p-adic Galois representations

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Need analytic continuation and functional equation!

Elliptic curves

Let E be an elliptic curve defined over $\mathbb{Q}$ $\rightsquigarrow$ L-function

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Analytic continuation in this case: Taniyama–Shimura–Weil conjecture (proven by Wiles and Taylor (1993)).

Varieties $\rightsquigarrow$ Galois representations

Let X be a smooth projective variety defined over $\mathbb{Q}$ and put $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For any prime ℓ and integer $i \geq 0$ we have an action of $G_{\mathbb{Q}}$ on the ℓ-adic cohomology group

$$H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) := \left(\varprojlim_r H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell^r\mathbb{Z}) \right) [\ell^{-1}] \, .$$

Reason for finite coefficients:
$H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell^r\mathbb{Z}) \cong H^i_{sing}(X(\mathbb{C}), \mathbb{Z}/\ell^r\mathbb{Z})$. Need to pass to characteristic 0 in order to define L-functions $\rightsquigarrow$ ℓ-adic representations!

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- $X = \{*\}, i = 0 \rightsquigarrow$ trivial Galois representation.
- $X = E$, $i = 1 \rightsquigarrow H^1_{et}(E_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell^r\mathbb{Z}) \cong E[\ell^r](1)$.

Galois representations $\rightsquigarrow$ L-functions

Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ for any prime p (and also $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$).
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$$1 \to I_p \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \to 1$$

where $G_{\mathbb{F}_p} \cong \widehat{\mathbb{Z}}$ is topologically generated by the (arithmetic) Frobenius automorphism $\mathrm{Frob}_p \colon x \mapsto x^p$.

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where $G_{\mathbb{F}_p} \cong \widehat{\mathbb{Z}}$ is topologically generated by the (arithmetic) Frobenius automorphism $\mathrm{Frob}_p \colon x \mapsto x^p$. Now if

$$\rho \colon G_{\mathbb{Q}} \to \mathrm{GL}(V)$$

is a global Galois-representation on a finite dimensional vectorspace V over a field K of characteristic 0 (embedded into $\mathbb{C}$) then we defined the local polynomial at p as the characteristic polynomial

$$P_{\rho,p}(T) := \det(id - T\,\mathrm{Frob}_p \mid V^{I_p}) \in K[T] \, .$$

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The L-function attached to the Galois representation ρ is defined as

$$L(\rho, s) := \prod_{p \text{ prime}} \frac{1}{P_{\rho,p}(p^{-s})} \qquad (\mathrm{Re}(s) \gg 0) \ .$$

In case of $X = \{*\}$, $i = 0$ this specializes to Riemann ζ and in case $X = E$, $i = 1$ to the L-function of the elliptic curve as above.

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Fundamental open questions in the theory:

- Analytic continuation and functional equation $\rightsquigarrow$ modularity

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- Analytic continuation and functional equation $\rightsquigarrow$ modularity
- Which Galois representations arise from geometry, ie. as a subquotient of the étale cohomology of a smooth projective variety?

The above 2 questions are closely related.

Geometric Galois representations

Fontaine–Mazur conjecture (1995)

An irred. ℓ-adic Galois representation $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ comes from geometry if and only if the following two conditions hold:

(*i*) ρ is unramified (ie. $\rho(I_p) = \{1\}$) at all but finitely many primes p.

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The "only if" part of the above conjecture is known: (i) by Grothendieck (note that in the case of elliptic curves those primes ramify at which the curve has bad reduction: criterion of Néron–Ogg–Shafarevich—in particular, there are finitely many). Assertion (ii) ("p-adic de Rham comparison isomorphism") was first proven by Faltings and by Tsuji and reproven recently by Beilinson (survey: Szamuely–Z) and by Scholze.

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Classical comparison isomorphism

Let X be a smooth projective variety over $\mathbb{C}$. Classical Poincaré lemma $\rightsquigarrow$

$$H^n_{sing}(X(\mathbb{C}), \mathbb{C}) = H^n_{dR}(X^{an}, \mathbb{C})$$

where the right hand side is computed by the Hodge–to–de Rham spectral sequence

$$E_1^{p,q} := H^q(X^{an}, \Omega^p_{X^{an}}) \Rightarrow H^{p+q}_{dR}(X^{an}, \mathbb{C})$$

where $\Omega^p_{X^{an}}$ stands for the sheaf of holomorphic p-forms on the analytic manifold X^{an}.

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- In case of algebraic de Rham cohomology coefficients lie in K!

p-adic comparison isomorphism

So we take $K = \mathbb{Q}_p$. Associated to the algebraic de Rham complex

$$\Omega_X^\bullet : \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \to \cdots$$

of sheaves (in the Zariski topology) of Kähler-differentials there is a Hodge–to–de Rham spectral sequence

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For a p-adic Poincaré lemma to hold, one has to pass to a big field $\mathbf{B}_{dR}$ (which is a discretely valued field with residue field $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ admitting an action of $G_{\mathbb{Q}_p}$) so one has an isomorphism (Faltings)

$$H_{dR}^i(X/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{dR} \xrightarrow{\sim} H_{et}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{dR}$$

compatible with the filtration and the Galois action on both sides.

de Rham representations

Taking $G_{\mathbb{Q}_p}$-invariants of the isomorphism above one obtains

$$H^i_{dR}(X/\mathbb{Q}_p) \cong \left(H^i_{et}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{dR}\right)^{G_{\mathbb{Q}_p}}$$

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using the fact $\mathbf{B}_{dR}^{G_{\mathbb{Q}_p}} = \mathbb{Q}_p$. By GAGA the two sides have the same dimension therefore we define a local p-adic Galois-representation V to be de Rham if we have $\dim_{\mathbb{Q}_p} D_{dR}(V) = \dim_{\mathbb{Q}_p} V$ where

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Problem: We cannot recover V from $D_{dR}(V)$!
(even if V is de Rham)

Galois representation in characteristic p

Let E be a perfect field of characteristic p and V be a finite dimensional representation of $G_E := \mathrm{Gal}(\overline{E}/E)$ over $\mathbb{F}_p$. By Hilbert's Theorem 90 we can trivialize V over $\overline{E}$, ie.

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as G_E-modules. In particular, $D(V) := \left(\overline{E} \otimes_{\mathbb{F}_p} V\right)^{G_E}$ has dimension $\dim_{\mathbb{F}_p} V$ over E.

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Put $\varphi := \mathrm{Frob}_p \otimes id_V \colon \overline{E} \otimes_{\mathbb{F}_p} V \to \overline{E} \otimes_{\mathbb{F}_p} V$ so we have $V = \left(\overline{E} \otimes_E D(V)\right)^{\varphi = id}$.

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Definition

Let K be a field that is complete with respect to a nonarchimedean *nondiscrete* valuation $|\cdot| \colon K \to \mathbb{R}^{\geq 0}$. We say that K is *perfectoid* if the p-Frobenius map $\mathrm{Frob}_p \colon \mathcal{O}_K/(p) \to \mathcal{O}_K/(p)$ is surjective.

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Examples: $\mathbb{C}_p$, $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$, $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$, $\widehat{\mathbb{F}_p((T^{1/p^\infty}))}$ but not $\mathbb{Q}_p$ (valuation is discrete!).

Tilting equivalence

Let K be a perfectoid field. The perfectoid field $K^\flat := \mathrm{Frac}(\mathcal{O}_{K^\flat})$ of characteristic p is called the *tilt* of K where

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Theorem (Tilting equivalence of Scholze)

Let K be a perfectoid field. Then the functor $\flat\colon L \mapsto L^\flat$ gives an equivalence of categories between perfectoid extensions of K and perfectoid extensions of $K^\flat$. Moreover, if L/K is finite separable then L is perfectoid (baby case of almost purity).

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Corollary

We have $G_K \cong G_{K^\flat}$ and if K is the completion of a Galois extension of $\mathbb{Q}_p$ then we have $\mathrm{Gal}(K/\mathbb{Q}_p) \hookrightarrow \mathrm{Aut}(K^\flat)$.

p-adic local Galois reps and perfect (φ, Γ)-modules

Let K be a perfectoid field (of char 0)

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New feature (Scholze): There is a geometric object $\mathrm{Spd}(\mathbb{Q}_p)$ in characteristic p with étale fundamental group $G_{\mathbb{Q}_p}$: formal orbit space of Γ-action on $\mathrm{Spa}(\widehat{\mathbb{Q}_p(\mu_{p^\infty})}^\flat)$ in the category of *diamonds*.

Imperfect (φ, Γ)-modules

We have $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}^\flat = \mathbb{F}_p(\widehat{(T^{1/p^\infty}})\!)$—one could, for most purposes, work with (φ, Γ)-modules over these. But e.g. for the p-adic Langlands programme one needs *imperfect* ground fields.

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Observation: we have $G_{\mathbb{F}_p(\!(T)\!)} \cong G_{\mathbb{F}_p(\!(\widehat{T^{1/p^\infty}})\!)} \rightsquigarrow$

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and

$$\{p\text{-adic reps of } G_{\mathbb{Q}_p}\} \leftrightarrow \{\text{étale } (\varphi, \Gamma)\text{-modules } /\mathcal{E}\}$$

where we put $\mathcal{E} := \mathcal{O}_\mathcal{E}[p^{-1}]$ and $\mathcal{O}_\mathcal{E} := \varprojlim_n \mathbb{Z}/(p^n)(\!(T)\!)$. Étale means: $id \otimes \varphi \colon \mathcal{E} \otimes_{\mathcal{E},\varphi} D \to D$ is bijective (note: $\mathrm{Frob}_p \colon \mathbb{F}_p(\!(T)\!) \to \mathbb{F}_p(\!(T)\!)$ is no longer bijective!) This is Fontaine's equivalence of categories (1990).

p-adic Hodge theory via (φ, Γ)-modules

Let V be a p-adic representation of $G_{\mathbb{Q}_p}$ and put $D(V)$ for the corresponding (φ, Γ)-module over $\mathcal{E}$. In order to recover $D_{dR}(V) = \left(\mathbf{B}_{dR} \otimes_{\mathbb{Q}_p} V\right)^{G_{\mathbb{Q}_p}}$ from $D(V)$ one first has to pass to coefficient rings converging p-adically at least somewhere.

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$$\mathcal{R}^{(r,1)} := \{f(T) = \sum_{i=-\infty}^{\infty} a_i T^i \mid a_i \in \mathbb{Q}_p,\ f \text{ converges if } r < |T|_p < 1\}$$

$$\mathcal{R} := \bigcup_{0<r<1} \mathcal{R}^{(r,1)} \qquad \mathcal{E}^\dagger := \{f \in \mathcal{R} \mid \limsup_{|T|_p \to 1} |f(T)|_p < \infty\}$$

Note: $\mathcal{E}^\dagger$ embeds into $\mathcal{E}$ (but $\mathcal{R}$ does note).

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Theorem (Cherbonnier–Colmez: overconergence)

$D(V)$ descends to an étale (φ, Γ)-module $D^\dagger(V)$ over $\mathcal{E}^\dagger$.

p-adic Hodge theory via (φ, Γ)-modules

Theorem (Berger)

Put $D^{rig}(V) := \mathcal{R} \otimes_{\mathcal{E}^\dagger} D^\dagger(V)$ and $t := \log(1+T) = \sum_{k=1}^\infty (-1)^{k+1} \frac{T^k}{k} \in \mathcal{R}$. Then there exists a p-adic differential equation ($\mathbb{Q}_p(\mu_{p^\infty})[[t]]$-module with Γ-action) $D^{dif}(V)$ associated to $D^{rig}(V)$ such that we have

$$D_{dR}(V) = D^{dif}(V)^\Gamma \ .$$

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Most applications use: for $? = rig, \dagger$, or empty Herr's complex below computes Galois cohomology:

$$0 \to D^?(V) \stackrel{(\varphi - id, \gamma - id)}{\to} D^?(V) \oplus D^?(V) \stackrel{(id - \gamma, \varphi - id)}{\to} D^?(V) \to 0 \ .$$

Motivation: generalize Colmez' functors

Main observation of Colmez when constructing a p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$:

$$1 + T \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \varphi \leftrightarrow \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \qquad \Gamma \leftrightarrow \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$$

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If there is a generalization to groups of higher rank (e.g. $\mathrm{GL}_n(\mathbb{Q}_p)$ with $n > 2$) it is natural to expect that "multivariable" objects come into picture. Hint (Breuil–Herzig–Schraen): A generalized Colmez functor applied to the automorphic $\mathrm{GL}_n(\mathbb{Q}_p)$-representation attached to a mod p (global) Galois representation ρ (the corresponding Hecke-isotypical component in the cohomology of a Shimura-variety) should *not* give ρ back but $\bigotimes_{i=1}^n \bigwedge^i \rho$.

Theorem (Z, Carter–Kedlaya–Z)

Let Δ be a finite set and put $G_{\mathbb{Q}_p,\Delta} := \prod_{\alpha\in\Delta} G_{\mathbb{Q}_p}$. There is an equivalence of categories

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where $\varphi_\Delta = (\varphi_\alpha \mid \alpha \in \Delta)$ (one Frobenius lift for each variable), $\Gamma_\Delta := \prod_{\alpha\in\Delta} \Gamma$, $\mathcal{E}_\Delta := \mathcal{O}_{\mathcal{E}_\Delta}[p^{-1}]$ and $\mathcal{O}_{\mathcal{E}_\Delta} := \varprojlim_n \mathbb{Z}/(p^n)[[T_\alpha \mid \alpha \in \Delta]][\prod_{\alpha\in\Delta} T_\alpha^{-1}]$.

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Theorem (Z)

There is a right exact functor compatible with parabolic induction and tensor products from the category of smooth mod p^n representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ to the category of mod p^n representations of $G_{\mathbb{Q}_p}^{n-1} \times \mathbb{Q}_p^\times$. In case of $n = 2$ this agrees with Colmez' functor realizing p-adic Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$.

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Let $X_1, \ldots, X_n$ be connected schemes of finite type $/\mathbb{F}_p$ and put $X := X_1 \times \cdots \times X_n$. Let $\varphi_i = 1 \times \cdots \times \varphi_{X_i} \times \cdots \times 1\colon X \to X$ be the ith partial Frobenius and denote by $\mathbf{FEt}(X/\Phi)$ the category of finite étale maps $Y \to X$ equipped with commuting isomorphisms $\beta_i\colon Y \to \varphi_i^* Y$ such that the "composite" $\beta_n \circ \cdots \circ \beta_1$ is the relative Frobenius $\varphi_{Y/X}$.

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Drinfeld's lemma for schemes

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but the rational subspace defined by

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Holds also for possibly distinct finite extensions $K_1, \dots, K_n$ of $\mathbb{Q}_p$.

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Future directions:

- Pass to the Robba ring and construct Bloch–Kato exponential maps and Perrin-Riou's big exponential maps in this product situation $\overset{?}{\rightsquigarrow}$ prove classical ε-isomorphisms (etc.?) for p-adic representations of the form $V_1 \otimes_{\mathbb{Q}_p} V_2$ if it is known for both V_1 and V_2.

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- Relate these notions to Berger's Lubin–Tate multivariable (φ, Γ)-modules $\overset{?}{\rightsquigarrow}$ better structural properties of the latter

Thanks for your attention!