_+. The action φ_t of each element $t \in T_+$ such that $\alpha(t) \in \mathbb{Z}_p^*$ is invertible. We show that the action $T_+ \times D \to D$ of T_+ on $D \in \mathcal{M}^{et}_{\mathcal{O}_{\mathcal{E}_\alpha}}(T_+)$ is continuous for the weak topology on D; the canonical action of the inverse T_- of T is also continuous.

Theorem 2.1. *The base change functors $\mathcal{O}_\mathcal{E} \otimes_{\ell_\alpha} -$ and $\Lambda_{\ell_\alpha}(N_0) \otimes_{\iota_\alpha} -$ induce quasi-inverse equivalences of categories*

$$\mathbb{D} : \mathcal{M}^{et}_{\Lambda_{\ell_\alpha}(N_0)}(T_+) \to \mathcal{M}^{et}_{\mathcal{O}_{\mathcal{E}_\alpha}}(T_+) \ , \ \mathbb{M} : \mathcal{M}^{et}_{\mathcal{O}_{\mathcal{E}_\alpha}}(T_+) \to \mathcal{M}^{et}_{\Lambda_{\ell_\alpha}(N_0)}(T_+) \ .$$

Using this theorem, we show that the action of T_+ and of the inverse monoid T_- (given by the operators ψ) on an étale T_+-module over $\Lambda_{\ell_\alpha}(N_0)$ is continuous for the weak topology.

2.3 B-equivariant sheaves on $\mathcal{C}$

The o-algebra $C^\infty(N_0, o)$ is naturally an étale $o[B_+]-$module, and the monoid B_+ acts on the o-algebra $\operatorname{End}_o M$ by $(b, F) \mapsto \varphi_b \circ F \circ \psi_b$. We show that there exists a unique $o[B_+]$-linear map

$$\mathrm{res} : C^\infty(N_0, o) \to \operatorname{End}_o M$$

sending the characteristic function 1_{N_0} of N_0 to the identity id_M; moreover res is an algebra homomorphism which sends $1_{b.N_0}$ to $\varphi_b \circ \psi_b$ for all $b \in B_+$ acting on $x \in N_0$ by $(b, x) \mapsto b.x$.

For the sake of simplicity, we denote now by the same letter a group defined over $\mathbb{Q}_p$ and the group of its $\mathbb{Q}_p$-rational points.

Let M^B be the $o[B]$-module induced by the canonical action of the inverse monoid B_- of B_+ on M; as a representation of N, it is isomorphic to the representation induced by the action of N_0 on M. The value at 1, denoted by $\mathrm{ev}_0 : M^B \to M$, is B_--equivariant, and admits a B_+-equivariant splitting $\sigma_0 : M \to M^B$ sending $m \in M$ to the function equal to $n \mapsto nm$ on N_0 and vanishing on $N - N_0$. The $o[B]$-submodule M_c^B of M^B generated by $\sigma_0(M)$ is naturally isomorphic to $A[B] \otimes_{A[B_+]} M$. When $M = C^\infty(N_0, o)$ then $M_c^B = C_c^\infty(N, o)$ and $M^B = C^\infty(N, o)$ with the natural $o[B]-$module structure. We have the natural o-algebra embedding

$$F \mapsto \sigma_0 \circ F \circ \mathrm{ev}_0 : \operatorname{End}_o M \to \operatorname{End}_o M^B \ .$$

sending id_M to the idempotent $R_0 = \sigma_0 \circ ev_0$ in $\operatorname{End}_o M^B$.

Proposition 2.2. *There exists a unique $o[B]$-linear map*

$$\mathrm{Res} : C_c^\infty(N, o) \to \operatorname{End}_o M^B$$

sending 1_{N_0} to R_0; moreover Res *is an algebra homomorphism.*

The topology of N is totally disconnected and by a general argument, the functor of compact global sections is an equivalence of categories from the B-equivariant sheaves on $N \simeq \mathcal{C}$ to the non-degenerate modules on the skew group ring

$$C_c^\infty(N, o)\#B \ = \oplus_{b \in B} b C_c^\infty(N, o) \ .$$

in which the multiplication is determined by the rule $(b_1 f_1)(b_2 f_2) = b_1 b_2 f_1^{b_2} f_2$ for $b_i \in B$, $f_i \in C_c^\infty(N, o)$ and $f_1^{b_2}(.) = f_1(b_2.)$.

Theorem 2.3. *The functor of sections over $N_0 \simeq \mathcal{C}_0$ from the B-equivariant sheaves on $N \simeq \mathcal{C}$ to the étale $o[B_+]$-modules is an equivalence of categories.*

The space of global sections of a B-equivariant sheaf $\mathcal{S}$ on $\mathcal{C}$ is $\mathcal{S}(C) = \mathcal{S}(C_0)^B$.

2.4 Generalities on G-equivariant sheaves on G/B

The functor of global sections from the G-equivariant sheaves on G/B to the modules on the skew group ring $\mathcal{A}_{G/B} = C^\infty(G/B, o)\#G$ is an equivalence of categories. We have the intermediate ring $\mathcal{A}$

$$\mathcal{A}_\mathcal{C} = C_c^\infty(\mathcal{C}, o)\#B \ \subset \ \mathcal{A} = \oplus_{g \in G} g C_c^\infty(g^{-1}\mathcal{C} \cap \mathcal{C}, o) \ \subset \ \mathcal{A}_{G/B},$$

and the o-module

$$\mathcal{Z} = \oplus_{g\in G} g C_c^\infty(\mathcal{C}, o)$$

which is a left ideal of $\mathcal{A}_{G/B}$ and a right $\mathcal{A}$-submodule.

Proposition 2.4. *The functor*

$$Z \mapsto Y(Z) = \mathcal{Z} \otimes_{\mathcal{A}} Z$$

from the non-degenerate $\mathcal{A}$-modules to the $\mathcal{A}_{G/B}$-modules is an equivalence of categories; moreover the G-sheaf on G/B corresponding to $Y(Z)$ extends the B-equivariant sheaf on $\mathcal{C}$ corresponding to $Z|_{\mathcal{A}_{\mathcal{C}}}$.

Given an étale $o[B_+]$-module M, we consider the problem of extending to $\mathcal{A}$ the o-algebra homomorphism

$$\mathrm{Res} : A_{\mathcal{C}} \to \mathrm{End}_o(M_c^B) \quad , \quad \sum_{b\in B} b f_b \mapsto b \circ \mathrm{Res}(f_b) \ .$$

We introduce the subrings

$$\mathcal{A}_0 = 1_{\mathcal{C}_0}\mathcal{A}1_{\mathcal{C}_0} = \bigoplus_{g\in G} gC^\infty(g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0, o) \ \subset \ \mathcal{A} \ ,$$

$$\mathcal{A}_{\mathcal{C}0} = 1_{\mathcal{C}_0}\mathcal{A}_{\mathcal{C}}1_{\mathcal{C}_0} = \bigoplus_{b\in B} bC^\infty(b^{-1}\mathcal{C}_0 \cap \mathcal{C}_0, o) \ \subset \ \mathcal{A}_{\mathcal{C}} \ .$$

The skew monoid ring $\mathcal{A}_{\mathcal{C}_0} = C^\infty(\mathcal{C}_0, o)\#B_+ = \oplus_{b\in B_+} bC^\infty(\mathcal{C}_0, o)$ is contained in $\mathcal{A}_{\mathcal{C}0}$. The intersection $g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$ is not 0 if and only if $g \in N_0\overline{B}N_0$. The subring $\mathrm{Res}(\mathcal{A}_{\mathcal{C}0})$ of $\mathrm{End}_o(M^B)$ necessarily lies in the image of $\mathrm{End}_o(M)$.

The group B acts on $\mathcal{A}$ by $(b, y) \mapsto (b1_{G/B})y(b1_{G/B})^{-1}$ for $b \in B$, and the map $b \otimes y \mapsto (b1_{G/B})y(b1_{G/B})^{-1}$ gives $o[B]$ isomorphisms

$$o[B] \otimes_{o[B_+]} \mathcal{A}_0 \to \mathcal{A} \quad \text{and} \quad o[B] \otimes_{o[B_+]} \mathcal{A}_{\mathcal{C}0} \to \mathcal{A}_{\mathcal{C}} \ .$$

Proposition 2.5. *Let M be an étale $o[B_+]$-module. We suppose given, for any $g \in N_0\overline{B}N_0$, an element $\mathcal{H}_g \in \mathrm{End}_o(M)$. The map*

$$\mathcal{R}_0 : \mathcal{A}_0 \to \mathrm{End}_o(M) \quad , \quad \sum_{g\in N_0\overline{B}N_0} gf_g \mapsto \sum_{g\in N_0\overline{B}N_0} \mathcal{H}_g \circ \mathrm{res}(f_g)$$

is a B_+-equivariant o-algebra homomorphism which extends $\mathrm{Res}|_{\mathcal{A}_{\mathcal{C}0}}$ if and only if, for all $g, h \in N_0\overline{B}N_0$, $b \in B \cap N_0\overline{B}N_0$, and all compact open subsets $\mathcal{V} \subset \mathcal{C}_0$, the relations

H1. $\mathrm{res}(1_{\mathcal{V}}) \circ \mathcal{H}_g = \mathcal{H}_g \circ \mathrm{res}(1_{g^{-1}\mathcal{V}\cap\mathcal{C}_0})$,

H2. $\mathcal{H}_g \circ \mathcal{H}_h = \mathcal{H}_{gh} \circ \mathrm{res}(1_{h^{-1}\mathcal{C}_0\cap\mathcal{C}_0})$,

H3. $\mathcal{H}_b = b \circ \mathrm{res}(1_{b^{-1}\mathcal{C}_0\cap\mathcal{C}_0})$.

hold true. In this case, the unique $o[B]$-equivariant map $\mathcal{R} : \mathcal{A} \to \mathrm{End}_A(M_c^B)$ extending $\mathcal{R}_0$ is multiplicative.

When these conditions are satisfied, we obtain a G-equivariant sheaf on G/B with sections on $\mathcal{C}_0$ equal to M.

2.5 $(s, \mathrm{res}, \mathfrak{C})$-integrals $\mathcal{H}_g$

Let M be an étale T_+-module M over $\Lambda_{\ell_\alpha}(N_0)$ with the weak topology. We denote by $\mathrm{End}_o^{cont}(M)$ the o-module of continuous o-linear endomorphisms of M, and for g in $N_0\overline{B}N_0$, by $U_g \subseteq N_0$ the compact open subset such that

$$U_g w_0 B/B = g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0 \ .$$

For $u \in U_g$, we have a unique element $\alpha(g,u) \in N_0T$ such that $guw_0N = \alpha(g,u)uw_0N$. We consider the map

$$\alpha_{g,0} : N_0 \to \mathrm{End}_o^{cont}(M)$$
$$\alpha_{g,0}(u) = \mathrm{Res}(1_{\mathcal{C}_0}) \circ \alpha(g,u) \circ \mathrm{Res}(1_{\mathcal{C}_0}) \text{ for } u \in U_g \text{ and } \alpha_{g,0}(u) = 0 \text{ otherwise.}$$

The module M is Hausdorff complete but not compact, also we introduce a notion of integrability with respect to a special family $\mathfrak{C}$ of compact subsets $C \subset M$, i.e. satisfying:

$\mathfrak{C}(1)$ Any compact subset of a compact set in $\mathfrak{C}$ also lies in $\mathfrak{C}$.

$\mathfrak{C}(2)$ If $C_1, C_2, \ldots, C_n \in \mathfrak{C}$ then $\bigcup_{i=1}^n C_i$ is in $\mathfrak{C}$, as well.

$\mathfrak{C}(3)$ For all $C \in \mathfrak{C}$ we have $N_0C \in \mathfrak{C}$.

$\mathfrak{C}(4)$ $M(\mathfrak{C}) := \bigcup_{C\in\mathfrak{C}} C$ is an étale $o[B_+]$-submodule of M.

A map from $M(\mathfrak{C})$ to M is called $\mathfrak{C}$-continuous if its restriction to any $C \in \mathfrak{C}$ is continuous. The o-module $\mathrm{Hom}_o^{\mathfrak{C}ont}(M(\mathfrak{C}), M)$ of $\mathfrak{C}$-continuous o-linear homomorphisms from $M(\mathfrak{C})$ to M with the $\mathfrak{C}$-open topology, is a topological complete o-module.

For $s \in T_{++}$, the open compact subgroups $N_k = s^k N_0 s^{-k} \subset N$ for $k \in \mathbb{Z}$, form a decreasing sequence of union N and intersection $\{1\}$. A map $F\colon N_0 \to \mathrm{Hom}_A^{\mathfrak{C}ont}(M(\mathfrak{C}), M)$ is called $(s, \mathrm{res}, \mathfrak{C})$-integrable if the limit

$$\int_{N_0} F d\,\mathrm{res} := \lim_{k\to\infty} \sum_{u\in J(N_0/N_k)} F(u) \circ \mathrm{res}(1_{uN_k}) \ ,$$

where $J(N_0/N_k) \subseteq N_0$, for any $k \in \mathbb{N}$, is a set of representatives for the cosets in N_0/N_k, exists in $\mathrm{Hom}_A^{\mathfrak{C}ont}(M(\mathfrak{C}), M)$ and is independent of the choice of the sets $J(N_0/N_k)$. We denote by $\mathcal{H}_{g,J(N_0/N_k)}$ the sum in the right hand side when $F = \alpha_{g,0}(.)|_{M(\mathfrak{C})}$.

Proposition 2.6. *For all $g \in N_0\overline{B}N_0$, the map $\alpha_{g,0}(.)|_{M(\mathfrak{C})}\colon N_0 \to \mathrm{Hom}_A^{\mathfrak{C}ont}(M(\mathfrak{C}), M)$ is (s,* res, *$\mathfrak{C}$)-integrable when*

$\mathfrak{C}(5)$ *For any $C \in \mathfrak{C}$ the compact subset $\psi_s(C) \subseteq M$ also lies in $\mathfrak{C}$.*

$\mathfrak{T}(1)$ *For any $C \in \mathfrak{C}$ such that $C = N_0C$, any open $A[N_0]$-submodule $\mathcal{M}$ of M, and any compact subset $C_+ \subseteq L_+$ there exists a compact open subgroup $B_1 = B_1(C, \mathcal{M}, C_+) \subseteq B_0$ and an integer $k(C, \mathcal{M}, C_+) \geq 0$ such that*

$$s^k(1 - B_1)C_+\psi_s^k \subseteq E(C, \mathcal{M}) \qquad \text{for any } k \geq k(C, \mathcal{M}, C_+) \ .$$

The integrals $\mathcal{H}_g$ of $\alpha_{g,0}(.)|_{M(\mathfrak{C})}$ satisfy the relations H1, H2, H3, when they belong to $\mathrm{End}_A(M(\mathfrak{C}))$, and when

$\mathfrak{C}(6)$ *For any $C \in \mathfrak{C}$ the compact subset $\varphi_s(C) \subseteq M$ also lies in $\mathfrak{C}$.*

$\mathfrak{T}(2)$ *Given a set $J(N_0/N_k) \subset N_0$ of representatives for cosets in N_0/N_k, for $k \geq 1$, for any $x \in M(\mathfrak{C})$ and $g \in N_0\overline{B}N_0$ there exists a compact A-submodule $C_{x,g} \in \mathfrak{C}$ and a positive integer $k_{x,g}$ such that $\mathcal{H}_{g,J(N_0/N_k)}(x) \subseteq C_{x,g}$ for any $k \geq k_{x,g}$.*

When $\mathfrak{C}$ satisfies $\mathfrak{C}(1), \ldots, \mathfrak{C}(6)$ and the technical properties $\mathfrak{T}(1), \mathfrak{T}(2)$ are true, we obtain a G-equivariant sheaf on G/B with sections on $\mathcal{C}_0$ equal to $M(\mathfrak{C})$.

2.6 Main theorem

Let M be an étale T_+-module M over $\Lambda_{\ell_\alpha}(N_0)$ with the weak topology and let $s \in T_{++}$. We have the natural T_+-equivariant quotient map

$$\ell_M : M \to D = \mathcal{O}_{\mathcal{E}_\alpha} \otimes_{\ell_\alpha} M \quad , \quad m \mapsto 1 \otimes m$$

from M to $D = \mathbb{D}(M) \in \mathcal{M}_{\mathcal{O}_{\mathcal{E}_\alpha}}(T_+)$, of T_+-equivariant section

$$\iota_D : D \to M = \Lambda_{\ell_\alpha}(N_0) \otimes_{\iota_\alpha} D \quad , \quad d \mapsto 1 \otimes d \ .$$

We note that $o[N_0]\iota_D(D)$ is dense in M. A lattice D_0 in D is a $\Lambda(\mathbb{Z}_p)$-submodule generated by a finite set of generators of D over $\mathcal{O}_{\mathcal{E}}$. When D is killed by a power of p, the o-module

$$M_s^{bd}(D_0) := \{m \in M \mid \ell_M(\psi_s^k(u^{-1}m)) \in D_0 \text{ for all } u \in N_0 \text{ and } k \in \mathbb{N}\}$$

of M is compact and is a $\Lambda(N_0)$-module. Let $\mathfrak{C}_s$ be the family of compact subsets of M contained in $M_s^{bd}(D_0)$ for some lattice D_0 of D, and let $M_s^{bd} = \cup_{D_0} M_s^{bd}(D_0)$ the union being taken over all lattices D_0 in D. In general, M is p-adically complete, M/p^nM is an étale T_+-module over $\Lambda_{\ell_\alpha}(N_0)$, and $D/p^nD = \mathbb{D}(M/p^nM)$. We denote by $p_n : M \to M/p^nM$ the reduction modulo p^n, and by $\mathfrak{C}_{s,n}$ the family of compact subsets constructed above for M/p^nM. We define the family $\mathfrak{C}_s$ of compact subsets $C \subset M$ such that $p_n(C) \in \mathfrak{C}_{s,n}$ for all $n \geq 1$, and the o-module M_s^{bd} of $m \in M$ such that the set of $\ell_M(\psi_s^k(u^{-1}m))$ for $k \in \mathbb{N}, u \in N_0$ is bounded in D for the weak topology.

By reduction to the easier case where M is killed by a power of p, we show that $\mathfrak{C}_s$ satisfies $\mathfrak{C}(1), \ldots, \mathfrak{C}(6)$ and that the technical properties $\mathfrak{T}(1), \mathfrak{T}(2)$ are true.

Proposition 2.7. *Let M be an étale T_+-module M over $\Lambda_{\ell_\alpha}(N_0)$ and let $s \in T_{++}$.*

(i) M_s^{bd} is a dense $\Lambda(N_0)[T_+]$-étale submodule of M containing $\iota_D(D)$.

(ii) For $g \in N_0\overline{B}N_0$, the $(s, \mathrm{res}, \mathfrak{C}_s)$-integrals $\mathcal{H}_{g,s}$ of $\alpha_{g,0}|_{M_s^{bd}}$ exist, lie in $\mathrm{End}_o(M_s^{bd})$, and satisfy the relations H1, H2, H3.

(iii) For $s_1, s_2 \in T_{++}$, there exists $s_3 \in T_{++}$ such that $M_{s_3}^{bd}$ contains $M_{s_1}^{bd} \cup M_{s_2}^{bd}$ and $\mathcal{H}_{g,s_1} = \mathcal{H}_{g,s_2}$ on $M_{s_1}^{bd} \cap M_{s_2}^{bd}$.

The endomorphisms $\mathcal{H}_{g,s} \in \mathrm{End}_o(M_s^{bd})$ induce endomorphisms of $\cap_{s \in T_{++}} M_s^{bd}$ and of $\cup_{s \in T_{++}} M_s^{bd} = \sum_{s \in T_{++}} M_s^{bd}$ satisfying the relations H1, H2, H3. Moreover $\cup_{s \in T_{++}} M_s^{bd}$ and $\cap_{s \in T_{++}} M_s^{bd}$ are $\Lambda(N_0)[T_+]$-étale submodules of M containing $\iota_D(D)$. Our main theorem is the following:

Theorem 2.8. *There are faithful functors*

$$\mathbb{Y}_\cap,\ (\mathbb{Y}_s)_{s\in T_{++}},\ \mathbb{Y}_\cup : \mathcal{M}^{et}_{\mathcal{O}_{\mathcal{E}_\alpha}}(T_+) \longrightarrow \ G\text{-equivariant sheaves on } G/B\ ,$$

sending $D = \mathbb{D}(M)$ to a sheaf with sections on $\mathcal{C}_0$ equal to the dense $\Lambda(N_0)[T_+]$-submodules of M

$$\bigcap_{s\in T_{++}} M_s^{bd}, \quad (M_s^{bd})_{s\in T_{++}},\ \text{and} \quad \bigcup_{s\in T_{++}} M_s^{bd}\ ,$$

respectively.

When $G = \mathrm{GL}_2(\mathbb{Q}_p)$, the sheaves $\mathbb{Y}_s(D)$ are all equal to the G-equivariant sheaf on $G/B \simeq \mathbb{P}^1(\mathbb{Q}_p)$ of global sections $D \boxtimes \mathbb{P}^1$ constructed by Colmez. When the root system of G is irreducible of rank > 1, we check that $\cup_{s\in T_{++}} M_s^{bd}$ is never equal to M.

3 Links between generalizations of Colmez's functor

Here we recall certain generalizations of Colmez's Montréal functor given by Schneider and Vigneras [34] and by Breuil [9] and the connection between them found [21] by the present author and his PhD student Márton Erdélyi. Unlike in the previous section where $s \in T_{++}$ is arbitrary we specify $s := \xi(p) \in T_{++}$ here (with $\xi\colon \mathbb{Q}_p^\times \to T$ being the cocharacter introduced in the introduction) and often denote the action of s by $\varphi = \varphi_s$.

3.1 The functor of Schneider and Vignéras

The approach by Schneider and Vigneras [34] starts with the set $\mathcal{B}_+(\pi)$ of generating B_+-subrepresentations $W \le \pi$. The Pontryagin dual $W^\vee = \mathrm{Hom}_o(W, K/o)$ of each W admits a natural action of the inverse monoid B_+^{-1}. Moreover, the action of $N_0 \le B_+^{-1}$ on $W^\vee$ extends to an action of the Iwasawa algebra $\Lambda(N_0) = o[[N_0]]$. For $W_1, W_2 \in \mathcal{B}_+(\pi)$ we also have $W_1 \cap W_2 \in \mathcal{B}_+(\pi)$ (Lemma 2.2 in [34]) therefore we may take the inductive limit $D_{SV}(\pi) := \varinjlim_{W\in\mathcal{B}_+(\pi)} W^\vee$. In general, $D_{SV}(\pi)$ does not have good properties: for instance it may not admit a canonical right inverse of the T_+-action making $D_{SV}(\pi)$ an étale T_+-module over $\Lambda(N_0)$.

However, by taking a resolution of π by compactly induced representations of B, one may consider the derived functors D^i_{SV} of D_{SV} for $i \ge 0$ producing étale T_+-modules $D^i_{SV}(\pi)$ over $\Lambda(N_0)$. Note that the functor D_{SV} is not exact in the middle, but takes surjective (resp. injective) maps to injective (resp. surjective). The fundamental open question of [34] whether the topological localizations $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D^i_{SV}(\pi)$ are finitely generated over $\Lambda_\ell(N_0)$ in case when π comes as a restriction of a smooth admissible representation of G of finite length. One can pass to usual 1-variable étale (φ, Γ)-modules—still not necessarily finitely generated—over $\mathcal{O}_\mathcal{E}$ via the map $\ell\colon \Lambda_\ell(N_0) \to \mathcal{O}_\mathcal{E}$ which step is an equivalence of categories for finitely generated étale (φ, Γ)-modules (Thm. 2.1 (Thm. 8.20 in [35])).

Instead of deriving the functor D_{SV} we propose to pass to an *étale hull* instead, in order to produce étale T_+-modules (see section 3.4).

3.2 Breuil's functor

More recently, Breuil [9] managed to find a different approach, producing a pseudocompact (ie. projective limit of finitely generated) (φ,Γ)-module $D^\vee_\xi(\pi)$ over $\mathcal{O}_\mathcal{E}$ when π is killed by a power ϖ^h of the uniformizer ϖ. In [9] (and also in [34]) ℓ is a *generic* Whittaker functional, namely ℓ is chosen to be the composite map

$$\ell\colon N_0 \to N_0/(N_0\cap[N,N]) \cong \prod_{\alpha\in\Delta} N_{\alpha,0} \overset{\sum\limits_{\alpha\in\Delta} u_\alpha^{-1}}{\longrightarrow} \mathbb{Z}_p \ .$$

Breuil passes right away to the space of H_0-invariants π^{H_0} of π where H_0 is the kernel of the group homomorphism $\ell\colon N_0\to\mathbb{Z}_p$. By the assumption that π is smooth, the invariant subspace π^{H_0} has the structure of a module over the Iwasawa algebra $\Lambda(N_0/H_0)/\varpi^h\cong o/\varpi^h[[X]]$. Moreover, it admits a semilinear action of F which is the Hecke action of $s:=\xi(p)$: For any $m\in\pi^{H_0}$ we define

$$F(m) := \mathrm{Tr}_{H_0/sH_0s^{-1}}(sm) = \sum_{u\in J(H_0/sH_0s^{-1})} usm \ .$$

So π^{H_0} is a module over the skew polynomial ring $\Lambda(N_0/H_0)/\varpi^h[F]$ (defined by the identity $FX=(sXs^{-1})F=((X+1)^p-1)F$). We consider those (i) finitely generated $\Lambda(N_0/H_0)/\varpi^h[F]$-submodules $M\subset\pi^{H_0}$ that are (ii) invariant under the action of Γ and are (iii) *admissible* as a $\Lambda(N_0/H_0)/\varpi^h$-module, ie. the Pontryagin dual $M^\vee=\mathrm{Hom}_o(M,o/\varpi^h)$ is finitely generated over $\Lambda(N_0/H_0)/\varpi^h$. Note that this admissibility condition (iii) is equivalent to the usual admissibility condition in smooth representation theory, ie. that for any (or equivalently for a single) open subgroup $N'\leq N_0/H_0$ the fixed points $M^{N'}$ form a finitely generated module over o. We denote by $\mathcal{M}(\pi^{H_0})$ the—via inclusion partially ordered—set of those submodules $M\leq\pi^{H_0}$ satisfying $(i),(ii),(iii)$. Note that whenever M_1,M_2 are in $\mathcal{M}(\pi^{H_0})$ then so is M_1+M_2. It is shown in [17] (see also Lemma 2.6 in [9]) that for $M\in\mathcal{M}(\pi^{H_0})$ the localized Pontryagin dual $M^\vee[1/X]$ naturally admits a structure of an étale (φ,Γ)-module over $o/\varpi^h((X))$. Therefore Breuil [9] defines

$$D^\vee_\xi(\pi) := \varprojlim_{M\in\mathcal{M}(\pi^{H_0})} M^\vee[1/X] \ .$$

By construction this is a projective limit of usual (φ,Γ)-modules. Moreover, $D^\vee_\xi$ is right exact and compatible with parabolic induction [9]. It can be characterized by the following universal property: For any (finitely generated) étale (φ,Γ)-module over $o/\varpi^h((X))\cong o/\varpi^h[[\mathbb{Z}_p]][([1]-1)^{-1}]$ (here $[1]$ is the image of the topological generator of $\mathbb{Z}_p$ in the Iwasawa algebra $o/\varpi^h[[\mathbb{Z}_p]]$) we may consider continuous $\Lambda(N_0)$-homomorphisms $\pi^\vee\to D$ via the map $\ell\colon N_0\to\mathbb{Z}_p$ (in the weak topology of D and the compact topology of $\pi^\vee$). These all factor through $(\pi^\vee)_{H_0}\cong(\pi^{H_0})^\vee$. So we may require these maps be ψ_s- and Γ-equivariant where $\Gamma=\xi(\mathbb{Z}_p\setminus\{0\})$ acts naturally on $(\pi^{H_0})^\vee$ and $\psi_s\colon(\pi^{H_0})^\vee\to(\pi^{H_0})^\vee$ is the dual of the Hecke-action $F\colon\pi^{H_0}\to\pi^{H_0}$ of s on π^{H_0}. Any such continuous ψ_s- and Γ-equivariant map f factors uniquely through $D^\vee_\xi(\pi)$. However, it is not known in general whether $D^\vee_\xi(\pi)$ is nonzero for smooth irreducible representations π of G (restricted to B).

3.3 Noncommutative variant of Breuil's functor

Our first result is the construction of a noncommutative multivariable version of $D_\xi^\vee(\pi)$. Let π be a smooth o-torsion representation of B such that $\varpi^h\pi = 0$. The idea here is to take the invariants π^{H_k} for a family of open normal subgroups $H_k \leq H_0$ with $\bigcap_{k\geq 0} H_k = \{1\}$. Now Γ and the quotient group N_0/H_k act on π^{H_k} (we choose H_k so that it is normalized by both Γ and N_0). Further, we have a Hecke-action of s given by $F_k := \mathrm{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot)$. As in [9] we consider the set $\mathcal{M}_k(\pi^{H_k})$ of finitely generated $\Lambda(N_0/H_k)[F_k]$-submodules of π^{H_k} that are stable under the action of Γ and admissible as a representation of N_0/H_k.

Proposition 3.1. *For any $M_k \in \mathcal{M}_k(\pi^{H_k})$ there is an étale (φ,Γ)-module structure on $M_k^\vee[1/X]$ over the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$.*

So the projective limit

$$D_{\xi,\ell,\infty}^\vee(\pi) := \varprojlim_{k\geq 0} \varprojlim_{M_k\in\mathcal{M}_k(\pi^{H_k})} M_k^\vee[1/X]$$

is an étale (φ,Γ)-module over $\Lambda_\ell(N_0)/\varpi^h = \varprojlim_k \Lambda(N_0/H_k)/\varpi^h[1/X]$. Moreover, we also give a natural isomorphism $D_{\xi,\ell,\infty}^\vee(\pi)_{H_0} \cong D_\xi^\vee(\pi)$ showing that $D_{\xi,\ell,\infty}^\vee(\pi)$ corresponds to $D_\xi^\vee(\pi)$ via (the projective limit of) the equivalence of categories in Thm. 2.1 (Thm. 8.20 in [35]). Further, the natural map $\pi^\vee \to D_{\xi,\ell}^\vee(\pi)$ factors through the projection map $D_{\xi,\ell,\infty}^\vee(\pi) \twoheadrightarrow D_{\xi,\ell}^\vee(\pi) = D_{\xi,\ell,\infty}^\vee(\pi)_{H_0}$. Note that this shows that $D_{\xi,\ell,\infty}^\vee(\pi)$ is naturally attached to π—not just simply via the equivalence of categories (loc. cit.)—in the sense that any ψ- and Γ-equivariant map from $\pi^\vee$ to an étale (φ,Γ)-module over $o/\varpi^h((X))$ factors uniquely through the corresponding multivariable (φ,Γ)-module. On of the key steps connecting D_{SV} to $D_{\xi,\ell,\infty}^\vee$ is to show that the natural map $\pi^\vee \to D_{\xi,\ell,\infty}^\vee(\pi)$ factors through the map $\pi^\vee \to D_{SV}(\pi)$. In particular, we obtain a natural map $\mathrm{pr}\colon D_{SV}(\pi) \to D_{\xi,\ell,\infty}^\vee(\pi)$. Further, this map has the following universal property:

Proposition 3.2. *Let D be an étale (φ,Γ)-module over $\Lambda_\ell(N_0)/\varpi^h$, and $f : D_{SV}(\pi) \to D$ be a continuous ψ_s and Γ-equivariant $\Lambda(N_0)$-homomorphism. Then f factors uniquely through* pr, *ie. there exists a unique ψ- and Γ-equi-variant $\Lambda(N_0)$-homomorphism $\hat{f} : D_{\xi,\ell,\infty}^\vee(\pi) \to D$ such that $f = \hat{f} \circ \mathrm{pr}$.*

One application is that Breuil's functor $D_\xi^\vee$ vanishes on compactly induced representations of B.

3.4 Étale hull and the comparison

In order to be able to compute $D_{\xi,\ell,\infty}^\vee(\pi)$ (hence also $D_\xi^\vee(\pi)$) from $D_{SV}(\pi)$ we introduce the notion of the *étale hull* of a $\Lambda(N_0)$-module with a ψ-action of T_+ (or of a submonoid $T_* \leq T_+$). Here a $\Lambda(N_0)$-module D with a ψ-action of T_+ is the analogue of a (ψ,Γ)-module over $o[[X]]$ in this multivariable noncommutative setting. The étale hull $\widetilde{D}$ of D (together with a canonical map $\iota\colon D \to \widetilde{D}$) is characterized by the following universal property:

Proposition 3.3. *For any $\Lambda(N_0)$-module D, with a ψ-action of T_* there exists an étale T_*-module $\widetilde{D}$ over $\Lambda(N_0)$ and a ψ-equivariant $\Lambda(N_0)$-homomor-phism $\iota\colon D \to \widetilde{D}$ with the* *following universal property: For any ψ-equivariant $\Lambda(N_0)$-homomorphism $f : D \to D'$ into an étale T_*-module D' we have a unique morphism $\widetilde{f} : \widetilde{D} \to D'$ of étale T_*-modules over $\Lambda(N_0)$ making the diagram*

$$\begin{array}{ccc} D & \xrightarrow{\iota} & \widetilde{D} \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \widetilde{f}} & \\ D' & & \end{array}$$

commutative. $\widetilde{D}$ is unique upto a unique isomorphism. If we assume the ψ-action on D to be nondegenerate then ι is injective.

The étale hull $\iota\colon D \to \widetilde{D}$ can be constructed as a direct limit $\varinjlim_{t \in T_+} \varphi_t^* D$ where $\varphi_t^* D = \Lambda(N_0) \otimes_{\varphi_t, \Lambda(N_0)} D$. The main result comparing the two functors is the following

Theorem 3.4. *$D^\vee_{\xi,\ell,\infty}(\pi)$ is the pseudocompact completion of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ in the category of étale (φ, Γ)-modules over $\Lambda_\ell(N_0)$, ie. we have*

$$D^\vee_{\xi,\ell,\infty}(\pi) \cong \varprojlim_{D} D$$

where D runs through the finitely generated étale (φ, Γ)-modules over $\Lambda_\ell(N_0)$ arising as a quotient of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ by a closed submodule. This holds in any *topology on $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ making both the maps $1 \otimes \iota\colon D_{SV}(\pi) \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$, $d \mapsto 1 \otimes \iota(d)$ and $1 \otimes \widetilde{\mathrm{pr}}\colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ continuous.*

Since the map $\mathrm{pr}\colon D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ is continuous, there exists such a topology on $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$. For instance we could take either the final topology of the map $D_{SV}(\pi) \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ or the initial topology of the map $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$.

The idea of the proof of Thm. 3.4 is to prove that both constructions have the same universal property.

3.5 Towards reconstructing π from $D^\vee_{\xi,\ell,\infty}(\pi)$

In order to go back to representations of G we need an étale action of T_+ on $D^\vee_{\xi,\ell,\infty}(\pi)$, not just of $\xi(\mathbb{Z}_p \setminus \{0\})$. This is only possible if $tH_0t^{-1} \leq H_0$ for all $t \in T_+$ which is not the case for generic ℓ. So we equip now $D^\vee_{\xi,\ell,\infty}(\pi)$ with an étale action of T_+ (extending that of $\xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+$) in case $\ell = \ell_\alpha$ is the projection of N_0 onto a root subgroup $N_{\alpha,0} \cong \mathbb{Z}_p$ for some simple root α in Δ. Moreover, we show that the map $\mathrm{pr}\colon D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ is ψ-equivariant for this extended action, too. Note that $D^\vee_{\xi,\ell,\infty}(\pi)$ may not be the projective limit of finitely generated étale T_+-modules over $\Lambda_\ell(N_0)$ as we do not necessarily have an action of T_+ on $M^\vee_\infty[1/X]$ for $M \in \mathcal{M}(\pi^{H_0})$, only on the projective limit. So the construction of a G-equivariant sheaf on G/B (in the spirit of section 2) with sections on $\mathcal{C}_0 = N_0 w_0 B/B \subset G/B$ isomorphic to a dense B_+-stable $\Lambda(N_0)$-submodule $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ of $D^\vee_{\xi,\ell,\infty}(\pi)$ is not immediate from the work [35] as only the case of finitely generated modules over $\Lambda_\ell(N_0)$ is treated in there. However, the most natural definition of bounded elements in $D^\vee_{\xi,\ell,\infty}(\pi)$ works: The $\Lambda(N_0)$-submodule $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ is defined as the union of ψ-invariant compact $\Lambda(N_0)$-submodules of $D^\vee_{\xi,\ell,\infty}(\pi)$. Moreover, we have

Proposition 3.5. *The image of* $\widetilde{\mathrm{pr}}\colon \widetilde{D_{SV}}(\pi) \to D^{\vee}_{\xi,\ell,\infty}(\pi)$ *is contained in* $D^{\vee}_{\xi,\ell,\infty}(\pi)^{bd}$.

Further, the constructions of [35] (section 2 of this thesis) can be carried over to this situation. We denote the resulting G-equivariant sheaf on G/B by $\mathfrak{Y} = \mathfrak{Y}_{\alpha,\pi}$.

Now consider the functors $(\cdot)^{\vee}\colon \pi \mapsto \pi^{\vee}$ and the composite

$$\mathfrak{Y}_{\alpha,\cdot}(G/B)\colon \pi \mapsto D^{\vee}_{\xi,\ell,\infty}(\pi) \mapsto \mathfrak{Y}_{\alpha,\pi}(G/B)$$

both sending smooth, admissible o/ϖ^h-representations of G of finite length to topological representations of G over o/ϖ^h. The main result of [21] is

Theorem 3.6. *The family of morphisms* $\beta_{G/B,\pi}$ *for smooth, admissible o-torsion representations* π *of* G *of finite length form a natural transformation between the functors* $(\cdot)^{\vee}$ *and* $\mathfrak{Y}_{\alpha,\cdot}(G/B)$. *Whenever* $D^{\vee}_{\xi,\ell}(\pi)$ *is nonzero, the map* $\beta_{G/B,\pi}$ *is nonzero either. In particular, if we further assume that* π *is irreducible then* $\beta_{G/B}$ *is injective.*

This generalizes Thm. IV.4.7 in [17]. The proof of this relies on the observation that the maps $\mathcal{H}_g\colon D^{\vee}_{\xi,\ell,\infty}(\pi)^{bd} \to D^{\vee}_{\xi,\ell,\infty}(\pi)^{bd}$ in fact come from the G-action on $\pi^{\vee}$. More precisely, for any $g \in G$ and $W \in \mathcal{B}_{+}(\pi)$ we have maps

$$(g\cdot)\colon (g^{-1}W \cap W)^{\vee} \to (W \cap gW)^{\vee}$$

where both $(g^{-1}W \cap W)^{\vee}$ and $(W \cap gW)^{\vee}$ are naturally quotients of $W^{\vee}$. These maps fit into a commutative diagram

$$\begin{array}{ccccc}
W^{\vee} & \longrightarrow & (g^{-1}W \cap W)^{\vee} & \xrightarrow{\ g\cdot\ } & (W \cap gW)^{\vee} \\
\big\downarrow \mathrm{pr}_W & & \big\downarrow & & \big\downarrow \\
D^{\vee}_{\xi,\ell,\infty}(\pi)^{bd} & \longrightarrow & \mathrm{res}^{\mathcal{C}_0}_{g^{-1}\mathcal{C}_0\cap\mathcal{C}_0}(D^{\vee}_{\xi,\ell,\infty}(\pi)^{bd}) & \xrightarrow{\ g\cdot\ } & \mathrm{res}^{\mathcal{C}_0}_{\mathcal{C}_0\cap g\mathcal{C}_0}(D^{\vee}_{\xi,\ell,\infty}(\pi)^{bd})
\end{array}$$

allowing us to construct the map $\beta_{G/B}$. The proof of Thm. 3.6 is similar to that of Thm. IV.4.7 in [17]. However, unlike that proof we do not need the full machinery of "standard presentations" in Ch. III.1 of [17] which is not available at the moment for groups other than $\mathrm{GL}_2(\mathbb{Q}_p)$.

4 Using multivariable (φ, Γ)-modules

We saw in section 3.2 that Breuil's functor $D^{\vee}_{\xi,\ell}$ has very promising properties: it is right exact and compatible with tensor products and with parabolic induction. Moreover, $D^{\vee}_{\xi,\ell}$ is exact and produces finitely generated objects on the category SP_A of finite length representations with all Jordan-Hölder factors appearing as a subquotient of principal series representations (ie. of $\mathrm{Ind}_B^G \chi$ for some character χ of T). Finally, $D^{\vee}_{\xi,\ell}$ is compatible with the conjectures in [10] made from a global point of view. The assumption on the genericity of ℓ is needed crucially for some of these properties, in particular for the exactness on SP_A

and for the compatibility with [10]. However, if ℓ is a generic Whittaker functional then the functor $D^\vee_{\xi,\ell}$ loses a lot of information, one cannot possibly recover the representation π from the attached (φ,Γ)-module $D^\vee_{\xi,\ell}(\pi)$ (by the methods developed in [35] or otherwise). This has also been predicted by the work of Breuil and Paškūnas [12]: when one moves beyond $\mathrm{GL}_2(\mathbb{Q}_p)$ then there are much more representations on the automorphic side than on the Galois side. So if we would like to have a bijection for some large class of representations on the reductive group side, we need to put additional data on our Galois-representations. One candidate is that we could perhaps equip the Galois representation with an additional character of the torus $T/\xi(\mathbb{Q}_p^\times)$ extending the action of φ and Γ. The heuristics for this is that even in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$ a central character appears naturally on the attached (φ,Γ)-module. However, if ℓ is generic then the action of φ and Γ on $D^\vee_{\xi,\ell}(\pi)$ cannot be extended to the dominant submonoid $T_+ \subset T$ since in this case the kernel $H_{gen} = \mathrm{Ker}(\ell\colon N \to \mathbb{Q}_p)$ is not invariant under the conjugation action of any larger subgroup of T than the product of the image of ξ and the centre. On the other hand, if we choose ℓ to be very far from being generic, ie. $\ell = \ell_\alpha$ is the projection onto a root subgroup N_α for some simple root $\alpha \in \Delta$ then we do have an additional action of T_+ on $D^\vee_{\xi,\ell}(\pi)$ as shown by the present author and Erdélyi [21] (see section 3 in this thesis). However, as mentioned above, for non-generic ℓ the functor $D^\vee_{\xi,\ell}$ does not have so good exactness and compatibility properties.

The goal in this section is to combine all the mentioned good properties of the above approaches. In order to do this we are going to use *multivariable* (φ,Γ)-modules in the variables X_α (α in the set Δ of simple roots). Apart from the last subsection—in which we relate these multivariable (φ,Γ)-modules to representations of the $|\Delta|$th direct product $G_{\mathbb{Q}_p,\Delta}$ of $G_{\mathbb{Q}_p}$ building on the paper [45]—the results in this section are proven in [44].

4.1 Commutative multivariable (φ,Γ)-modules

Consider the Laurent series ring $A((X_\alpha \mid \alpha \in \Delta)) := A[[X_\alpha, \alpha \in \Delta]][X_\alpha^{-1} \mid \alpha \in \Delta]$ (the variables are indexed by the finite set Δ of simple roots of the pair (G,B)) where we put $A := o/(\varpi^h)$ for the coefficient ring. Let $N_{\Delta,0} := N_0/H_{\Delta,0}$ where $H_{\Delta,0} = \prod_{\beta\in\Phi^+\setminus\Delta}(N_\beta \cap N_0)$. This group is isomorphic to the direct product of the groups $N_\alpha \cap N_0 \cong \mathbb{Z}_p$ for all simple roots α in Δ, in particular, it is commutative. Therefore the Iwasawa algebra $A[[N_{\Delta,0}]]$ can be identified with the ring $A[[X_\alpha, \alpha \in \Delta]]$ of power formal power series in $|\Delta|$ variables. In particular, the ring $A((N_{\Delta,0})) := A((X_\alpha \mid \alpha \in \Delta))$ comes equipped with the conjugation action of the monoid $T_+ := \{t \in T \mid \alpha(t) \in \mathbb{Z}_p \text{ for all } \alpha \in \Delta\}$. Like (φ,Γ)-modules in the appendix A.4, an étale T_+-module over $A((N_{\Delta,0}))$ is a finitely generated module D together with a semilinear action of the monoid T_+ such that for all $t \in T_+$ the map

$$\begin{array}{rcl} A((N_{\Delta,0})) \otimes_{\varphi_t, A((N_{\Delta,0}))} M & \to & M \\ \lambda \otimes m & \mapsto & \lambda\varphi_t(m) \end{array}$$

is bijective.

Even though the ring $A((N_{\Delta,0}))$ is not artinian, the existence of an action of the group T_0 improves its properties: by the nonexistence of T_0-invariant ideals in $\kappa((N_{\Delta,0}))$ it follows that any finitely generated module over $A((N_{\Delta,0}))$ admitting a semilinear action of T_0 has finite length (in the category of modules with semilinear T_0-action). This fact allows us to construct a functor $D^\vee_\Delta$ from the category of smooth A-representations of the Borel B to projective limits of finitely generated étale T_+-modules over $A(\!(N_{\Delta,0})\!)$ in an analogous way to Breuil's functor [9] (see section 3.2). More precisely, we consider the skew polynomial ring $A[\![N_{\Delta,0}]\!][F_\alpha \mid \alpha \in \Delta]$ where the variables F_α commute with each other and we have $F_\alpha \lambda = (t_\alpha \lambda t_\alpha^{-1}) F_\alpha$ for $\lambda \in A[\![N_{\Delta,0}]\!]$. For a smooth representation π of B over A we denote by $\mathcal{M}_\Delta(\pi^{H_{\Delta,0}})$ the set of finitely generated $A[\![N_{\Delta,0}]\!][F_\alpha \mid \alpha \in \Delta]$-submodules of $\pi^{H_{\Delta,0}}$ that are stable under the action of T_0 and admissible as a representation of $N_{\Delta,0} = N_0/H_{\Delta,0}$. Here F_α acts on $\pi^{H_{\Delta,0}}$ by the Hecke action of $t_\alpha \in T_+$, ie. $F_\alpha v := \mathrm{Tr}_{H_{\Delta,0}/t_\alpha H_{\Delta,0} t_\alpha^{-1}}(t_\alpha v)$ for $v \in \pi^{H_{\Delta},0}$. Then the functor $D_\Delta^\vee$ is defined by the projective limit

$$D_\Delta^\vee(\pi) := \varprojlim_{M \in \mathcal{M}_\Delta(\pi^{H_{\Delta,0}})} M^\vee[1/X_\Delta]$$

where $X_\Delta = \prod_{\alpha\in\Delta} X_\alpha$ is the product of all the variables $X_\alpha = n_\alpha - 1$ in the power series ring $A[\![N_{\Delta,0}]\!]$.

If we define

$$\ell\colon N \to N/[N,N] = \prod_{\alpha\in\Delta} N_\alpha \stackrel{\sum_{\alpha\in\Delta} u_\alpha^{-1}}{\longrightarrow} \mathbb{Q}_p$$

in a generic way and extend this to the Iwasawa algebra $A[\![N_{\Delta,0}]\!]$ then we find that $\ell(X_\alpha) = X$ for all $\alpha \in \Delta$ after the identification $A[\![\mathbb{Z}_p]\!] \cong A[\![X]\!]$. Therefore we may extend ℓ to a map $\ell\colon A(\!(N_{\Delta,0})\!) \to A(\!(X)\!)$ of Laurent series rings. Note that the kernel of ℓ is not stable under the action of T_+, but it is stable under the action of φ and Γ. So we obtain a reduction map $A(\!(X)\!) \otimes_{A(\!(N_{\Delta,0})\!),\ell} \cdot$ from étale T_+-modules to usual étale (φ,Γ)-modules. We show that this reduction map is faithful and exact which implies

Theorem 4.1. *The functor $D_\Delta^\vee$ is right exact.*

In particular, one has a natural transformation from Breuil's functor $D_{\xi,\ell}^\vee$ to the composite $A(\!(X)\!) \otimes_{A(\!(N_{\Delta,0})\!),\ell} D_\Delta^\vee$. When restricted to the category SP_A, this is an isomorphism. Moreover, this is also an isomorphism for objects obtained by parabolic induction from a subgroup with Levi component isomorphic to the product of copies of $\mathrm{GL}_2(\mathbb{Q}_p)$ and a split torus. For general π the map $D_{\xi,\ell}^\vee(\pi) \to A(\!(X)\!) \otimes_{A(\!(N_{\Delta,0})\!),\ell} D_\Delta^\vee(\pi)$ is always surjective and we conjecture it to be an isomorphism at least for admissible π.

4.2 Compatibility, finiteness, and exactness results

The functor $D_\Delta^\vee$ is compatible with products $G \times G'$ of groups with simple roots Δ, resp. Δ' in the following sense. The value of $D_{\Delta\cup\Delta'}^\vee$ on a tensor product $\pi \otimes_\kappa \pi'$ of representations π of G (resp. π' of G') is the completed tensor product $D_\Delta^\vee(\pi) \hat{\otimes}_\kappa D_{\Delta'}^\vee(\pi')$. Note that this is a module over a multivariate Laurent series ring $A(\!(N_{\Delta\cup\Delta',0})\!)$ in variables indexed by the union $\Delta \cup \Delta'$. Similarly, we have a compatibility result for parabolic induction: Let $P = L_P N_P$ be a parabolic subgroup containing B and π_P a smooth representation of L_P over A viewed as representation of the opposite parabolic P^-. Denote by $\Delta_P \subseteq \Delta$ the set of those simple roots whose root subgroups are contained in the Levi component L_P. We show

Theorem 4.2. *Let π_P be a smooth locally admissible representation of L_P over A which we view by inflation as a representation of P^-. We have an isomorphism*

$$D_\Delta^\vee\left(\mathrm{Ind}_{P^-}^G \pi_P\right) \cong A(\!(N_{\Delta,0})\!) \hat{\otimes}_{A(\!(N_{\Delta_P,0})\!)} D_{\Delta_P}^\vee(\pi_P)$$

in the category $\mathcal{D}^{pro-et}(T_+, A(\!(N_{\Delta,0})\!))$.

On one hand, the above result shows that $D^\vee_\Delta$ is nonzero and finitely generated on parabolically induced representations from products of copies of $\mathrm{GL}_2(\mathbb{Q}_p)$ and a torus unless one of the representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ is finite dimensional. Moreover, combined with the right exactness we also know this for extensions of representations of this type just like for Breuil's functor [9]. On the other hand, this might lead to another characterization of supercuspidal representations: it would be natural to expect that if π is an irreducible supercuspidal representation then $D^\vee_\Delta(\pi)$ cannot be induced from a T_+-module in less variables. However, showing this would require a better understanding of supercuspidals beyond GL_2.

Let SP_A be the category of smooth finite length representations of G whose Jordan-Hölder factors are subquotients of principal series (ie. representations of the form $\mathrm{Ind}_B^G \chi$ for some character $B \twoheadrightarrow T \to A^\times$. We have

Theorem 4.3. *The restriction of* $D^\vee_\Delta$ *to* SP_A *is exact and produces finitely generated objects.*

The proof of this builds on showing that the finitely generated $A[\![N_{\Delta,0}]\!][F_\alpha \mid \alpha \in \Delta]$-submodules of representations in SP_A are in fact finitely presented.

Corollary 4.4. *On the category* SP_A *Breuil's functor* $D^\vee_{\xi,\ell}$ *coincides with the composite functor* $A(\!(X)\!) \otimes_{\ell, A(\!(N_{\Delta,0})\!)} \cdot \circ D^\vee_\Delta$.

4.3 Lifting to noncommutative coefficients

We develop a noncommutative analogue of $D^\vee_\Delta$ as in [21] for Breuil's functor. The first step is the construction of the ring $A(\!(N_{\Delta,\infty})\!)$ as a projective limit $\varprojlim_k A(\!(N_{\Delta,k})\!)$ where the finite layers $A(\!(N_{\Delta,k})\!) := A[\![N_{\Delta,k}]\!][\varphi_s^{kn_0}(X_\alpha)^{-1}]$ (where $n_0 = n_0(G) \in \mathbb{N}$ is the maximum of the degrees of the algebraic characters $\beta \circ \xi \colon \mathbf{G}_m \to \mathbf{G}_m$ for all positive roots $\beta \in \Phi^+$) are defined as localisations of the Iwasawa algebra $A[\![N_{\Delta,k}]\!]$. Here the group $N_{\Delta,k} := N_0/H_{\Delta,k}$ is the extension of $N_{\Delta,0}$ by a finite p-group $H_{\Delta,0}/H_{\Delta,k}$ where $H_{\Delta,k}$ is the smallest normal subgroup in N_0 containing $s^k H_{\Delta,0} s^{-k}$. Note that unlike in the one variable localization $\Lambda_\ell(N_0)$ we do not have a section of the group homomorphism $N_{\Delta,k} \to N_{\Delta,0}$. However, restricting to the image of the conjugation by s^{kn_0}, we do: this allows us to build a functor $\mathbb{M}_{k,0}$ from the category $\mathcal{D}^{et}(T_+, A(\!(N_{\Delta,0})\!))$ of finitely generated étale T_+-modules over $A(\!(N_{\Delta,0})\!)$ to the category $\mathcal{D}^{et}(T_+, A(\!(N_{\Delta,k})\!))$ of finitely generated étale T_+-modules over $A(\!(N_{\Delta,k})\!)$. Putting $\mathbb{M}_{\infty,0} := \varprojlim_k \mathbb{M}_{k,0}$ and $\mathbb{D}_{0,\infty}$ to be the functor from the category $\mathcal{D}^{et}(T_+, A(\!(N_{\Delta,\infty})\!))$ of finitely generated étale T_+-modules over $A(\!(N_{\Delta,\infty})\!)$ to $\mathcal{D}^{et}(T_+, A(\!(N_{\Delta,0})\!))$ induced by the reduction map $A(\!(N_{\Delta,\infty})\!) \to A(\!(N_{\Delta,0})\!)$ we obtain

Theorem 4.5. *The functors* $\mathbb{M}_{\infty,0}$ *and* $\mathbb{D}_{0,\infty}$ *are quasi-inverse equivalences of categories between* $\mathcal{D}^{et}(T_+, A(\!(N_{\Delta,0})\!))$ *and* $\mathcal{D}^{et}(T_+, A(\!(N_{\Delta,\infty})\!))$.

By considering finitely generated $A[\![N_{\Delta,k}]\!][F_{\alpha,k} \mid \alpha \in \Delta]$-submodules of $\pi^{H_{\Delta,k}}$ that are stable under the action of T_0 and are admissible as representations of $N_{\Delta,k}$ we introduce the functors $D^\vee_{\Delta,k}$ analogous to $D^\vee_\Delta$ for all $k \geq 0$ and we put $D^\vee_{\Delta,\infty}(\pi) := \varprojlim_k D^\vee_{\Delta,k}(\pi)$ for a smooth representation π of B over A. This corresponds to $D^\vee_\Delta(\pi)$ via the extension of the equivalence of categories in Theorem 4.5 to pro-objects on both sides. The universal property of $D^\vee_{\Delta,\infty}$ leads to its alternative description via the Schneider–Vigneras functor $D_{SV}(\pi)$ (and via its étale hull $\widetilde{D_{SV}}(\pi)$):

Theorem 4.6. *We have*

$$D^\vee_{\Delta,\infty}(\pi) \cong \varprojlim_D D$$

where D runs through the finitely generated étale T_+-modules over $A((N_{\Delta,\infty}))$ arising as a quotient of $A((N_{\Delta,\infty})) \otimes_{A[\![N_0]\!]} \widetilde{D_{SV}}(\pi)$ such that the quotient map is continuous in the weak topology of D and the final topology on $A((N_{\Delta,\infty})) \otimes_{A[\![N_0]\!]} \widetilde{D_{SV}}(\pi)$ of the map $1 \otimes \iota \colon D_{SV}(\pi) \to A((N_{\Delta,\infty})) \otimes_{A[\![N_0]\!]} \widetilde{D_{SV}}(\pi)$.

4.4 Fully faithful property in a special case

Now we turn to the question of reconstructing the smooth representation π of G from $D^\vee_\Delta(\pi)$. This is certainly not possible in general, as for instance finite dimensional representations are in the kernel of $D^\vee_\Delta$ (unless the set Δ of simple roots is empty). However, using the ideas of [35] we show the following positive results in this direction. For an object $M \in \mathcal{M}_\Delta(\pi^{H_{\Delta,0}})$ we denote by $\widetilde{M^\vee_\infty}$ the étale hull of the image $M^\vee_\infty$ of the natural map from $\pi^\vee$ to the étale T_+-module $\mathbb{M}_{\infty,0}(M^\vee[1/X_\Delta])$.

Theorem 4.7. *For any smooth o-torsion representation π of G and any $M \in \mathcal{M}_\Delta(\pi^{H_{\Delta,0}})$ there exists a G-equivariant sheaf $\mathfrak{Y}_{\pi,M}$ on G/B with sections $\mathfrak{Y}_{\pi,M}(\mathcal{C}_0)$ on $\mathcal{C}_0$ isomorphic to $\widetilde{M^\vee_\infty}$ as an étale T_+-module over $A[\![N_0]\!]$. Moreover, we have a G-equivariant continuous map $\beta_{G/B,M}$ from the Pontryagin dual $\pi^\vee$ to the global sections $\mathfrak{Y}_{\pi,M}(G/B)$ that is natural in both π and M, and is nonzero unless $M^\vee[1/X_\Delta] = 0$.*

Here we in fact use the G-action on π in order to construct the sheaf $\mathfrak{Y}_{\pi,M}$ unlike in [35] where the operators $\mathcal{H}_g = \mathrm{res}(g\mathcal{C}_0 \cap \mathcal{C}_0) \circ (g\cdot)$ for the open cell $\mathcal{C}_0 := N_0\overline{B} \subset G/\overline{B} \cong G/B$ are constructed as a limit. Apparently the formulas defining this limit do not converge in the weak topology of the finitely generated $A((N_{\Delta,\infty}))$-module $M^\vee_\infty[1/X_\Delta]$. Nevertheless, if π is irreducible and $D^\vee_\Delta(\pi) \neq 0$ then we can realize $\pi^\vee$ as a subrepresentation of the global sections of a G-equivariant sheaf on G/B whose space of sections on $\mathcal{C}_0$ is "small" in the sense that it is contained in a finitely generated $A((N_{\Delta,\infty}))$-module. Let us denote by SP^0_A the full subcategory of SP_A containing those representations whose Jordan-Hölder factors are irreducible principle series. As an application of the methods above we prove

Theorem 4.8. *The restriction of $D^\vee_\Delta$ to the category SP^0_A is fully faithful.*

In particular, the forgetful functor restricting π to B is also fully faithful on SP^0_A as $D^\vee_\Delta$ factors through this.

4.5 Connections to the Galois side

In this section we relate the multivariable (φ,Γ)-modules to the Galois side generalizing Fontaine's theory (see appendix A.4). Note that under our standing assumption that the centre $Z(G)$ of G is connected, the monoid T_+ decomposes as a direct product of $|\Delta|$ copies of $\mathbb{Z}_p \setminus \{0\}$ and $Z(G)$. In this case the present author's results in [45] imply the following theorems.

Theorem 4.9. *The category of étale T_+-modules over $A((N_{\Delta,0}))$ is equivalent to the category of continuous representations of the group $G_{\mathbb{Q}_p,\Delta} \times Z(G)$ on finitely generated A-modules.*

Put $\mathcal{E}_\Delta := \left(\varprojlim_h o/(\varpi^h)((N_{\Delta,0}))\right)[\varpi^{-1}]$. Taking the limit with $h \to \infty$ in $A = o/(\varpi^h)$ we obtain

Theorem 4.10. *The category of étale T_+-modules over $\mathcal{E}_\Delta$ is equivalent to the category of continuous representations of the group $G_{\mathbb{Q}_p,\Delta} \times Z(G)$ on finite dimensional K-vectorspaces.*

The above results are proven by reducing the statement to the one-variable case using an integrality argument originating in the ideas of Colmez.

5 An optimistic conjecture

In this section we attempt to explain how the theory in the previous section should, conjecturally, fit into the global picture of the p-adic (or rather modulo p) Langlands programme.

Let F be an imaginary quadratic field in which the prime p splits. (More generally, we could take a *CM field* F—ie. a totally imaginary quadratic extension of a totally real field F^+—such that p splits completely in F.) Let $\mathbf{G}$ be a unitary group over $\mathbb{Z}[1/N]$ for some integer N, ie. an algebraic group such that we have $\mathbf{G}(\mathbb{R}) \cong U_n(\mathbb{R})$ and $\mathbf{G} \times_{\mathbb{Z}[1/N]} F \cong \mathrm{GL}_n/F$. By our assumption on the splitting of p the field $\mathbb{Q}_p$ contains F as a subfield whence we have $\mathbf{G}(\mathbb{Q}_p) \cong \mathrm{GL}_n(\mathbb{Q}_p)$. As in Emerton's compatibility result (see section 1.5.4) let $K_f^p \leq \mathbf{G}(\mathbb{A}^p_{\mathbb{Q},fin})$ be an open compact subgroup of the group of points of $\mathbf{G}$ over the ring $\mathbb{A}^p_{\mathbb{Q},fin}$ of finite adèles outside p. For any finite field $\mathbb{F}_q$ ($q = p^f$) of characteristic p let

$$S(K_f^p, \mathbb{F}_q) := \{f\colon \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}^p_{\mathbb{Q},fin})/K_f^p \to \mathbb{F}_q,\ \text{locally constant}\} = \\ = \varinjlim_{\substack{K_{f,p} \leq_o \mathrm{GL}_n(\mathbb{Q}_p) \\ \text{compact}}} \{f\colon \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}^p_{\mathbb{Q},fin})/K_f^p K_{f,p} \to \mathbb{F}_q\}$$

be the set of "modulo p automorphic forms". This is a smooth representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathbb{F}_q$. Further, let $\ell \neq p$ be a prime with the following properties

(*i*) ℓ is *unramified* for the group K_f^p, ie. $\mathbf{G}(\mathbb{Z}_\ell) \leq K_f^p$. This is satisfied for all but finitely many primes since K_f^p is open.

(*ii*) ℓ splits in F, ie. we have $(\ell) = \mathfrak{l}\bar{\mathfrak{l}}$ for some prime ideal $\mathfrak{l}$ in the ring of integers of F. In particular, $\mathbf{G}(\mathbb{Q}_\ell) \cong \mathrm{GL}_n(\mathbb{Q}_\ell)$.

(*iii*) $\ell \nmid N$ whence $\mathbb{Z}_\ell$ is a $\mathbb{Z}[1/N]$-algebra, so we can speak of $\mathbf{G}(\mathbb{Z}_\ell)$.

(*iv*) The isomorphism in (*ii*) restricts to an isomorphism $\mathbf{G}(\mathbb{Z}_\ell) \cong \mathrm{GL}_n(\mathbb{Z}_\ell)$.

For such a prime ℓ we have the Hecke operators $T_\ell^{(j)}$ ($1 \leq j \leq n$) associated to the double cosets

$$\mathrm{GL}_n(\mathbb{Z}_\ell)\begin{pmatrix} \mathbf{1}_{n-j} & 0 \\ 0 & \ell\mathbf{1}_j \end{pmatrix}\mathrm{GL}_n(\mathbb{Z}_\ell) = \coprod_i g_i \begin{pmatrix} \mathbf{1}_{n-j} & 0 \\ 0 & \ell\mathbf{1}_j \end{pmatrix}\mathrm{GL}_n(\mathbb{Z}_\ell)$$

acting on $S(K_f^p,\mathbb{F}_q)$ by sending a function $f\in S(K_f^p,\mathbb{F}_q)$ to the function $T_\ell^{(j)}f$ defined by putting

$$T_\ell^{(j)}f(g):=\sum_i f\left(gg_i\begin{pmatrix}\mathbf{1}_{n-j} & 0\\ 0 & \ell\mathbf{1}_j\end{pmatrix}\right)$$

for any $g\in\mathbf{G}(\mathbb{A}^p_{\mathbb{Q},fin})$. The Hecke operators $T_\ell^{(j)}$ commute with each other and with the action of $\mathrm{GL}_n(\mathbb{Q}_p)$.

Now consider a global modulo p Galois representation $\overline{\rho}\colon G_F=\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to\mathrm{GL}_n(\mathbb{F}_q)$ and the prime ℓ satisfies the above conditions and, in addition, the representation $\overline{\rho}$ is unramified at both primes $\mathfrak{l}$ and $\overline{\mathfrak{l}}$ dividing ℓ. Fix the prime $\mathfrak{l}\mid\ell$ in F dividing ℓ and let $X^n+a_\ell^{(1)}X^{n-1}+\dots+a_\ell^{(n-1)}X+a_\ell^{(n)}\in\mathbb{F}_q[X]$ be the characteristic polynomial of $\overline{\rho}(\mathrm{Frob}_\mathfrak{l})\in\mathrm{GL}_n(\mathbb{F}_q)$. The Hecke eigenspace $S(K_f^p,\mathbb{F}_q)[\mathfrak{m}_{\overline{\rho}}]$ corresponding to the representation $\overline{\rho}$ is the space of functions $f\in S(K_f^p,\mathbb{F}_q)$ such that

$$(-1)^j\ell^{\frac{j(j-1)}{2}}T_\ell^{(j)}(f)=a_\ell^{(j)}f$$

for all primes ℓ satisfying the above conditions ($(i)-(iv)$ and $\overline{\rho}$ being unramified at the primes above ℓ) and all integers $1\leq j\leq n$.

In order to state the conjectured value of the functor $D_\Delta^\vee$ (defined in section 4.1) at the smooth modulo p representation $S(K_f^p,\mathbb{F}_q)[\mathfrak{m}_{\overline{\rho}}]$ composed with the equivalence of categories provided by Thm. 4.9 note that for the group $G=\mathrm{GL}_n(\mathbb{Q}_p)$ the set Δ of simple roots has cardinality $n-1$ with elements $\alpha_i\in\Delta$ $(1\leq i\leq n-1)$ sending a diagonal matrix $\mathrm{diag}(a_1,\dots,a_n)\in\mathrm{GL}_n(\mathbb{Q}_p)$ to $\frac{a_i}{a_{i+1}}$. So we index the copies of $G_{\mathbb{Q}_p}$ in $G_{\mathbb{Q}_p,\Delta}$ accordingly. We denote by $\omega\colon G_{\mathbb{Q}_p}\to\mathbb{F}_p^\times\leq\mathbb{F}_q^\times$ the modulo p cyclotomic character: $g\zeta=\zeta^{\omega(g)}$ for a pth root of unity $\zeta\in\overline{\mathbb{Q}_p}$.

Conjecture 5.1. *In the situation discribed above there exists an integer $d\geq 1$ depending on K_f^p and $\overline{\rho}$ such that the value of $D_\Delta^\vee$ composed by the equivalence of categories in Thm. 4.9 at the smooth representation $S(K_f^p,\mathbb{F}_q)[\mathfrak{m}_{\overline{\rho}}]$ is the $\mathbb{F}_q$-dual of*

$$\left(\left(\bigotimes_{i=1}^{n-1}\left(\omega^{\frac{i^2-i}{2}}\otimes\bigwedge^i\overline{\rho}\right)\right)\otimes\left(\omega^{\frac{n^2-n}{2}}\otimes\bigwedge^n\overline{\rho}\right)\right)^{\oplus d}$$

where the ith term in the above tensor product is viewed as a representation of the ith copy of $G_{\mathbb{Q}_p}$ in $G_{\mathbb{Q}_p,\Delta}$ and the last term—being a one-dimensional representation of $G_{\mathbb{Q}_p}$—is viewed as a representation of $\mathbb{Q}_p^\times\cong Z(\mathrm{GL}_n(\mathbb{Q}_p))$ via local class field theory.

So far there is not much evidence supporting conjecture 5.1 apart from the case $n=2$. However, in case $\overline{\rho}$ is an n-dimensional successive extension of 1-dimensional representations, Breuil and Herzig [10] constructed a smooth representation $\Pi(\overline{\rho})^{ord}$ of $G=\mathrm{GL}_n(\mathbb{Q}_p)$ with the following properties. Under mild assumptions on $\overline{\rho}$ the representation $(\Pi(\overline{\rho})^{ord}\otimes(\omega^{n-1}\circ\det))^{\oplus d}$ injects [11] into $S(K_f^p,\mathbb{F}_q)[\mathfrak{m}_{\overline{\rho}}]$ for K_f^p small enough and some integer $d>0$ (see also Thm. 4.4.8 in [10] for a p-adic version of this). Moreover, the dual of the $G_{\mathbb{Q}_p,\Delta}\times Z(G)$ representation corresponding to $D_\Delta^\vee(\Pi(\overline{\rho})^{ord}\otimes(\omega^{n-1}\circ\det))$ has the right value expected by conjecture 5.1 (a certain well-described subrepresentation of the above representation) at least if we restrict to the action of $G_{\mathbb{Q}_p}$ via the diagonal embedding $G_{\mathbb{Q}_p}\hookrightarrow G_{\mathbb{Q}_p,\Delta}\hookrightarrow G_{\mathbb{Q}_p,\Delta}\times Z(G)$. This latter result is a combination of Cor. 9.8 in [9] and Cor. 4.4 above.

Appendix A Galois side

In this appendix we collect some background information on the Galois side of the (p-adic) Langlands programme.

A.1 The structure of local and global Galois groups

We denote by $\overline{\mathbb{Q}}$ the field of algebraic numbers and by $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the group of its all field automorphisms, ie. the absolute Galois group of the field $\mathbb{Q}$ of rational numbers. This is naturally a profinite group (ie. the projective limit of finite groups): we have the isomorphism

$$G_{\mathbb{Q}} \cong \varprojlim_{K/\mathbb{Q} \text{ finite Galois}} \mathrm{Gal}(K/\mathbb{Q}) \ .$$

Therefore $G_{\mathbb{Q}}$ is equipped with the projective limit topology of these finite discrete groups (called the Krull topology) with respect to which $G_{\mathbb{Q}}$ is compact.

There exists an absolute value $|\cdot|_p$ on the field $\mathbb{Q}$ defined by the formula $|p^k\frac{a}{b}|_p := p^{-k}$ whenever $p \nmid a, b \in \mathbb{Z}$, $k \in \mathbb{Z}$ and $|0|_p := 0$. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is the field $\mathbb{Q}_p$ of p-adic numbers. Now we choose an embedding $\iota_p \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ into the algebraic closure of $\mathbb{Q}_p$. This is only unique upto an automorphism of $\overline{\mathbb{Q}}$, ie. an element of $G_{\mathbb{Q}}$. One can extend the p-adic absolute value to $\overline{\mathbb{Q}_p}$ and the absolute Galois group

$$G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) = \varprojlim_{K/\mathbb{Q}_p \text{ finite Galois}} \mathrm{Gal}(K/\mathbb{Q}_p)$$

equals the group of continuous automorphisms of $\overline{\mathbb{Q}_p}$. Moreover, the image of ι_p is dense in $\overline{\mathbb{Q}_p}$, so the natural group homomorphism $G_{\mathbb{Q}_p} \to G_{\mathbb{Q}}$ defined by the restriction of automorphisms to $\overline{\mathbb{Q}}$ is injective. Hence we may identify $G_{\mathbb{Q}_p}$ with a subgroup of $G_{\mathbb{Q}}$ that is unique upto conjugation. Similarly, fixing an embedding $\iota_\infty \colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ gives rise to an injective homomorphism $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \hookrightarrow G_{\mathbb{Q}}$.

While the structure of $G_{\mathbb{Q}}$ is a mistery (one could say that algebraic number theory is nothing else, but the study of this group), we understand *local* Galois groups, such as $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and $G_{\mathbb{Q}_p}$ rather well. For instance, $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ consists of only two elements (the identity and the complex conjugation), so we can describe ι_∞ by just picking "the complex conjugation" in $G_{\mathbb{Q}}$ which is unique upto conjugation. The case of non-archimedean places is a little more complicated as $\overline{\mathbb{Q}_p}/\mathbb{Q}_p$ is an infinite extension. One can show, however, that each automorphism $g \in G_{\mathbb{Q}_p}$ fixes the p-adic absolute value on $\overline{\mathbb{Q}_p}$ therefore also the closed unit ball $\overline{\mathbb{Z}_p} = \{x \in \overline{\mathbb{Q}_p} \mid |x|_p \leq 1\}$. What makes the p-adic world a little different from the real is that $\overline{\mathbb{Z}_p}$ is a *subring* in $\overline{\mathbb{Q}_p}$: it is closed under addition by the ultrametric inequality. Further, the open unit ball $\overline{m_p} = \{x \in \overline{\mathbb{Q}_p} \mid |x|_p < 1\}$ is an *ideal* in $\overline{\mathbb{Z}_p}$ and the quotient $\overline{\mathbb{Z}_p}/\overline{m_p}$ is isomorphic to the algebraic closure $\overline{\mathbb{F}_p}$ of the field $\mathbb{F}_p$ of p elements. Therefore $g \in G_{\mathbb{Q}_p}$ induces an automorphism of $\overline{\mathbb{F}_p}$, too. This way we obtain a group homomorphism $G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} := \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ that is onto by Hensel's Lemma and whose kernel is called the inertia subgroup I_p. Finally the field $\overline{\mathbb{F}_p}$ has a distinguished automorphism: namely p-Frobenius Frob_p which sends an element $\alpha \in \overline{\mathbb{F}_p}$ to its pth power α^p. We choose a (non-unique) lift of Frob_p to $G_{\mathbb{Q}_p}$ which we still denote by Frob_p by an abuse of notation. The group $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$

is topologically generated by Frob_p and is isomorphic to $\widehat{\mathbb{Z}} := \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$. All in all we have a short exact sequence

$$1 \to I_p \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \cong \widehat{\mathbb{Z}} \to 1\ .$$

The fixed field $\overline{\mathbb{Q}_p}^{I_p}$ of the inertia group is called *the maximal unramified extension* $\mathbb{Q}_p^{ur}$ of $\mathbb{Q}_p$. This is nothing else but the cyclotomic field $\bigcup_{p\nmid n} \mathbb{Q}_p(\mu_n)$ adjoining to $\mathbb{Q}_p$ all the roots of unity of order prime to p.

The structure of the inertia subgroup I_p is also fairly well-understood. There is a closed pro-p (ie. projective limit of finite p-groups) normal (and even characteristic) subgroup $W_p \lhd I_p$, called the wild inertia with quotient $I_p/W_p \cong \prod_{\ell\neq p \text{ prime}} \mathbb{Z}_\ell$ called the tame inertia. The fixed field $\overline{\mathbb{Q}_p}^{W_p}$ of W_p is called *the maximal tamely ramified extension* $\mathbb{Q}_p^{tm}$ of $\mathbb{Q}_p$ and can be obtained by adjoining all the nth roots of p to $\mathbb{Q}_p^{ur}$ for all n coprime to p.

A.2 Étale cohomology

Étale cohomology is a kind of sheaf cohomology theory. Roughly speaking, a sheaf is a tool for systematically tracking locally defined data attached to the open sets of a topological space. More precisely a presheaf Γ (of abelian groups) on a topological space X associates an abelian group $\Gamma(U)$ (whose elements are referred to as "sections of Γ on U") to each open subset $U \subseteq X$ and for each pair $U \subseteq V$ of open sets a *restriction map* $\mathrm{res}_U^V \colon \Gamma(V) \to \Gamma(U)$ such that $\mathrm{res}_U^U = \mathrm{id}_U$ and whenever $U \subseteq V \subseteq W$ then we have $\mathrm{res}_U^W = \mathrm{res}_U^V \circ \mathrm{res}_V^W$. A presheaf is a sheaf if it satisfies the following two conditions:

(i) Localty: For any open space U and open covering $U = \bigcup_{i\in I} U_i$ and elements $s, t \in \Gamma(U)$ we have $s = t$ if $\mathrm{res}_{U_i}^U(s) = \mathrm{res}_{U_i}^U(t)$ for all $i \in I$.

(ii) Glueing: For any open space U and open covering $U = \bigcup_{i\in I} U_i$ and elements $s_i \in \Gamma(U_i)$ such that $\mathrm{res}_{U_i\cap U_j}^{U_i}(s_i) = \mathrm{res}_{U_i\cap U_j}^{U_j}(s_j)$, there exists a (by localty unique) section $s \in \Gamma(U)$ with $s_i = \mathrm{res}_{U_i}^U(s)$ for all $i \in I$.

Now sheaves of abelian groups on a topological space X form an abelian category from wich "taking global sections" (sending the sheaf Γ to $\Gamma(X)$) is a functor to the category of abelian groups. This functor is known to be left exact, ie. having a short exact sequence $0 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 0$ on X the sequence

$$0 \to \Gamma_1(X) \to \Gamma_2(X) \to \Gamma_3(X) \tag{1}$$

is also exact. The higher sheaf cohomology groups are furnished by a general method in homological algebra (they are the right derived functors of the global sections functor) so that one can continue the exact sequence (1) further to the right in order to obtain a long exact sequence.

The idea of Grothendieck formulating étale cohomology theory is that one does not quite need a topological space in order to define sheaves, ie. open subsets need not be subsets. What one really needs are the *coverings* which can be axiomatized allowing open subsets to be non-injective maps into the set X. Such a generalized topology is called a *Grothendieck topology* and a space equipped with a Grothendieck topology is called a *site*. On the *étale site* X_{et} the coverings are the étale surjective maps of schemes. These maps are the geometric analogues of (unramified) coverings in topology. The "open subsets" can be thought of as usual open subsets of finite unbranched covers of the space. Now *étale cohomology* is the sheaf cohomology of sheaves on the étale site.

One may wonder what this has to do with usual cohomology theories in topology. If X is a proper smooth variety over $\mathbb{C}$ then one can indeed compare the (singular) cohomology of the topological space $X(\mathbb{C})$ and the étale cohomology of X_{et}—but only with finite coefficients. More precisely, consider the *constant sheaf* with coefficients in $\mathbb{Z}/n\mathbb{Z}$ on X_{et}, ie. we adjoin the abelian group $\mathbb{Z}/n\mathbb{Z}$ to each étale open subset of X. Then the cohomology of this sheaf is isomorphic to the singular cohomology of $X(\mathbb{C})$ with coefficients in $\mathbb{Z}/n\mathbb{Z}$:

$$H^i(X_{et}, \mathbb{Z}/n\mathbb{Z}) \cong H^i_{sing}(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z})$$

for all $i \geq 0$. The above formula is no longer valid if we replace $\mathbb{Z}/n\mathbb{Z}$ with a field of characteristic 0, so this way there is no right analogue of Betti numbers. However, one can choose $n = \ell^r$ for some prime ℓ and let r tend to infinity in order to obtain $\mathbb{Z}_\ell$- and then invert ℓ to obtain $\mathbb{Q}_\ell$ coefficients:

$$H^i(X_{et}, \mathbb{Q}_\ell) := \left(\varprojlim_r H^i(X_{et}, \mathbb{Z}/\ell^r\mathbb{Z}) \right) [\ell^{-1}] \ .$$

The reason why this is useful in arithmetic situations is the following. If X is defined over $\mathbb{Q}$ (ie. the equations have rational coefficients) then we have an action of $G_\mathbb{Q}$ on the algebraic points $X(\overline{\mathbb{Q}})$ on X. Further, this extends to an action on étale coverings: one can pull back the covering via the automorphism $g \in G_\mathbb{Q}$. So the group $G_\mathbb{Q}$ acts on the whole étale site X_{et} and since étale cohomology is independent of the algebraically closed ground field (ie. we may take $\overline{\mathbb{Q}}$ instead of $\mathbb{C}$), we obtain an action of $G_\mathbb{Q}$ on the abelian groups $H^i(X_{et}, \mathbb{Z}/n\mathbb{Z})$ hence also on the vectorspace $H^i(X_{et}, \mathbb{Q}_\ell)$.

A.3 p-adic Hodge theory

In topology one learns that classical cohomology theories (such as de Rahm, singular, simplicial, etc.) all essentially lead to the same cohomology groups. To us the most important of all is the comparison between de Rahm and singular cohomology theories. In the heart of the proof of this isomorphism lies the Poincaré Lemma, stating, informally, that locally each closed differential form is exact. Therefore having the class of a closed n-form ω in $H^n_{dR}(X/\mathbb{C})$ (for some differentiable manifold X), one can integrate ω on each n-cycle in X: If σ_n is a smooth map from the standard n-simplex Δ_n to X then one can pull ω back to Δ_n along σ_n and integrate it on the whole Δ_n. One extends this linearly to formal linear combination of singular n-simplices. This way we attached a function on n-cycles, ie. an n-cocycle to ω whose class in $H^n_{sing}(X, \mathbb{C})$ is well-defined and is the image of ω under the comparison map. The integral of a form on a cycle is usually called a *period*.

In some sense this has an analogue in the p-adic world, called p-adic Hodge theory. However, while the periods in the complex case lie simply in $\mathbb{C}$, it turns out that the p-adic periods are only contained in a very big field $\mathbf{B}_{dR}$ (the field of p-adic periods after Fontaine). We are not going to define precisely what $\mathbf{B}_{dR}$ is (see [24] or [36] for a definition), but briefly recall some of its properties. At first we denote by $\mathbb{C}_p$ the completion of the algebraic closure $\overline{\mathbb{Q}_p}$

of the field $\mathbb{Q}_p$ of p-adic numbers with respect to the p-adic absolute value. This is an algebraically closed field of characteristic 0 that is complete with respect to the p-adic absolute value. Now $\mathbf{B}_{dR}$ is a complete discretely valued field with prime elemenet usually denoted by $t \in \mathbf{B}_{dR}$ and residue field isomorphic to $\mathbb{C}_p$. The valuation ring inside $\mathbf{B}_{dR}$ is denoted by $\mathbf{B}_{dR}^+$ which is a complete DVR. This, in particular, yields a filtration on $\mathbf{B}_{dR}$ indexed by the integers: $\mathrm{Fil}^i \mathbf{B}_{dR} := t^i \mathbf{B}_{dR}^+$. Further, the absolute Galois group $G_{\mathbb{Q}_p}$ acts on $\mathbf{B}_{dR}$ by field automorphisms respecting this filtration. Note that as an abstract field (forgetting the action of $G_{\mathbb{Q}_p}$) the field $\mathbf{B}_{dR}$ is (non-canonically) isomorphic to the field $\mathbb{C}_p((t))$ of formal Laurent series. However, it might be misleading to interpret $\mathbf{B}_{dR}$ like this as there does not exist such a Galois-equivariant isomorphism: one cannot even embed $\mathbb{C}_p$ into $\mathbf{B}_{dR}$ in a Galois equivariant way (even though one can embed $\overline{\mathbb{Q}_p} \dots$).

If X is a smooth projective variety defined over $\mathbb{Q}_p$ then there is an algebraic de Rahm cohomology group $H^i_{dR}(X/\mathbb{Q}_p)$ for all $i \geq 0$. This comes equipped with a Hodge filtration coming from the Hodge–to–de Rahm spectral sequence

$$E_1^{p,q} = H^p(X, \Omega^q) \Rightarrow H^{p+q}_{dR}(X/\mathbb{Q}_p)$$

where Ω^q denotes the (Zariski-) sheaf of algebraic q-forms on X. On the other hand, one has the étale cohomology groups $H^i(X_{et} \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}, \mathbb{Q}_p)$ defined in section A.2 for all $i \geq 0$ that admit an action of $G_{\mathbb{Q}_p}$. The p-adic de Rahm comparison isomorphism (conjectured by Fontaine and first proven by Faltings [22]) states the existence of an isomorphism

$$H^i_{dR}(X/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{dR} \xrightarrow{\sim} H^i(X_{et} \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{dR} \tag{2}$$

that is compatible with the filtration and the Galois action on both sides. Note that we put the trivial filtration on the étale cohomology $H^i(X_{et} \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}, \mathbb{Q}_p)$ and the trivial Galois action on the de Rahm cohomology $H^i_{dR}(X/\mathbb{Q}_p)$. Without these extra structures the statement would be much easier as both sides are finite dimensional vectorspaces over the field $\mathbf{B}_{dR}$, so the statement would just be the equality of dimensions. One can, in particular, recover the de Rahm cohomology groups from $H^i(X_{et} \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}, \mathbb{Q}_p)$ by taking $G_{\mathbb{Q}_p}$-invariants of the right hand side of (2) and using Tate's theorem that the $G_{\mathbb{Q}_p}$-invariant part of $\mathbf{B}_{dR}$ is just $\mathbb{Q}_p$. Consequently, a finite dimensional continuous representation V of $G_{\mathbb{Q}_p}$ over the coefficient field $\mathbb{Q}_p$ is called *de Rahm* if the space $(V \otimes_{\mathbb{Q}_p} \mathbf{B}_{dR})^{G_{\mathbb{Q}_p}}$ of $G_{\mathbb{Q}_p}$-invariants has dimension $\dim_{\mathbb{Q}_p} V$ over $\mathbb{Q}_p = \mathbf{B}_{dR}^{G_{\mathbb{Q}_p}}$. In particular, if X is a smooth projective variety over $\mathbb{Q}_p$ then its étale cohomology groups $H^i(X_{et} \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}, \mathbb{Q}_p)$ are de Rahm representations of $G_{\mathbb{Q}_p}$ by the above result of Faltings.

On the other hand, one cannot recover the Galois representation $H^i(X_{et} \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}, \mathbb{Q}_p)$ from just the Hodge filtration on $H^i_{dR}(X/\mathbb{Q}_p)$. There are, however, more sophisticated comparison isomorphisms under certain assumptions on the reduction of X modulo p. More precisely, if X has a smooth (resp. semistable) model over $\mathbb{Z}_p$, ie. the reduction modulo p is good (resp. semistable) then one can put extra structures on the de Rahm cohomology (apart from the existing filtration): an action of a Frobenius lift φ, and in case of semistable reduction a monodromy operator, too. From this datum one can rebuild the Galois representation $H^i(X_{et} \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}, \mathbb{Q}_p)$. Informally, a semistable model of X is a scheme $\mathcal{X}$ over $\mathbb{Z}_p$ whose generic fibre is X and whose singularity in the special fibre is not too bad: locally given by equations of the form $x_1 \dots x_r - p = 0$ where $x_1, \dots, x_n$ are the local coordinates and $1 \leq r \leq n$. (Note that the spectrum $\mathrm{Spec}(\mathbb{Z}_p)$ consists of two points: the generic point $(0) \lhd \mathbb{Z}_p$ and the special point $(p) \lhd \mathbb{Z}_p$, this is why a scheme $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z}_p)$ over $\mathbb{Z}_p$ has two fibres with respective names.)

A.4 Fontaine's (φ, Γ)-modules

The main reference for this section is the book [24] by Fontaine and Ouyang in preparation.

In the previous section we saw that one can use p-adic Hodge theory in order to attach invariants (such as filtered vectorspaces, possibly together with an extra action of a lift of Frobenius) to p-adic local Galois representations coming from geometry. However, not all Galois representations come from geometry and in case we wish to study p-adic families of Galois representations, we need to understand those, too, that are not geometric (there are only countably many geometric Galois representations even though the number of *all* continuous representations of $G_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$ is uncountable, so in a p-adically parametrized family most representations will be non-geometric). The systematic way of doing so is provided by Fontaine's theory of (φ, Γ)-modules. To describe what these are and where they come from let us start with a field E of characteristic p with absolute Galois group $G_E := \mathrm{Gal}(E^{sep}/E)$ (soon enough we will choose E to be the field $\mathbb{F}_p((T))$ of formal Laurent series over the field $\mathbb{F}_p$ with p elements). Given a finite dimensional representation V of G_E over the finite field $\mathbb{F}_p$, a variant of Hilbert's Theorem 90 implies that V trivializes if we basechange to the separable closure E^{sep}: the E^{sep}-vectorspace $V \otimes_{\mathbb{F}_p} E^{sep}$ has a basis fixed by the group G_E. (Here the Galois group acts on both terms V and E^{sep} in the tensor product.) On the other hand, one can recover the prime field $\mathbb{F}_p$ inside the rather big field E^{sep} as these are the solutions of the equation $x^p = x$, ie. the fixed points of the p-Frobenius endomorphism $\varphi\colon x \mapsto x^p$. In particular, we can also recover V from $V \otimes_{\mathbb{F}_p} E^{sep}$ as these are the fixed points of φ (if we define the action of φ trivially on V). Further, since the action of G_E is trivialized, there is no harm in taking G_E-invariants of $V \otimes_{\mathbb{F}_p} E^{sep}$ to obtain a $\dim_{\mathbb{F}_p} V$-dimensional vectorspace D over $E = (E^{sep})^{G_E}$ together with a Frobenius-semilinear operator $\varphi\colon D \to D$. Note that the matrix of φ is always invertible as we started with the trivial action of φ on V. Such a datum is called an étale φ-module over E. This way one can pass back and forth from V to D and from D to V. In other words, we obtain an equivalence of categories between representations of G_E over $\mathbb{F}_p$ and étale φ-modules over E given by the functors

$$V \mapsto D := \left(V \otimes_{\mathbb{F}_p} E^{sep}\right)^{G_E} \quad ; \qquad D \mapsto V := (D \otimes_E E^{sep})^{\varphi=\mathrm{id}} \quad .$$

However, we are interested in representations of $G_{\mathbb{Q}_p}$, and not of G_E. The way to pass from absolute Galois groups of p-adic fields (of characteristic 0) to absolute Galois groups of fields of characteristic p is the *fields of norms* functor invented by Fontaine and Winterberger [23]. Their main result is that whenever the field extension $K/\mathbb{Q}_p$ is ramified enough then the absolute Galois group $G_K = \mathrm{Gal}(\overline{\mathbb{Q}_p}/K)$ is isomorphic to G_E where $E = \mathbb{F}_p((T))$. For instance, one could take the field $K := \mathbb{Q}_p(\mu_{p^\infty})$ adjoining the p-power roots of unity to $\mathbb{Q}_p$. Now having a representation V of $G_{\mathbb{Q}_p}$ over $\mathbb{F}_p$ we can trade the action of the subgroup $G_{\mathbb{Q}_p(\mu_{p^\infty})} \leq G_{\mathbb{Q}_p}$ on V for the Frobenius-semilinear action of a single operator φ on a vectorspace D over $\mathbb{F}_p((T))$. Fortunately, the field of norms functor also allows one to embed the Galois group $\Gamma := \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ (ie. the quotient of $G_{\mathbb{Q}_p}$ by the normal subgroup $G_{\mathbb{Q}_p(\mu_{p^\infty})}$) into the group of field automorphisms $\mathrm{Aut}(\mathbb{F}_p((T)))$. More concretely, note that the cyclotomic character χ

gives an isomorphism of Γ with the group $\mathbb{Z}_p^\times$ of units in $\mathbb{Z}_p$ and an element $\gamma \in \Gamma$ sends the variable T to $(T+1)^{\chi(\gamma)} - 1$. Putting these together one obtains an equivalence of categories between representations V of $G_{\mathbb{Q}_p}$ on vectorspaces over $\mathbb{F}_p$ and the so-called (φ, Γ)-*modules* D over $\mathbb{F}_p((T))$, ie. étale φ-modules D over $\mathbb{F}_p((T))$ together with a semilinear action of the group Γ that commutes with the action of φ. Note that unless $p = 2$, the group Γ is topologically generated by a single element ("there exists a primitive root modulo any odd prime power"), therefore the Galois representation V can be described this way by two matrices: that of φ and of a topological generator γ of Γ.

There are versions of the above results of Fontaine with modulo p^N coefficients instead of $\mathbb{F}_p$-coefficients. Taking the projective limit with respect to N one obtains an equivalence of categories for continuous representations of $G_{\mathbb{Q}_p}$ with coefficients in $\mathbb{Z}_p$ and étale (φ, Γ)-modules over $\mathcal{O}_{\mathcal{E}} := \widehat{\mathbb{Z}_p((T))} = \varprojlim \mathbb{Z}/p^N\mathbb{Z}((T))$. Finally, inverting p on both sides (ie. passing to the isogeny category) one obtains an equivalence of categories between continuous representations of $G_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$ and étale (φ, Γ)-modules over $\mathcal{E} := \mathcal{O}_{\mathcal{E}}[p^{-1}]$.

Note that the field $\mathcal{E}$ consists of Laurent series in T (infinite in both directions) with bounded coefficients such that the coefficients of T^n tend p-adically to 0 as $n \to -\infty$. These power series may not converge for any choice of a p-adic T. However, it is fundamental result of Cherbonnier and Colmez [14] that each étale (φ, Γ)-module D over $\mathcal{E}$ admits a basis over $\mathcal{E}$ such that the matrices of the operators φ and $\gamma \in \Gamma$ have entries in the subfield $\mathcal{E}^\dagger \subset \mathcal{E}$ of Laurent series that converge on the annulus $\rho < |T|_p < 1$ for some real number $0 < \rho < 1$ (depending on the actual Laurent series). In other words there exists an étale (φ, Γ)-module $D^\dagger$ over $\mathcal{E}^\dagger$ such that $D \cong \mathcal{E} \otimes_{\mathcal{E}^\dagger} D^\dagger$. Further, one can pass to the Robba ring $\mathcal{R}$ which consists of all Laurent series that converge in some annulus $\rho < |T|_p < 1$ (note that elements in $\mathcal{E}^\dagger$ have, in addition, bounded coefficients). $D \mapsto D^{rig} := \mathcal{R} \otimes_{\mathcal{E}^\dagger} D^\dagger$ is a fully faithful functor with well-described image by a combination of Kedlaya's p-adic monodromy theorem [27] and Berger's [1] p-adic differential equations attached to (φ, Γ)-modules. We say that D is *trianguline* if D^{rig} is a successive extension of rank 1 (φ, Γ)-modules over $\mathcal{R}$.

Appendix B Automorphic side

B.1 From modular forms to automorphic representations

Our main reference for this section is Bump's book [13].

Let N be a positive integer and let $\Gamma_0(N) \leq \mathrm{SL}_2(\mathbb{Z})$ (resp. $\Gamma_1(N)$) be the subgroup of those matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $N \mid c$ (resp. $N \mid c$ and $a \equiv d \equiv 1 \pmod{N}$). This group, as a subgroup of $\mathrm{SL}_2(\mathbb{R})$, acts on the complex upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ by fractional linear maps. Further, we fix a Dirichlet character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ such that $\chi(-1) = (-1)^k$. By a modular form (on $\mathcal{H}$) of weight $0 < k \in \mathbb{Z}$, level N with character χ we mean a holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ with the following properties:

(*i*) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \chi(d) f(z)$ for all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\Gamma_0(N)$;

(*ii*) f is holomorphic at the cusps of $\Gamma_1(N)\backslash\mathcal{H}$.

By the *cusps* of the orbit space $\Gamma_1(N)\backslash\mathcal{H}$ we mean the following. Note that $\Gamma_1(N)$ also acts on the projective line $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ by fractional linear transformations hence we obtain an action on the topological space $\mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ where the neighbourhood of a point in $\mathcal{H}$ is the usual one and a neighbourhood of $a \in \mathbb{P}^1(\mathbb{Q})$ contains a and the intersection of an open subset around a inside $\mathbb{P}^1(\mathbb{C})$ with $\mathcal{H} \subset \mathbb{P}^1(\mathbb{C})$. It can be shown (see 1.2 in [13]) that $\Gamma_1(N)\backslash\mathcal{H}^*$ is compact if we equip it with the quotient topology. The *cusps* of $\Gamma_1(N)\backslash\mathcal{H}$ are the elements of the complement of $\Gamma_1(N)\backslash\mathcal{H}$ in $\Gamma_1(N)\backslash\mathcal{H}^*$, ie. the orbits of $\Gamma_1(N)$ acting on $\mathbb{P}^1(\mathbb{Q})$. These form a finite set as $\Gamma_1(N)$ has finite index in $\mathrm{SL}_2(\mathbb{Z})$ which acts transitively on $\mathbb{P}^1(\mathbb{Q})$.

We denote by $\mathcal{M}_k(\Gamma_0(N),\chi)$ the $\mathbb{C}$-vector space of modular forms on $\mathcal{H}$ of weight k, level N, character χ. We say that $f \in \mathcal{M}_k(\Gamma_0(N),\chi)$ is a *cusp form* if it also vanishes at the cusps—the space of cusp forms is denoted by $\mathcal{S}_k(\Gamma_0(N),\chi)$.

Now we recall how to *adelize* a modular form f and how to associate with it an "automorphic representation". At first we define a function $F\colon \mathrm{GL}_2(\mathbb{R})^+ \to \mathbb{C}$ attached to f. For an invertible real matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with positive determinant we put

$$F(g) := \left(\frac{-ci+d}{|ci+d|}\right)^k \mathrm{Im}\left(\frac{ai+b}{ci+d}\right)^{k/2} f\left(\frac{ai+b}{ci+d}\right) .$$

Since f is a modular form, it follows that $F(\gamma g) = \chi(\gamma)F(g)$ for any $\gamma \in \Gamma_0(N)$ where $\chi(\gamma)$ is defined as $\chi(d)$ where d is the bottom right entry in γ (see 3.2 in [13]). The obtained function F is called an *automorphic form* on $\mathrm{GL}_2(\mathbb{R})^+$ (for the—somewhat technical—definition of which we refere the reader to 3.2 in [13]).

The ring $\mathbb{A}_\mathbb{Q}$ of adèles is defined as $\mathbb{A}_{\mathbb{Q},fin} \times \mathbb{R}$ where the ring $\mathbb{A}_{\mathbb{Q},fin}$ of finite adèles is the restricted direct product

$$\mathbb{A}_{\mathbb{Q},fin} := \widehat{\prod_{p \text{ prime}}}^{\mathbb{Z}_p} \mathbb{Q}_p$$

where the notation $\widehat{\prod\limits_{p \text{ prime}}}^{\mathbb{Z}_p}$ means that all but finitely many coordinates have to lie in the subring $\mathbb{Z}_p \subset \mathbb{Q}_p$. We have a diagonal embedding of $\mathbb{Q}$ into $\mathbb{A}_\mathbb{Q}$. On the other hand, to $\Gamma_0(N)$ we associate the subgroup $K_0(N)$ in $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},fin})$ consisting of matrices whose component $\begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix}$ at a prime p lies in $\mathrm{GL}_2(\mathbb{Z}_p)$ and satisfies that $c_p \in N\mathbb{Z}_p$ for all primes p. Note that this latter condition is empty unless $p \mid N$. We associate to $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ a character λ of the group $K_0(N)$ as follows. If $N = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ is the decomposition of N into prime factors then by the Chinese Remainder Theorem we have $(\mathbb{Z}/N\mathbb{Z})^\times \cong \prod_{i=1}^r (\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^\times$, so the character decomposes as $\chi = \chi_1 \dots \chi_r$ where χ_i is a character of $(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^\times$ for $i = 1, \dots, r$. Now if the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $K_0(N)$ has component $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ in $\mathrm{GL}_r(\mathbb{Z}_{p_i})$ then we define $\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \prod_{i=1}^r \chi_i(a_i \mod p_i^{\alpha_i})$.

The special case of the Strong Approximation Theorem (Thm. 3.3.1 in [13]) in this situation states that we have an equality

$$\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})\,\mathrm{GL}_2(\mathbb{R})^+ K_0(N) .$$

The function $\phi\colon \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$ associated to the automorphic form F is defined as

$$\phi(\gamma g_\infty k_0) := F(g_\infty)\lambda(k_0)$$

where $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $g_\infty \in \mathrm{GL}_2(\mathbb{R})^+$, and $k_0 \in K_0(N)$. It can be shown (see 3.6 in [13]) that this is well-defined, ie. does not depend on the decomposition of an element in $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ as such a product. By the left invariance under $\mathrm{GL}_2(\mathbb{Q})$ we in fact obtained a function on the coset space $\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. The automorphic representations are certain invariant subspaces of the square-intagrable functions $L^2(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}))$. If we assume that we started with an eigenfunction f of certain "Hecke operators" then the resulting function ϕ will be contained in an irreducible component of $L^2(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}))$ which we call the automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ attached to f. Such a representation decomposes as a restricted tensor product $\pi_\infty \otimes \bigotimes_p' \pi_p$ of representations π_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ for all primes p and π_∞ of $\mathrm{GL}_2(\mathbb{R})$. The representation π_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ is infinite dimensional in general. However, it is *smooth* (the stabilizer of each vector is open in $\mathrm{GL}_2(\mathbb{Q}_p)$) and *admissible* (the fixed space of each compact open subgroup in $\mathrm{GL}_2(\mathbb{Q}_p)$ is finite dimensional). There is a *local L-function* (see (3) in the next section for the definition in a special, but generic case) $L(\pi_p, s)$ corresponding to each smooth admissible representation π_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ and the L-function of π is defined as the product $L(\pi, s) := \prod_p L(\pi_p, s)$.

There is an analogous theory for automorphic representations of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ for $n > 2$ and also for other reductive groups over $\mathbb{Q}$ and over finite extensions of $\mathbb{Q}$. These representations also split as a restricted tensor product over all primes which gives rise to L-functions.

B.2 Classical representation theory of p-adic groups

As we have seen in the previous section, the (global) automorphic representations of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ can be written as restricted tensor products of smooth representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ for varying primes p. Now our goal is to define these local objects. A (classical) *smooth representation* of $\mathrm{GL}_n(\mathbb{Q}_p)$ is a vectorspace V over $\mathbb{C}$ together with a linear action of the group $\mathrm{GL}_n(\mathbb{Q}_p)$, ie. a group homomorphism $\mathrm{GL}_n(\mathbb{Q}_p) \to \mathrm{GL}(V)$ such that for each $v \in V$ there exists an open subgroup $H \leq \mathrm{GL}_n(\mathbb{Q}_p)$ (in the p-adic topology on $\mathrm{GL}_n(\mathbb{Q}_p)$) such that $hv = v$ for all $h \in H$. These representations can be rather huge, so we need to impose finitenes conditions on them. The assumption on V having finite dimension over $\mathbb{C}$ would leave us with too few interesting examples—certainly not containing all the representations arising from automorphic forms. The right finiteness condition turns out to be admissibility that is defined as follows. We say that a smooth representation V is *admissible* if for each open subgroup $H \leq \mathrm{GL}_n(\mathbb{Q}_p)$ the subspace $V^H := \{v \in V \mid hv = v \text{ for all } h \in H\}$ is finite dimensional over $\mathbb{C}$. The reprentations arising from automorphic forms are indeed admissible.

Now if π is an irreducible global automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ written as a restricted tensor product $\bigotimes_p' \pi_p$ for representations π_p of $\mathrm{GL}_n(\mathbb{Q}_p)$, then it can be shown that π_p is an unramified principal series representation for all but finitely many primes p. We define the local L-function of π_p in this special case. To define these let $B \leq G := \mathrm{GL}_n(\mathbb{Q}_p)$

denote the subgroup of invertible upper triangular matrices and consider the character

$$\begin{array}{rcl} \chi_z\colon B & \to & \mathbb{C}^\times \\ \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} & \mapsto & |a_1|_p^{z_1}\cdots|a_n|_p^{z_n} \end{array}$$

for an n-tuple $z=(z_1,\dots,z_n)\in\mathbb{C}^n$. Now we define the induced representation $\operatorname{Ind}_B^G\chi_z$ as the space of *uniformly locally constant* functions $f\colon G\to\mathbb{C}$ s.t. $f(bg)=\delta^{\frac{1}{2}}(b)\chi_z(b)f(g)$ where

$$\begin{array}{rcl} \delta\colon B & \to & \mathbb{C}^\times \\ \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} & \mapsto & \prod_{j=1}^n |a_j|_p^{n-2j+1} = |a_1|_p^{n-1}|a_2|_p^{n-3}\cdots|a_{n-1}|_p^{3-n}|a_n|_p^{1-n} \end{array}$$

is the so-called *modular function* of B and appears in the picture for technical reasons (note that this only amounts to the change of the parametrization of the character χ_z). The condition on f being *uniformly locally constant* is that there exists an open compact subgroup $K\leq G$ (depending on f) such that $f(gk)=f(g)$ for all $k\in K$ and $g\in G$. Now $\operatorname{Ind}_B^G\chi_z$ admits an action of the group G by *right translations*: $(gf)(h):=f(hg)$ for $f\in\operatorname{Ind}_B^G\chi_z$ and $g,h\in G$ and the representation of G. By a theorem of Casselman, $\operatorname{Ind}_B^G\chi_z$ is irreducible as a representation of G if we assume that the complex numbers s_j-s_l $(1\leq j<l\leq n)$ are not conguent to $0,\pm1$ modulo $\frac{2\pi i}{\log p}$. Moreover, as a special case of Langlands' classification this representation always has a unique irreducible quotient $\pi_{z,p}$ which we call the *unramified principal series*. These are precisely the irreducible smooth admissible representations of G whose restriction to $\mathrm{GL}_n(\mathbb{Z}_p)$ contains the trivial representation of $\mathrm{GL}_n(\mathbb{Z}_p)$. The local L-function of the unramified principal series representation $\pi_{z,p}$ coming from the character χ_z is

$$L(\pi_{z,p},s):=\det(1-\sigma_{z,p}p^{-s})^{-1} \tag{3}$$

where $\sigma_{z,p}\in\mathrm{GL}_n(\mathbb{C})$ is (the semisimple conjugacy class of) the diagonal matrix

$$\sigma_{z,p}:=\begin{pmatrix} p^{-z_1} & & 0 \\ & \ddots & \\ 0 & & p^{-z_n} \end{pmatrix}.$$

The advantage of considering smooth representations is that they do not depend on the topology of the coefficient field. In fact, smooth representations are exactly those for which the map $\mathrm{GL}_n(\mathbb{Q}_p)\times V\to V$ sending the pair (g,v) to gv is continuous if we equip V with the discrete topology. Now the algebraic closure $\overline{\mathbb{Q}_\ell}$ of the field $\mathbb{Q}_\ell$ is abstractly isomorphic to $\mathbb{C}$, so the smooth ℓ-adic representations that are more apparent in geometry can be thought of as $\mathbb{C}$-representations. However, the ℓ-adic representations have the following advantage noted by Vignéras [38]: On each smooth admissible representation V of $\mathrm{GL}_n(\mathbb{Q}_p)$ over (a finite extension of) $\mathbb{Q}_\ell$ (for primes $\ell\neq p$) there exists an ℓ-adic norm $\|\cdot\|$ on V that is invariant under the action of $\mathrm{GL}_n(\mathbb{Q}_p)$. The completion of V with respect to $\|\cdot\|$ is a unitary (ie. norm-preserving) Banach space representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathbb{Q}_\ell$. Moreover, one can formulate [38] the local Langlands correspondence for $\mathrm{GL}_n(\mathbb{Q}_p)$ using such unitary ℓ-adic Banach space representations instead of smooth representations on the automorphic side.

B.3 Mod p and p-adic representations of p-adic groups

The main reference for this section is the lecture notes [33] of Schneider and Teitelbaum.

The representation theory of groups like $\mathrm{GL}_n(\mathbb{Q}_p)$ over fields of characteristic p is easier than the p-adic theory. As one can hardly think of any useful topology on finite fields other than the discrete topology, it is most natural to consider smooth representations of $G = \mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathbb{F}_p$, too, just like in the case of $\mathbb{C}$ as coefficient field: a representation of G on a vectorspace V over $\mathbb{F}_p$ is *smooth*, if the stabilizer of each vector $v \in V$ is open in G. One can define admissibility the same way as in the classical case, ie. for each open compact subgroup $H \leq G$ we require that V^H be finite dimensional (over $\mathbb{F}_p$). The unexpected extra feature in the characteristic p theory is the following: smooth representations of pro-p groups always have a nonzero fixed vector! This follows essentially from the fact that the intersection of the centre of a finite p-group with any nontrivial normal subgroup is always nontrivial. In particular, the admissibility of a representation V can be tested on a single choice of an open pro-p subgroup, for example on the first congruence subgroup $U^{(1)} := \{A \in \mathrm{GL}_n(\mathbb{Z}_p) \mid A \equiv I \pmod{p}\}$. Moreover, one has Pontryagin duality for the representations V: the dual $V^\vee := \mathrm{Hom}_{\mathbb{F}_p}(V, \mathbb{F}_p)$ is compact and still admits a continuous action of G. Therefore $V^\vee$ can be made into a module over mod p the Iwasawa algebra (ie. certain completed group algebra) $\mathbb{F}_p[[\mathrm{GL}_n(\mathbb{Z}_p)]] := \varprojlim_N \mathbb{F}_p[\mathrm{GL}_n(\mathbb{Z}/p^N\mathbb{Z})]$. Moreover, the dual of V^H is simply the space $V^\vee_H$ of coinvariants. Therefore by a variant of Nakayama's Lemma one finds that the smooth representation V of G is admissible if and only of $V^\vee$ is finitely generated as a module over $\mathbb{F}_p[[\mathrm{GL}_n(\mathbb{Z}_p)]]$. The latter is a a noetherian noncommutative ring, so one can apply the theory of these (eg. homological algebra methods) to derive properties of V (or at least of its restriction to the compact subgroup $\mathrm{GL}_n(\mathbb{Z}_p) \leq G$).

The main goal in the p-adic representation theory of G is to find a class of linear representations that is wide enough to contain all the interesting examples, but restrictive enough to avoid all the pathologies and to have a conceptual theory. There are two such theories: one being the continuous representation theory on Banach spaces over $\mathbb{Q}_p$ and the other being the locally analytic representation theory on locally convex vectorspaces over $\mathbb{Q}_p$. There is a functor in one direction: if V is a Banach space representation then one can take the space V^{an} of *locally analytic vectors* in V, ie. the vectors $v \in V$ such that the map $G \ni g \mapsto gv \in V$ can be written locally as a convergent power series in the entries of $g \in G = \mathrm{GL}_n(\mathbb{Q}_p)$. The advantage of the locally analytic representation theory is that it contains both the smooth (ie. locally constant) representations and the finite dimensional algebraic representations of G as an algebraic group. However, for the purpose of the p-adic Langlands programme, the Banach space representations are more important, so we describe these in more detail.

A Banach space V over $\mathbb{Q}_p$ is a topological vectorspace over $\mathbb{Q}_p$ such that the topology on V is given by a norm $\|\cdot\|\colon V \to \mathbb{R}^{\geq 0}$ such that V is complete with respect to $\|\cdot\|$, ie. any Cauchy sequence is convergent. A *Banach space representation* of G over $\mathbb{Q}_p$ is a Banach space V together with a linear and continuous action of G. We say that the representation V is *unitary* if for all $v \in V$ and $g \in G$ we have $\|gv\| = \|v\|$. If V is a unitary Banach space representation then the unit ball $V^{\leq 1} := \{v \in V \mid \|v\| \leq 1\}$ is a subrepresentation of V over $\mathbb{Z}_p$. This allows one to reduce V modulo p: the reduction $\overline{V}$ is the quotient $V^{\leq 1}/V^{<1}$ (where $V^{<1} := \{v \in V \mid \|v\| < 1\}$) which is a smooth representation of G over $\mathbb{F}_p$. Inspired by the modulo p theory, we define admissibility of unitary Banach space representations of G as follows. Consider the (strong) continuous dual space $V' := \mathrm{Hom}^{cont}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)$. Now the action of

$\mathrm{GL}_n(\mathbb{Z}_p)$ on V can be extended to an action of the completed group algebra $\mathbb{Z}_p[[\mathrm{GL}_n(\mathbb{Z}_p)]] := \varprojlim_N \mathbb{Z}/p^N\mathbb{Z}[\mathrm{GL}_n(\mathbb{Z}/p^N\mathbb{Z})]$ and therefore also to $\mathbb{Q}_p[[\mathrm{GL}_n(\mathbb{Z}_p)]] := \mathbb{Z}_p[[\mathrm{GL}_n(\mathbb{Z}_p)]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. By a theorem of Lazard, this is a noetherian ring, so we define V to be *admissible* if V' is a finitely generated module over $\mathbb{Q}_p[[\mathrm{GL}_n(\mathbb{Z}_p)]]$. In fact, it is not so hard to show that V is admissible if and only if $\overline{V}$ is admissible as a smooth modulo p representation.

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