(φ, Γ)-modules over noncommutative overconvergent and Robba rings

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5th April 2013

Abstract

We construct noncommutative multidimensional versions of overconvergent power series rings and Robba rings. We show that the category of étale (φ, Γ)-modules over certain completions of these rings are equivalent to the category of étale (φ, Γ)-modules over the corresponding classical overconvergent, resp. Robba rings (hence also to the category of p-adic Galois representations of $\mathbb{Q}_p$). Moreover, in the case of Robba rings, the assumption of étaleness is not necessary, so there exists a notion of trianguline objects in this sense.

1 Introduction

In recent years it has become increasingly clear that some kind of p-adic version of the local Langlands correspondence should exist. In fact, Colmez [8, 9] constructed such a correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. His construction is done in several steps using (φ, Γ)-modules (the category of which is well-known [11] to be equivalent to the category of p-adic Galois representations of $\mathbb{Q}_p$). We briefly recall Colmez's correspondence here. Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers o_K and uniformizer p_K.

The so-called "Montreal-functor" associates to a smooth o_K-torsion representation of the standard Borel subgroup $B_2(\mathbb{Q}_p)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ an o_K-torsion (φ, Γ)-module over Fontaine's ring $\mathcal{O}_{\mathcal{E}}$. If we are given a unitary Banach space representation Π over the field K of the group $\mathrm{GL}_2(\mathbb{Q}_p)$ then it admits an o_K-lattice $L(\Pi)$ which is invariant under $\mathrm{GL}_2(\mathbb{Q}_p)$. Hence $L(\Pi)/p_K^r$ is a smooth o_K-torsion representation that we restrict now to $B_2(\mathbb{Q}_p)$. The (φ, Γ)-module associated to Π is the projective limit (as $r \to \infty$) of the (φ, Γ)-modules associated to $L(\Pi)/p_K^r$ via the Montreal functor. This is generalized in [18] to general reductive groups over $\mathbb{Q}_p$.

The reverse direction, how one adjoins a unitary continuous p-adic representation to a 2-dimensional (φ, Γ)-module D over Fontaine's ring, is even more subtle. One first constructs a unitary p-adic Banach space representation $\Pi(D)$ to each 2-dimensional *trianguline* (φ, Γ)-module D over $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[p^{-1}]$ using some kind of parabolic induction. This Banach space is well described as a quotient of the space of p-adic functions satisfying certain properties by a certain $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant subspace (see [7] and [4] for details), however, a priori it is not clear whether or not it is nontrivial. On the other hand, there is a general construction of a (somewhat bigger) $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $D \boxtimes_\delta \mathbb{P}^1$ which is in fact the space of global sections of a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant sheaf $U \mapsto D \boxtimes_\delta U$ ($U \subseteq \mathbb{P}^1$ open) on the projective space $\mathbb{P}^1(\mathbb{Q}_p) \cong \mathrm{GL}_2(\mathbb{Q}_p)/B_2(\mathbb{Q}_p)$ for any (not necessarily 2-dimensional) (φ, Γ)-module D and any unitary character $\delta\colon \mathbb{Q}_p^\times \to o_K^\times$. This sheaf has the following properties: (i) the centre of $\mathrm{GL}_2(\mathbb{Q}_p)$ acts via δ on $D\boxtimes_\delta \mathbb{P}^1$; ($ii$) we have $D\boxtimes_\delta \mathbb{Z}_p\cong D$ as a module over the monoid $\begin{pmatrix}\mathbb{Z}_p\setminus\{0\} & \mathbb{Z}_p\\ 0 & 1\end{pmatrix}$ (where we regard $\mathbb{Z}_p$ as an open subspace in $\mathbb{P}^1=\mathbb{Q}_p\cup\{\infty\}$). (See [19] for a generalization of this construction to general reductive groups.) Then Colmez shows that in case D is 2-dimensional and trianguline, then there exists a unitary character δ (namely $\delta=\chi^{-1}\det D$ where χ is the cyclotomic character and $\det D$ is the character associated to the 1-dimensional (φ,Γ)-module $\bigwedge^2 D$ via Fontaine's equivalence composed with class field theory) such that a certain subspace $D^\natural\boxtimes_\delta \mathbb{P}^1$ (for the definition see [9]) of $D\boxtimes_\delta\mathbb{P}^1$ is isomorphic to the dual of the Banach space representation $\Pi(\check{D})$ associated earlier to the dual (φ,Γ)-module $\check{D}$—therefore showing in particular that the previous construction is nonzero. This subspace makes sense also in case D is not trianguline (nor of rank 2), but a priori only known to be $B_2(\mathbb{Q}_p)$-invariant. Moreover, whenever D is indecomposable and 2-dimensional, then the above δ is unique [15], and whenenever D is absolutely irreducible and ≥ 3-dimensional, then there does not exist [15] such a character δ (so that the subspace $D^\natural\boxtimes_\delta\mathbb{P}^1$ is $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant). Since the construction of $D\mapsto D^\natural\boxtimes_\delta\mathbb{P}^1$ behaves well in families (see chapter II in [8]) and the trianguline Galois-representations are Zariski-dense in the deformation space of 2-dimensional (φ,Γ)-modules with given reduction mod p [14], Colmez [8] shows that this subspace is not only $B_2(\mathbb{Q}_p)$, but also $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant for general 2-dimensional (φ,Γ)-modules. Moreover, for $\delta=\chi^{-1}\det D$ (in this case we omit the subscript δ from the notation) we have a short exact sequence

$$0\to\Pi(\check{D})\to D\boxtimes\mathbb{P}^1\to\Pi(D)\to 0$$

where $\Pi(D)$ is the unitary Banach-space representation associated to D via the p-adic Langlands correspondence.

Colmez ([8], chapter V and VI) also identifies the space $\Pi(D)^{an}$ of locally analytic and the space $\Pi(D)^{alg}$ of locally algebraic vectors in the Banach-space representation $\Pi(D)$. These play a crucial role in the proof of the compatibility of the p-adic and the classical local Langlands correspondence. In fact, we have $\Pi(D)^{an}=(D^\dagger\boxtimes\mathbb{P}^1)/K\cdot(D^\natural\boxtimes\mathbb{P}^1)$ where $D^\dagger\boxtimes\mathbb{P}^1$ is the subspace of elements $x\in D\boxtimes\mathbb{P}^1$ such that both $\mathrm{Res}^{\mathbb{P}^1}_{\mathbb{Z}_p}(x)$ and $\mathrm{Res}^{\mathbb{P}^1}_{\mathbb{Z}_p}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}x\right)$ lie in the subspace of overconvergent elements $D^\dagger\subset D\cong D\boxtimes\mathbb{Z}_p$. $D^\dagger$ is an étale (φ,Γ)-module over the ring $\mathcal{E}^\dagger$ of overconvergent power series with coefficients in K such that $D\cong\mathcal{E}\otimes_{\mathcal{E}^\dagger}D^\dagger$ [6].

Let now G be the group of $\mathbb{Q}_p$-points of a connected $\mathbb{Q}_p$-split reductive group and $P=TN$ a Borel subgroup of G. Further denote by Φ^+ the set of positive roots with respect to P and $\Delta\subset\Phi^+$ the set of simple roots. The above noted generalizations of Colmez's work ([18] and [19]) both use a certain microlocalisation $\Lambda_\ell(N_0)$ (constructed originally in [17]) of the Iwasawa algebra $\Lambda(N_0)$ of a compact open subgroup N_0 of N. This can be thought of as the noncommutative analogue of Fontaine's ring $\mathcal{O}_\mathcal{E}$. On the other hand, Colmez's p-adic Langlands correspondence heavily relies on the theory of trianguline (φ,Γ)-modules. A (φ,Γ)-module over the Robba ring is a free module $D^\dagger_{rig}$ over $\mathcal{R}$ together with commuting semilinear actions of the operator φ and the group Γ such that φ takes a basis of the free module to another basis. Such a (φ,Γ)-module $D^\dagger_{rig}$ is said to be étale (or of slope 0) if there is a basis of $D^\dagger_{rig}$ such that the matrix of φ in this basis is an invertible matrix over the subring $\mathcal{O}^\dagger_\mathcal{E}\subset\mathcal{R}$ of overconvergent Laurent series. An étale (φ,Γ)-module over $\mathcal{R}$ is *trianguline* if it admits a filtration of (not necessarily étale) (φ,Γ)-modules over $\mathcal{R}$ with subquotients of rank 1 possibly after a finite base change $E\otimes_K\cdot$. The fact that the Robba ring and the ring of overconvergent

Laurent series play such a role in the construction of the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and also in the identification of the locally analytic vectors is the motivation for the construction of noncommutative analogues of these rings—as they will most probably be needed for a future correspondence for reductive groups other than $\mathrm{GL}_2(\mathbb{Q}_p)$.

The motivation of this paper is twofold. On one hand, we reinterpret the ring $\Lambda_\ell(N_0)$ as follows. Instead of localising and completing the Iwasawa algebra $\Lambda(N_0)$, one may construct $\Lambda_\ell(N_0)$ as the projective limit of certain skew group rings over $\mathcal{O}_\mathcal{E}$. The only assumptions on the ring $R = \mathcal{O}_\mathcal{E}$ such that this new construction of $\Lambda_\ell(N_0)$ can be carried out are that R admits an inclusion $\chi\colon \mathbb{Z}_p \to R^\times$ of the additive group $\mathbb{Z}_p$ into its group of invertible elements and an étale action of an operator φ that is compatible with χ. The noncommutative ring that is constructed is a completed skew group ring $R[[H_1, \ell]]$ of a closed normal subgroup H_1 of a pro-p group H_0 such that $\ell\colon H_0 \twoheadrightarrow \mathbb{Z}_p$ is a homomorphism with kernel H_1 (hence $H_0/H_1 \cong \mathbb{Z}_p$). The main result in this direction is Prop. 3.1 showing that the category of φ-modules over R is equivalent to the category of φ-modules over the completed skew group ring $R[[H_1, \ell]]$. This can be applied also to the ring $R = \mathcal{O}_\mathcal{E}^\dagger$ of overconvergent Laurent series with coefficients in o_K and the Robba ring $\mathcal{R}$. The other motivation (probably also the more important one) is the construction of the right noncommutative analogues of $\mathcal{O}_\mathcal{E}^\dagger$ and $\mathcal{R}$. The elements of the rings $\mathcal{R}[[H_1, \ell]]$ and $\mathcal{O}_\mathcal{E}^\dagger[[H_1, \ell]]$, however, are not necessarily convergent in any open annulus since they are obtained by taking an inverse limit. Therefore in section 4.2 we construct the rings $\mathcal{R}(H_1, \ell)$ and $\mathcal{R}^{int}(H_1, \ell)$ as direct limits of certain microlocalisations of the distribution algebra. The elements of these are convergent in a region of the form

$$\left\{\rho_2 < |b_\alpha| < 1,\ |b_\beta| < |b_\alpha|^r \text{ for } \beta \in \Phi^+ \setminus \{\alpha\}\right\}$$

for some $p^{-1} < \rho_2 < 1$ and $1 \leq r \in \mathbb{Z}$. In section 4.4 we show that $\mathcal{R}[[N_1, \ell]]$ (resp. $\mathcal{O}_\mathcal{E}^\dagger[[N_1, \ell]]$) is a certain completion of $\mathcal{R}(N_1, \ell)$ (resp. $\mathcal{R}^{int}(N_1, \ell)$). Note that although the natural map $j_{int}\colon \mathcal{R}^{int}(N_1, \ell) \to \mathcal{O}_\mathcal{E}^\dagger[[N_1, \ell]]$ is injective, the map $j\colon \mathcal{R}(N_1, \ell) \to \mathcal{R}[[N_1, \ell]]$ is not. Both rings $\mathcal{R}(N_1, \ell)$ and its integral version admit an étale action of the monoid $T_+ = \{t \in T \mid tN_0t^{-1} \subseteq N_0\}$. However, it is an open question whether the categories of étale T_+-modules over these rings are equivalent to the étale T_+-modules over their completions.

In my opinion, the right noncommutative analogues of the ring $\mathcal{R}$ (resp. $\mathcal{O}_\mathcal{E}^\dagger$) is $\mathcal{R}(N_1, \ell)$ (resp. $\mathcal{R}^{int}(N_1, \ell)$) in the context of $\mathbb{Q}_p$-split reductive groups G over $\mathbb{Q}_p$ as both rings admit an étale action of the monoid T_+ and their elements converge in certain polyannuli. However, it might still be useful to also consider the rings $\mathcal{R}[[H_1, \ell]]$ and $\mathcal{O}_\mathcal{E}^\dagger[[H_1, \ell]]$, as they can help us compare the category of usual (φ, Γ)-modules with the category of T_+-modules over $\mathcal{R}(N_1, \ell)$ (resp. over $\mathcal{R}^{int}(N_1, \ell)$) using the equivalence of categories in Proposition 3.1. Note that only one variable is inverted in these rings in contrast to the rings constructed in [20]. The reasons for this are the following: (i) this way $\mathcal{R}^{int}(N_1, \ell)$ is a subring of $\Lambda_\ell(N_0)$; (ii) the equivalence of categories in Proposition 3.1 holds for rings in which only one variable is inverted; (iii) all the usual (φ, Γ)-modules are overconvergent, ie. descend to $\mathcal{O}_\mathcal{E}^\dagger$ already in one variable. However, if $\mathbb{Q}_p$ is replaced by a finite unramified extension F then one might have to consider Lubin-Tate (φ, Γ_F)-modules (with $\Gamma_F \cong o_F^\times$) instead so that the monoid $\varphi^\mathbb{N}\Gamma_F$ is isomorphic to $o_F \setminus \{0\}$. These (φ, Γ_F)-modules are not overconvergent in general but they might still correspond to objects over certain multivariable Robba rings (in which all the variables are inverted). For a first result in this direction see [3]. It is plausible to expect that for general reductive groups G over F one has to invert exactly $|F : \mathbb{Q}_p|$ ($\mathbb{Q}_p$-)variables that correspond to the root subgroup $N_\alpha \cong F \cong \mathbb{Q}_p^{|F:\mathbb{Q}_p|}$ for a given simple root α.

1.1 Acknowledgements

This research was supported by the Hungarian OTKA grant K-100291. The proof of Prop. 3.1 is a direct generalization of Thm. 8.20 in [19] and grew out of related discussions with Marie-France Vigneras and Peter Schneider. I express my gratitude to both of them for allowing me to include this proof here and for other valueable discussions. My debt is especially due to Peter Schneider from whom I learnt the most that I know about p-adic representation theory. I would like to thank Torsten Schoeneberg for reading the manuscript and for his comments. I also benefited from enlightening conversations with Kiran Kedlaya. Finally, I thank the referee for a careful reading of the manuscript.

2 Completed skew group rings

Let R be a commutative ring (with identity) with the following properties:

(*i*) There exists a group homomorphism $\chi\colon \mathbb{Z}_p \hookrightarrow R^\times$.

(*ii*) The ring R admits an étale action of the p-Frobenius φ that is compatible with χ. More precisely there is an injective ring homomorphism $\varphi\colon R \hookrightarrow R$ such that $\varphi(\chi(x)) = \chi(px)$ and
$$R = \bigoplus_{i=0}^{p-1} \chi(i)\varphi(R).$$
In particular, R is free of rank p over $\varphi(R)$.

We remark first of all that one may iterate (*ii*) c times for any positive integer c to obtain

$$R = \bigoplus_{i=0}^{p^c-1} \chi(i)\varphi^c(R)\ . \tag{1}$$

Indeed, by induction we may assume that (1) holds for $c-1$ and obtain

$$R = \bigoplus_{k=0}^{p^{c-1}-1} \chi(k)\varphi^{c-1}(R) = \bigoplus_{k=0}^{p^{c-1}-1} \chi(k)\varphi^{c-1}\left(\bigoplus_{j=0}^{p-1}\chi(j)\varphi(R)\right) = \bigoplus_{k=0}^{p^{c-1}-1}\bigoplus_{j=0}^{p-1}\chi(k+p^{c-1}j)\varphi^c(R)$$

since φ^{c-1} takes direct sums to direct sums as it is injective. Now the claim follows from noting that any integer $0 \leq i \leq p^c-1$ can be uniquely written in the form $i = k + p^{c-1}j$ with $0 \leq k \leq p^{c-1}-1$ and $0 \leq j \leq p-1$.

Moreover, for any $x \in \mathbb{Z}_p$ we have $\chi(p^c x) = \varphi^c(\chi(x)) \in \varphi^c(R)^\times$. Hence $\chi(i)\varphi^c(R) = \chi(i+p^c x)\varphi^c(R)$ and we may replace each value of i in the formula (1) by any element in the coset $i + p^c\mathbb{Z}_p$.

Definition 2.1. *We call a ring R with the above properties* (*i*) *and* (*ii*) *a φ-ring over $\mathbb{Z}_p$ or often just a φ-ring.*

For example, if $K/\mathbb{Q}_p$ is a finite extension with ring of integers o and uniformizer p_K then the Iwasawa algebra $o[[T]]$ is a φ-ring with the homomorphism

$$\begin{aligned}\chi\colon \mathbb{Z}_p &\to o[[T]]\\ 1 &\mapsto 1+T\end{aligned}$$

and Frobenius $\varphi(T) = (T+1)^p - 1$. Similarly with the same χ and φ, Fontaine's ring $\mathcal{O}_\mathcal{E}$, its field of fractions $\mathcal{E}$, the Robba ring $\mathcal{R}$ and the rings $\mathcal{E}^\dagger$, $\mathcal{O}_\mathcal{E}^\dagger$ of overconvergent power series are also φ-rings (for the definitions see the paragraph before Lemma 2.13 (for $\mathcal{O}_\mathcal{E}$ and $\mathcal{E}$), section 4.2 (for $\mathcal{R}$, $\mathcal{O}_\mathcal{E}^\dagger$, and $\mathcal{E}^\dagger$)).

Lemma 2.2. *For any positive integer c we have a ring isomorphism*

$$\begin{array}{rcl} \varphi^c(R)[X]/(X^{p^c} - \chi(p^c)) & \xrightarrow{\sim} & R \\ X & \longmapsto & \chi(1) \ . \end{array}$$

Proof. Since the polynomial ring $\varphi^c(R)[X]$ is a free object in the category of commutative $\varphi^c(R)$-algebras, we may extend the natural inclusion homomorphism $f\colon \varphi^c(R) \hookrightarrow R$ given by (ii) to a ring homomorphism $\tilde{f}\colon \varphi^c(R)[X] \to R$ by any free choice for the value $\tilde{f}(X)$, in particular such that $\tilde{f}(X) := \chi(1) \in R$ and, of course, $\tilde{f}_{|\varphi^c(R)} := f$. We need to show that $\tilde{f}$ is surjective with kernel equal to the ideal generated by $X^{p^c} - \chi(p^c)$. Note that $\chi(p^c) = \varphi^c(\chi(1))$ lies in $\varphi^c(R)$ so the claim makes sense.

By (i) the map χ is a group homomorphism, so $\chi(r) = \chi(1)^r = \tilde{f}(X)^r = \tilde{f}(X^r)$ lies in the image of $\tilde{f}$ for any positive integer r. Hence we obtain the surjectivity from (1) by noting that $\varphi^c(R)$ also lies in the image of $\tilde{f}$.

Using again $\chi(r) = \tilde{f}(X^r)$ with the choice of $r = p^c$ we see immediately that $X^{p^c} - \chi(p^c)$ lies in the kernel of $\tilde{f}$. Moreover, note that $\varphi^c(R)[X]/(X^{p^c} - \chi(p^c))$ is a free module of rank p^c over $\varphi^c(R)$ with generators the classes of $\{X^r\}_{r=0}^{p^c-1}$ in the quotient. On the other hand, R is also a free module of rank p^c with generators $\{\chi(r)\}_{r=0}^{p^c-1}$ by (1) and these two sets of generators correspond to each other under the map $\tilde{f}$ hence the isomorphism. □

Let H_0 be a pro-p group of finite rank (therefore a compact p-adic Lie group by Corollary 4.3 and Theorem 8.18 in [10]) without elements of order p admitting a continuous surjective group homomorphism $\ell\colon H_0 \twoheadrightarrow \mathbb{Z}_p$ with kernel $H_1 := \mathrm{Ker}(\ell)$. We assume further the following

(A) H_0 also admits an injective group endomorphism $\varphi\colon H_0 \hookrightarrow H_0$ with finite cokernel and compatible with ℓ in the sense that $\ell(\varphi(h)) = \varphi(\ell(h)) = p\ell(h)$. In particular, we have $\varphi(H_1) \subseteq H_1$.

(B) $\bigcap_{n\geq 1} \varphi^n(H_0) = \{1\}$ and the subgroups $\varphi^n(H_0)$ form a system of neighbourhoods of 1 in H_0.

We remark first of all that by a Theorem of Serre (Thm. 1.17 in [10]) any finite index subgroup in H_0 is open. Hence the homomorphism φ is automatically continuous and the subgroups $\varphi^n(H_0)$ are open.

Note that H_1 is a closed subgroup of H_0 hence it is also a pro-p group of finite rank. By assumption (B) we also have in particular that the subgroups $\varphi^n(H_1)$ form a system of open neighbourhoods of 1 in H_1. Note that the subgroups $\varphi^n(H_1)$ may not be normal in either H_1 or H_0. Hence we define the normal subgroups $H_k \lhd H_0$ as the normal subgroup of H_0 generated by $\varphi^{k-1}(H_1)$. Since H_1 is normal in H_0 we automatically have $H_k \subseteq H_1$ for any $k \geq 1$. Moreover, since the p-adic Lie group H_1 has a system of neighbourhoods of 1 containing only characteristic subgroups, the H_k also form a system of neighbourhoods of 1 in H_1. On the other hand, we have by definition that $\varphi(H_k) \subseteq H_{k+1} \subseteq H_k$ for each $k \geq 1$. In particular, we have an induced φ action on the quotient group H_0/H_k. This is, of course, no longer injective.

Since the group $\mathbb{Z}_p$ is topologically generated by one element, we may find a splitting $\iota\colon \mathbb{Z}_p \hookrightarrow H_0$ for the group homomorphism ℓ. We fix this splitting ι, too. Assume further that

(C) the group homomorphism ι is φ-equivariant, ie. we have $\iota(\varphi(x)) = \varphi(\iota(x))$ for all $x \in \mathbb{Z}_p$.

We define the skew group ring $R[H_1/H_k, \ell, \iota]$ as follows. We put

$$R[H_1/H_k, \ell, \iota] := \bigoplus_{h \in H_1/H_k} Rh \; . \tag{2}$$

as left R-modules. Note that since H_1 is a normal subgroup in H_0, we also have $H_1/H_k \lhd H_0/H_k$. Therefore we obtain a conjugation action of $\mathbb{Z}_p$ on H_1/H_k given by

$$\begin{aligned} \rho\colon \mathbb{Z}_p &\to \operatorname{Aut}(H_1/H_k) \\ z &\mapsto (h \mapsto \iota(z) h \iota(z)^{-1}, h \in H_1/H_k) \; . \end{aligned}$$

Since H_1/H_k is a finite p-group, $|\operatorname{Aut}(H_1/H_k)| < \infty$ and we have an integer $c_k \geq 1$ such that $p^{c_k}\mathbb{Z}_p \subseteq \operatorname{Ker}(\rho)$. The multiplication is defined so that $\varphi^{c_k}(R)$ commutes with elements h in H_1/H_k and $\chi(i)$ acts on H_1/H_k via $\iota \circ \chi^{-1}$ and conjugation. More precisely for $r_1, r_2 \in R$ and $h_1, h_2 \in H_1/H_k$ we may write

$$r_2 = \sum_{i=0}^{p^{c_k}-1} \chi(i) \varphi^{c_k}(r_{i,2}) \tag{3}$$

and put

$$(r_1 h_1)(r_2 h_2) := \sum_{i=0}^{p^{c_k}-1} r_1 \chi(i) \varphi^{c_k}(r_{i,2}) \left((\iota(i)^{-1} h_1 \iota(i)) h_2 \right) \in \bigoplus_{h \in H_1/H_k} Rh. \tag{4}$$

Note that in case $r_2 = 1$ we have $(r_1 h_1) h_2 = r_1(h_1 h_2)$ and in case $h_1 = 1$ we have $r_1(r_2 h_2) = (r_1 r_2) h_2$. Moreover, by the choice of c_k, $\iota(p^{c_k}\mathbb{Z}_p)$ lies in the centre of H_0/H_k. So we may use any set of representatives of $\mathbb{Z}_p/p^{c_k}\mathbb{Z}_p$ instead of $\{0, 1, \dots, p^{c_k} - 1\}$ in (3) in order to compute (4). Indeed, if $i \equiv i' \pmod{p^{c_k}}$ then we have $\chi(i)\varphi^{c_k}(r_{i,2}) = \chi(i')\varphi^{c_k}(\chi(\frac{i-i'}{p^{c_k}}) r_{i,2})$ and $\iota(i)^{-1} h_1 \iota(i) = \iota(i')^{-1} h_1 \iota(i')$.

Lemma 2.3. *The multipliction* (4) *equips* $R[H_1/H_k, \ell, \iota]$ *with a ring structure.*

Proof. There exists an easy, but rather long computation showing this. However, there is another, more conceptual description of the ring $R[H_1/H_k, \ell, \iota]$ pointed out by Torsten Schoeneberg that proves this lemma without any computations. Let S be the group ring $S := \varphi^{c_k}(R)[H_1/H_k]$ and σ be the automorphism of S trivial on $\varphi^{c_k}(R)$ and acting by conjugation with $\iota(1)$ on H_1/H_k, ie. for $h \in H_1/H_k$ put $\sigma(h) := \iota(1)^{-1} h \iota(1)$. Now define the skew polynomial ring $S[X, \sigma]$ by the relation $aX = X\sigma(a)$ for $a \in S$. Note that by the definition of σ, the subring $\varphi^{c_k}(R)$ lies in the centre of $S[X, \sigma]$ therefore so does $\chi(p^{c_k}) = \varphi^{c_k}(\chi(1)) \in \varphi^{c_k}(R)$. On the other hand, we have $aX^{p^{c_k}} = X^{p^{c_k}} \sigma^{p^{c_k}}(a) = X^{p^{c_k}} a$ for all $a \in S$ since $\sigma^{p^{c_k}}$ is the conjugation by the central element $\iota(1)^{p^{c_k}} = \iota(p^{c_k}) \in H_0/H_k$ on H_1/H_k and is trivial by definition on $\varphi^{c_k}(R)$ hence $\sigma^{p^{c_k}} = \mathrm{id}_S$. This shows that $X^{p^{c_k}} - \chi(p^{c_k})$ is central and that

$S[X,\sigma](X^{p^{c_k}} - \chi(p^{c_k})) = (X^{p^{c_k}} - \chi(p^{c_k}))S[X,\sigma]$ is a two-sided ideal in $S[X,\sigma]$. So we may form the quotient ring and compute (as left $\varphi^{c_k}(R)$-modules)

$$S[X,\sigma]/(X^{p^{c_k}} - \chi(p^{c_k})) \cong \left(\bigoplus_{r=0}^{\infty} \bigoplus_{h\in H_1/H_k} X^r\varphi^{c_k}(R)h\right)/(X^{p^{c_k}} - \chi(p^{c_k})) \cong$$
$$\cong \bigoplus_{h\in H_1/H_k} \left(\varphi^{c_k}(R)[X]/(X^{p^{c_k}} - \chi(p^{c_k}))\right) h \cong \bigoplus_{h\in H_1/H_k} Rh$$

using Lemma 2.2 in the middle. Note that on the component $h = 1$ in the above direct sum the identification is even multiplicative as Lemma 2.2 gives an isomorphism of rings, not just $\varphi^{c_k}(R)$-modules. Hence $S[X,\sigma]/(X^{p^{c_k}} - \chi(p^{c_k}))$ contains R as a subring and the isomorphism above is an isomorphism of left R-modules. The transport of ring structure gives back the definition (4) of multiplication on the right hand side. Indeed, we have

$$(r_1h_1)(r_2h_2) = \sum_{i=0}^{p^{c_k}-1} r_1h_1\chi(i)\varphi^{c_k}(r_{i,2})h_2 = \sum_{i=0}^{p^{c_k}-1} r_1h_1X^i\varphi^{c_k}(r_{i,2})h_2 =$$
$$= \sum_{i=0}^{p^{c_k}-1} r_1X^i\sigma^i(h_1)\varphi^{c_k}(r_{i,2})h_2 = \sum_{i=0}^{p^{c_k}-1} r_1\chi(i)\varphi^{c_k}(r_{i,2})\left((\iota(i)^{-1}h_1\iota(i))h_2\right)$$

since $\chi(i)$ corresponds to X^i under the isomorphism in Lemma 2.2. □

We further have a natural action of φ on $R[H_1/H_k,\ell,\iota]$ coming from the φ-action on both R and H_1/H_k by putting $\varphi(rh) := \varphi(r)\varphi(h)$ for $r \in R$ and $h \in H_1/H_k$.

Lemma 2.4. *The map $\varphi\colon R[H_1/H_k,\ell,\iota] \to R[H_1/H_k,\ell,\iota]$ defined above is a ring homomorphism.*

Proof. The additivity is clear, so it suffices to check the multiplicativity. Using (4) we compute

$$\varphi((r_1h_1)(r_2h_2)) = \sum_{i=0}^{p^{c_k}-1} \varphi\left(r_1\chi(i)\varphi^{c_k}(r_{i,2})\right)\varphi\left((\iota(i)^{-1}h_1\iota(i))h_2\right) =$$
$$= \sum_{i=0}^{p^{c_k}-1} \varphi(r_1)\chi(pi)\varphi^{c_k+1}(r_{i,2})\left((\iota(pi)^{-1}\varphi(h_1)\iota(pi))\varphi(h_2)\right) =$$
$$= \sum_{i=0}^{p^{c_k}-1} \varphi(r_1)\varphi^{c_k+1}(r_{i,2})\varphi(h_1)\chi(pi)\varphi(h_2) = \varphi(r_1)\varphi(h_1)\sum_{i=0}^{p^{c_k}-1} \chi(pi)\varphi^{c_k+1}(r_{i,2})\varphi(h_2) =$$
$$= \varphi(r_1h_1)\varphi\left(\sum_{i=0}^{p^{c_k}-1} \chi(i)\varphi^{c_k}(r_{i,2})h_2\right) = \varphi(r_1h_1)\varphi(r_2h_2)\ .$$

□

Moreover, the map χ and the inclusion of the group H_1/H_k in the multiplicative group of $R[H_1/H_k,\ell,\iota]$ are compatible in the sense that they glue together to a φ-equivariant group homomorphism $\chi_k\colon H_0 \to R[H_1/H_k, \ell, \iota]^\times$ (with kernel $\mathrm{Ker}\chi_k = H_k$) making the diagram

$$\begin{array}{ccc} \mathbb{Z}_p & \xrightarrow{\ \iota\ } & H_0 \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi_k} \\ R & \xrightarrow{\ \iota_{R,k}\ } & R[H_1/H_k, \ell] \end{array} \tag{5}$$

commutative where $\iota_{R,k}$ is the natural inclusion of R in $R[H_1/H_k, \ell]$. Indeed, $H_0 \cong \iota(\mathbb{Z}_p) \ltimes H_1$, so we put $\chi_k(\iota(i)h) := \chi(i)(hH_k)$ for $i \in \mathbb{Z}_p$, $h \in H_1$ and compute

$$\begin{aligned}\chi_k(\iota(i_1)h_1\iota(i_2)h_2) = \chi_k(\iota(i_1+i_2)\iota(i_2)^{-1}h_1\iota(i_2)h_2) = \chi(i_1+i_2)(\iota(i_2)^{-1}h_1\iota(i_2)h_2)H_k = \\ = \chi(i_1)(h_1H_k)\chi(i_2)(h_2H_k) = \chi_k(\iota(i_1)h_1)\chi_k(\iota(i_2)h_2)\end{aligned}$$

showing that χ_k is indeed a group homomorphism. The commutativity of the diagram (5) is clear by definition. Moreover, χ_k is φ-equivariant, since we have

$$\chi_k \circ \varphi(\iota(i)h) = \chi_k(\iota(pi)\varphi(h)) = \chi(pi)\varphi(h)H_k = \varphi(\chi(i)hH_k) = \varphi \circ \chi_k(\iota(i)h)\ .$$

Lemma 2.5. *The above definition of $R[H_1/H_k, \ell, \iota]$ does not depend on the choice of the section ι up to natural isomorphism.*

Proof. Let $\iota'\colon \mathbb{Z}_p \hookrightarrow H_0$ be another section of ℓ. Note that the integer c_k depends on ι but we also have another integer c'_k such that $\iota'(p^{c'_k})$ acts trivially by conjugation on H_1/H_k, ie. $\iota'(p^{c'_k})$ lies in the centre of H_0/H_k. On the other hand, we may choose $m_k \geq 0$ so that $H_1^{p^{m_k}} \subseteq H_k$ since H_1/H_k is a finite p-group. From $\ell \circ \iota = \mathrm{id}_{\mathbb{Z}_p} = \ell \circ \iota'$ we see that $\iota^{-1}\iota'(\mathbb{Z}_p) \subseteq \mathrm{Ker}(\ell) = H_1$ hence for any $x \in \mathbb{Z}_p$ we have

$$\begin{aligned}\iota^{-1}(p^{m_k+\max(c_k,c'_k)}x)\iota'(p^{m_k+\max(c_k,c'_k)}x) = \iota^{-1}(p^{\max(c_k,c'_k)}x)^{p^{m_k}}\iota'(p^{\max(c_k,c'_k)}x)^{p^{m_k}} = \\ = \left(\iota^{-1}(p^{\max(c_k,c'_k)}x)\iota'(p^{\max(c_k,c'_k)}x)\right)^{p^{m_k}} \in H_1^{p^{m_k}} \subseteq H_k\ .\end{aligned}$$

Therefore for $m \geq m_k + \max(c_k, c'_k)$ the map

$$\begin{array}{rcl}\iota'_k\colon R & \hookrightarrow & R[H_1/H_k, \ell, \iota] \\ \iota'_k\left(\displaystyle\sum_{i=0}^{p^m-1}\chi(i)\varphi^m(r_i)\right) & := & \displaystyle\sum_{i=0}^{p^m-1}\chi(i)\varphi^m(r_i)(\iota(i)^{-1}\iota'(i))\end{array} \tag{6}$$

extends to an isomorphism

$$\begin{array}{rcl}\iota'_k\colon R[H_1/H_k, \ell, \iota'] & \to & R[H_1/H_k, \ell, \iota] \\ rh & \mapsto & \iota'_k(r)h\end{array}$$

of φ-rings. Indeed, the map ι'_k is clearly additive and bijective. We claim that it is multiplicative and φ-equivariant. We first show the latter statement and compute

$$\begin{aligned}\varphi \circ \iota'_k\left(\sum_{i=0}^{p^m-1}\chi(i)\varphi^m(r_i)\right) = \sum_{i=0}^{p^m-1}\varphi(\chi(i)\varphi^m(r_i)(\iota(i)^{-1}\iota'(i))) = \\ \sum_{i=0}^{p^m-1}\chi(pi)\varphi^{m+1}(r_i)(\iota(pi)^{-1}\iota'(pi)) = \iota'_k\left(\sum_{i=0}^{p^m-1}\chi(pi)\varphi^{m+1}(r_i)\right) = \iota'_k \circ \varphi\left(\sum_{i=0}^{p^m-1}\chi(i)\varphi^m(r_i)\right)\ .\end{aligned}$$

Note that since $m \geq \max(c_k, c'_k)$ the subring $\varphi^m(R)$ lies in the centre of both $R[H_1/H_k, \ell, \iota]$ and $R[H_1/H_k, \ell, \iota']$. Therefore—in view of the associativity (Lemma 2.3)—we may compute the multiplication (4) by expanding elements of R to degree m. So we write

$$r_1 = \sum_{j=0}^{p^m-1} \chi(j)\varphi^m(r_{j,1}) \ , \quad r_2 = \sum_{i=0}^{p^m-1} \chi(i)\varphi^m(r_{i,2}) \ .$$

Moreover, we may compute (6) using any set of representatives of $\mathbb{Z}_p/p^m\mathbb{Z}_p$ (e.g. $\{j, j+1, \dots, j+p^m-1\}$ instead of $\{0, 1, \dots, p^m-1\}$) since $\iota^{-1}\iota'(p^m\mathbb{Z}_p) \subseteq H_k$. Hence we obtain

$$\begin{aligned}
\iota'_k\left((r_1h_1)(r_2h_2)\right) = \iota'_k\left(\sum_{i=0}^{p^m-1} r_1\chi(i)\varphi^m(r_{i,2})(\iota'(i)^{-1}h_1\iota'(i)h_2)\right) = \\
= \iota'_k\left(\sum_{i,j=0}^{p^m-1} \chi(j)\varphi^m(r_{j,1})\chi(i)\varphi^m(r_{i,2})(\iota'(i)^{-1}h_1\iota'(i)h_2)\right) = \\
= \sum_{i=0}^{p^m-1} \iota'_k\left(\sum_{j=0}^{p^m-1} \chi(i+j)\varphi^m(r_{j,1}r_{i,2})\right)\iota'(i)^{-1}h_1\iota'(i)h_2 = \\
= \sum_{i,j=0}^{p^m-1} \chi(i+j)\varphi^m(r_{j,1}r_{i,2})\iota(i+j)^{-1}\iota'(i+j)\iota'(i)^{-1}h_1\iota'(i)h_2 = \\
= \sum_{i,j=0}^{p^m-1} \chi(i+j)\varphi^m(r_{j,1}r_{i,2})\iota(i+j)^{-1}\iota'(j)h_1\iota'(i)h_2) = \\
= \sum_{i,j=0}^{p^m-1} \chi(j)\varphi^m(r_{j,1})\chi(i)\varphi^m(r_{i,r})\iota(i)^{-1}(\iota(j)^{-1}\iota'(j)h_1)\iota(i)(\iota(i)^{-1}\iota'(i)h_2) = \\
= \left(\sum_{j=0}^{p^m-1} \chi(j)\varphi^m(r_{j,1})(\iota(j)^{-1}\iota'(j)h_1)\right)\left(\sum_{i=0}^{p^m-1} \chi(i)\varphi^m(r_{i,2})(\iota(i)^{-1}\iota'(i)h_2)\right) = \\
\left(\sum_{j=0}^{p^m-1} \varphi^m(r_{j,1})\iota'(j)h_1\right)\left(\sum_{i=0}^{p^m-1} \varphi^m(r_{i,2})\iota'(i)h_2\right) = \iota'_k(r_1h_1)\iota'_k(r_2h_2) \ .
\end{aligned}$$

□

In view of the above Lemma we omit ι from the notation from now on. This construction is compatible with the natural surjective homomorphisms $H_1/H_{k+1} \twoheadrightarrow H_1/H_k$ therefore the rings $R[H_1/H_k, \ell]$ form an inverse system for the induced maps. So we may define the completed skew group ring $R[[H_1, \ell]]$ as the projective limit

$$R[[H_1, \ell]] := \varprojlim_k R[H_1/H_k, \ell].$$

We denote by I_k the kernel of the canonical surjective homomorphism from $R[[H_1, \ell]]$ to $R[H_1/H_k, \ell]$.

Whenever R is a topological ring we equip $R[[H_1, \ell]]$ with the projective limit topology of the product topologies on each $\bigoplus_{h \in H_1/H_k} Rh$.

The augmentation map $H_1 \twoheadrightarrow 1$ induces a ring homomorphism $\ell := \ell_R\colon R[[H_1,\ell]] \twoheadrightarrow R$. This also has a section $\iota := \iota_R = \varprojlim \iota_{R,k}\colon R \hookrightarrow R[[H_1,\ell]]$ (whenever clear we omit the subscript $_R$), ie. $\ell_R \circ \iota_R = \mathrm{id}_R$. Moreover, by (5) the group homomorphism $\chi\colon \mathbb{Z}_p \to R^\times$ extends to a group homomorphism $\chi_{H_0}\colon H_0 \to R[[H_1,\ell]]^\times$ making the diagram

$$\begin{array}{ccc} \mathbb{Z}_p & \xrightarrow[\iota]{} & H_0 \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi_{H_0}} \\ R & \xrightarrow[\iota_R]{} & R[[H_1,\ell]] \end{array}$$

commutative.

The operator φ acts naturally on this projective limit. If R is a topological ring and φ acts continuously on R then φ also acts continuously on each $R[H_1/H_k,\ell]$ and by taking the limit also on $R[[H_1,\ell]]$. For an open subgroup H' of a profinite group H we use the notation $J(H/H')$ for a set of representatives of the left cosets of H' in H. Similarly, we use $J(H'\setminus H)$ for a set of representatives of the right cosets $H'\setminus H$.

Lemma 2.6. *a) Let $L \leq K \leq H$ be groups. Then the set $J(H/K)J(K/L)$ (resp. $J(L\setminus K)J(K\setminus H)$) is a set of representatives for the cosets H/L (resp. for $L\setminus H$).*

b) Let $K \leq H$ be groups and $N \lhd H$ a normal *subgroup. Then $J((K\cap N)\setminus N)$ is also a set of representatives for $K\setminus KN$.*

Proof. These are well-known facts in group theory, however, for the convenience of the reader we recall their proofs here. Note that in *b*) we need N to be a normal subgroup so that KN is a subgroup of H. Also note that $J(K\setminus KN)$ might not lie in N in general.

a) Let $h_1, h_2 \in J(H/K)$ and $k_1, k_2 \in J(K/L)$. Suppose we have $h_1k_1L = h_2k_2L$. Then we also have $h_1^{-1}h_2 \in K$ so $h_1 = h_2$, whence $k_1^{-1}k_2 \in L$ so $k_1 = k_2$. So the elements of the set $J(H/K)J(K/L)$ are in distinct left cosets of L. On the other hand, if $hL \in H/L$ is a left coset, then we may first choose $h_1 \in J(H/K)$ so that $h_1^{-1}h \in K$ and then $k_1 \in J(K/L)$ so that $k_1^{-1}h_1^{-1}h \in L$, ie. $hL = h_1k_1L$.

b) If $n_1 \neq n_2 \in J((K\cap N)\setminus N)$ are distinct then $Kn_1 \neq Kn_2$ as $n_1n_2^{-1}$ does not lie in $K\cap N$, but it lies in N. On the other hand, if $kn \in KN$ then we may find $n_1 \in J((K\cap N)\setminus N)$ such that $nn_1^{-1} \in K\cap N$ hence $knn_1^{-1} \in K$. □

Proposition 2.7. *The map $\varphi\colon R[[H_1,\ell]] \to R[[H_1,\ell]]$ is injective. Moreover, we have*

$$R[[H_1,\ell]] = \bigoplus_{h\in J(\varphi(H_0)\setminus H_0)} \varphi(R[[H_1,\ell]])h.$$

In particular, $R[[H_1,\ell]]$ is a free (left) module of rank $[H_0:\varphi(H_0)]$ over itself via φ.

Proof. Step 1. Let k be an integer and denote by A_k the kernel of the map $\varphi\colon R[H_1/H_k,\ell] \to R[H_1/H_k,\ell]$ so that we have a short exact sequence of abelian groups

$$0 \longrightarrow A_k \longrightarrow R[H_1/H_k,\ell] \xrightarrow{\varphi} \varphi(R[H_1/H_k,\ell]) \longrightarrow 0.$$

We are going to show that the sequence A_k satisfies the trivial Mittag-Leffler condition. From this the injectivity of φ follows, and we obtain

$$\varprojlim_k \varphi(R[H_1/H_k,\ell]) \cong \varphi(\varprojlim_k R[H_1/H_k,\ell]) = \varphi(R[[H_1,\ell]]). \tag{7}$$

Take a fixed positive integer k. Since $\varphi\colon H_1 \to H_1$ is an open map (bijective and continuous between the compact sets H_1 and $\varphi(H_1)$ hence a homeomorphism) and the subgroups H_l form a system of neighbourhoods we find an integer $l > k$ such that $H_k \supseteq \varphi^{-1}(H_l)$. In view of Lemma 2.6 we put

$$J(H_1/H_l) := J(H_1/\varphi^{-1}(H_l))J(\varphi^{-1}(H_l)/H_l)$$

for $J(H_1/\varphi^{-1}(H_l))$ and $J(\varphi^{-1}(H_l)/H_l)$ arbitrarily fixed sets of representatives for the cosets of $H_1/\varphi^{-1}(H_l)$ and of $\varphi^{-1}(H_l)/H_l$, respectively.

Let now $\sum_{h\in J(H_1/H_l)} r_h\chi_l(h)$ be an element in A_l and denote by $f_{k,l}$ the natural surjection from $R[H_1/H_l,\ell] \twoheadrightarrow R[H_1/H_k,\ell]$. We have

$$0 = \varphi\left(\sum_{h\in J(H_1/H_l)} r_h\chi_l(h)\right) = \sum_{h_1\in J(H_1/\varphi^{-1}(H_l))}\ \sum_{h_2\in J(\varphi^{-1}(H_l)/H_l)} \varphi(r_{h_1h_2})\chi_l(\varphi(h_1h_2)) =$$

$$= \sum_{h_1}\sum_{h_2}\varphi(r_{h_1h_2})\chi_l(\varphi(h_1)) = \sum_{h_1\in J(H_1/\varphi^{-1}(H_l))} \varphi\left(\sum_{h\in J(H_1/H_l)\cap h_1\varphi^{-1}(H_l)} r_h\right)\chi_l(\varphi(h_1)).$$

Note that for $h_1 \neq h_1' \in J(H_1/\varphi^{-1}(H_l))$ we have $\varphi(h_1)H_l \neq \varphi(h_1')H_l$. Since $R[H_1/H_l,\ell]$ is defined as a direct sum, we obtain

$$\varphi\left(\sum_{h\in J(H_1/H_l)\cap h_1\varphi^{-1}(H_l)} r_h\right) = 0\ ,\ \text{whence}\ \sum_{h\in J(H_1/H_l)\cap h_1\varphi^{-1}(H_l)} r_h = 0$$

for any fixed $h_1 \in J(H_1/\varphi^{-1}(H_l))$ as φ is injective on R. On the other hand, we have

$$f_{k,l}\left(\sum_{h\in J(H_1/H_l)} r_h\chi_l(h)\right) = \sum_{h_1\in J(H_1/H_k)} (\sum_{h\in J(H_1/H_l)\cap h_1H_k} r_h)\chi_k(h_1) = \sum_{h_1\in J(H_1/H_k)} 0\chi_k(h_1) = 0$$

as h_1H_k is a disjoint union of cosets of $\varphi^{-1}(H_l)$ by the choice of l. This shows that $f_{k,l}(A_l) = 0$ as claimed. Therefore (7) follows as discussed above.

Step 2. Since $\varphi(H_0)\cap H_1$ is open in H_1, there exists an integer $k_0 \geq 2$ such that for $k \geq k_0$ we have $H_k \subseteq \varphi(H_0)$. (We remark here that we may not be able to take $k_0 = 2$ because H_k is the *normal* subgroup generated by $\varphi(H_1)$ which does have elements outside $\varphi(H_0)$ in general.) We claim now the decomposition

$$R[H_1/H_k,\ell] = \bigoplus_{h\in J(\varphi(H_0)\setminus H_0)} \varphi(R[H_1/H_k,\ell])\chi_k(h) \tag{8}$$

for $k \geq k_0$. Note that since H_k is a normal subgroup of H_0 contained in $\varphi(H_0)$ the elements $\chi_k(h)$ above are distinct.

For the proof of (8) we apply Lemma 2.6 *b)* in the situation $K := \varphi(H_0)$, $N := H_1$, and $H := H_0$ to be able to choose $J(\varphi(H_0)\setminus\varphi(H_0)H_1) := J((\varphi(H_0)\cap H_1)\setminus H_1)$. Moreover, by the injectivity of φ on H_0/H_1 we see that $\varphi(H_0)\cap H_1 = \varphi(H_1)$. On the other hand, $\iota(\{0,1,\dots,p-1\})$ is a set of representatives for the cosets $H_1\varphi(H_0)\setminus H_0$. Therefore (using Lemma 2.6 *a)* with $L := \varphi(H_0)$, $K := \varphi(H_0)H_1$, and $H := H_0$) we may choose

$$J(\varphi(H_0)\setminus H_0) := J(\varphi(H_0)\setminus\varphi(H_0)H_1)J(\varphi(H_0)H_1\setminus H_0) = J(\varphi(H_1)\setminus H_1)\iota(\{0,1,\dots,p-1\})\ .$$

We are going to use this specific set $J(\varphi(H_0)\setminus H_0)$ in order to compute the right hand side of (8). Let $\sum_{h\in J(H_1/H_k)} r_h\chi_k(h)$ be an arbitrary element in $R[H_1/H_k,\ell]$. By the étaleness of the action of φ on R (noting that R is commutative) we may uniquely decompose

$$r_h=\sum_{i=0}^{p-1}\chi(i)\varphi(r_{i,h})=\sum_{i=0}^{p-1}\varphi(r_{i,h})\chi(i)\ .$$

On the other hand, we write $\iota(i)h\iota(i)^{-1}=\varphi(u_{i,h})v_{i,h}$ with unique $u_{i,h}\in H_1$ and $v_{i,h}\in J(\varphi(H_1)\setminus H_1)$. Therefore we have

$$\sum_{h\in J(H_1/H_k)} r_h\chi_k(h)=\sum_{h\in J(H_1/H_k)}\sum_{i=0}^{p-1}\varphi(r_{i,h})\chi(i)\chi_k(\iota(i)^{-1}\varphi(u_{i,h})v_{i,h}\iota(i))=$$
$$=\sum_{h\in J(H_1/H_k)}\sum_{i=0}^{p-1}\varphi(r_{i,h}\chi_k(u_{i,h}))\chi_k(v_{i,h}\iota(i))\in\sum_{h\in J(\varphi(H_0)\setminus H_0)}\varphi(R[H_1/H_k,\ell])\chi_k(h)$$

as $\chi(i)=\chi_k(\iota(i))$ and $\chi_k\circ\varphi=\varphi\circ\chi_k$ by (5).

It remains to show that the sum in (8) is indeed direct. For this we may expand any element $x_{i,h}\in R[H_1/H_k,\ell]$ as

$$x_{i,h}=\sum_{m\in J(H_1/H_k)} r_{i,h,m}\chi_k(m)$$

and compute

$$\sum_{i=0}^{p-1}\sum_{h\in J(\varphi(H_1)\setminus H_1)}\varphi(x_{i,h})\chi_k(h\iota(i))=\sum_{i,h,m}\varphi(r_{i,h,m})\chi_k(\varphi(m)h\iota(i))=\tag{9}$$
$$=\sum_{i,h,m}\chi(i)\varphi(r_{i,h,m})\chi_k(\iota(i)^{-1}\varphi(m)h\iota(i))=$$
$$=\sum_{i,h}\sum_{m_0\in J(\varphi(H_1)/H_k)}\chi(i)\left(\sum_{m\in J(H_1/H_k)\cap\varphi^{-1}(m_0H_k)}\varphi(r_{i,h,m})\right)\chi_k(\iota(i)^{-1}m_0h\iota(i)).$$

Assume now that the left hand side of (9) is 0. The set $J(\varphi(H_1)/H_k)J(\varphi(H_1)\setminus H_1)$ is a set of representatives of $H_k\setminus H_1$ because H_k is normal in H_1 whence $\varphi(H_1)/H_k=H_k\setminus\varphi(H_1)$. This shows that the elements m_0h are distinct in H_1/H_k on the right hand side of (9). Moreover, the conjugation by $\iota(i)$ is an automorphism of H_1/H_k therefore the elements $\iota(i)^{-1}m_0h\iota(i)$ are also distinct for any fixed $i\in\{0,1,\dots,p-1\}$. On the other hand, by the étaleness of φ on R and by (2) we obtain

$$R[H_1/H_k,\ell]=\bigoplus_{i=0}^{p-1}\bigoplus_{h_1\in H_1/H_k}\chi(i)\varphi(R)h\ .$$

Hence we have

$$\sum_{m\in J(H_1/H_k)\cap\varphi^{-1}(m_0H_k)}\varphi(r_{i,h,m})=0$$

for any fixed m_0, i, and h. In particular, we also have

$$\varphi(x_{i,h}) = \sum_{m\in J(H_1/H_k)} \varphi(r_{i,h,m})\chi_k(\varphi(m)) =$$

$$= \sum_{m_0\in J(\varphi(H_1)/H_k)} \left(\sum_{m\in J(H_1/H_k)\cap\varphi^{-1}(m_0 H_k)} \varphi(r_{i,h,m}) \right) \chi_k(m_0) = 0$$

showing that the sum in (8) is direct.

Step 3. The result follows by taking the projective limit of (8) using (7). □

Remark 2.8. *The above lemma holds for replacing left and right, as well, ie. we also have*

$$R[\![H_1,\ell]\!] = \bigoplus_{h\in J(H_0/\varphi(H_0))} h\varphi(R[\![H_1,\ell]\!]).$$

Let S be a (not necessarily commutative) ring (with identity) with the following properties:

(i) There exists a group homomorphism $\chi\colon H_0 \hookrightarrow S^\times$.

(ii) The ring S admits an étale action of the p-Frobenius φ that is compatible with χ. More precisely there is an injective ring homomorphism $\varphi\colon S \hookrightarrow S$ such that $\varphi(\chi(x)) = \chi(\varphi(x))$ and

$$S = \bigoplus_{h\in\varphi(H_0)\backslash H_0} \varphi(S)\chi(h) = \bigoplus_{h\in H_0/\varphi(H_0)} \chi(h)\varphi(S)\ .$$

In particular, S is free of rank $|H_0 : \varphi(H_0)|$ as a left, as well as a right module over $\varphi(S)$.

Definition 2.9. *We call a ring S with the above properties (i) and (ii) a φ-ring over H_0.*

Corollary 2.10. *The map $R \mapsto R[\![N_1,\ell]\!]$ is a functor from the category of φ-rings over $\mathbb{Z}_p$ to the category of φ-rings over H_0.*

Remark 2.11. *We have $\varphi(I_k) \subseteq I_{k+1}$ for all $k \geq 1$.*

Proof. Take $x \in I_k$ and write $x+I_{k+1} \in R[H_1/H_{k+1},\ell]$ as $x+I_{k+1} = \sum_{h\in J(H_1/H_{k+1})} r_h\chi_{k+1}(h)$. Since $x \in I_k$ we have

$$0 = \sum_{h\in J(H_1/H_{k+1})} r_h\chi_k(h) = \sum_{h_1\in J(H_1/H_k)} \sum_{h\in J(H_1/H_{k+1})\cap h_1H_k} r_h\chi_k(h_1)$$

hence $\sum_{h\in J(H_1/H_{k+1})\cap h_1H_k} r_h = 0$ for any fixed $h_1 \in J(H_1/H_k)$. So we compute

$$\varphi(x) + I_{k+1} = \sum_{h_1\in J(H_1/H_k)} \sum_{h\in J(H_1/H_{k+1})\cap h_1H_k} \varphi(r_h)\varphi(\chi_k(h)) =$$

$$= \sum_{h_1\in J(H_1/H_k)} \sum_{h\in J(H_1/H_{k+1})\cap h_1H_k} \varphi(r_h)\chi_k(\varphi(h_1)) = \sum_{h_1\in J(H_1/H_k)} 0\chi_k(\varphi(h_1)) = 0$$

since $\varphi(H_k) \subseteq H_{k+1}$ whence $\varphi(h_1) = \varphi(h)$ above. □

Recall that Fontaine's ring $\mathcal{O}_{\mathcal{E}} := \varprojlim_n (o[[T]][T^{-1}])/p_K^n$ is defined as the p-adic completion of the ring of formal Laurent-series over o. It is a complete discrete valuation ring with maximal ideal $p_K\mathcal{O}_{\mathcal{E}}$, residue field $k((T))$, and field of fractions $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[p_K^{-1}]$. We are going to show that the completed skew group ring $\mathcal{O}_{\mathcal{E}}[[H_1, \ell]]$ is isomorphic to the previously constructed ([17], see also section 8 of [18], [19], [21]) microlocalized ring $\Lambda_\ell(H_0)$ of the Iwasawa algebra $\Lambda(H_0)$. ($H_0 = N_0$ in the notations of [18], [19], and [21].) For the convenience of the reader we recall the definition here. Let $\Lambda(H_0) := o[[H_0]]$ be the Iwasawa algebra of the pro-p group H_0. It is shown in [5] that $S := \Lambda(H_0) \setminus (p_K, H_1 - 1)$ is a left and right Ore set in $\Lambda(H_0)$ so that the localization $\Lambda(H_0)_S$ exists. The ring $\Lambda_\ell(H_0)$ is defined as the $(p_K, H_1 - 1)$-adic completion of $\Lambda(H_0)_S$ (the so called "microlocalization"). Note that since $\varphi\colon H_0 \to H_0$ is a continuous group homomorphism, it induces a continuous ring homomorphism $\varphi\colon \Lambda(H_0) \to \Lambda(H_0)$ of the Iwasawa algebra. Moreover, since $\varphi(S) \subset S$, φ extends to a ring homomorphism $\varphi\colon \Lambda(H_0)_S \to \Lambda(H_0)_S$ and by continuity to its completion $\Lambda_\ell(H_0)$ (see section 8 of [18] for more details).

Remark 2.12. *Let R be a φ-ring containing (as a φ-subring) the Iwasawa algebra $o[[T]] \cong \Lambda(\mathbb{Z}_p)$. Then using* (1) *we compute*

$$\begin{aligned} R[H_1/H_k, \ell] &\cong (R \otimes_{\Lambda(\mathbb{Z}_p)} \Lambda(\mathbb{Z}_p))[H_1/H_k, \ell] \cong R \otimes_{\Lambda(\mathbb{Z}_p)} (\Lambda(\mathbb{Z}_p)[H_1/H_k, \ell]) \\ &\cong R \otimes_{\Lambda(\mathbb{Z}_p),\iota} \Lambda(H_0/H_k) \cong (\varphi^{c_k}(R) \otimes_{\varphi^{c_k}(\Lambda(\mathbb{Z}_p))} \Lambda(\mathbb{Z}_p)) \otimes_{\Lambda(\mathbb{Z}_p),\iota} \Lambda(H_0/H_k) \\ &\cong \varphi^{c_k}(R) \otimes_{\varphi^{c_k}(\Lambda(\mathbb{Z}_p)),\iota} \Lambda(H_0/H_k) \end{aligned}$$

for any $k \geq 1$.

Lemma 2.13. *We have a φ-equivariant ring-isomorphism $\mathcal{O}_{\mathcal{E}}[[H_1, \ell]] \cong \Lambda_\ell(H_0)$.*

Proof. The ring $\Lambda_\ell(H_0)$ is complete and Hausdorff with respect to the filtration by the ideals generated by $(H_k - 1)$ since these ideals are closed with intersection zero in the pseudocompact ring $\Lambda_\ell(H_0)$ (cf. Thm. 4.7 in [17]). So it remains to show that $\Lambda_\ell(H_0/H_k)$ is naturally isomorphic to the skew group ring $\mathcal{O}_{\mathcal{E}}[H_1/H_k, \ell]$. At first we show that $\Lambda(H_0/H_k) \cong \Lambda(\mathbb{Z}_p)[H_1/H_k, \ell]$. Both sides are free modules of rank $|H_1/H_k|$ over $\Lambda(\mathbb{Z}_p)$ with generators $h \in H_1/H_k$ so there is an obvious isomorphism between them as $\Lambda(\mathbb{Z}_p)$-modules. Moreover, $\varphi^{c_k}(\Lambda(\mathbb{Z}_p))$ lies in the centre of both rings. However, the obvious map above is also multiplicative since the multiplication on $\Lambda(\mathbb{Z}_p)[H_1/H_k, \ell]$ is uniquely determined by (4) so that (5) is satisfied and $\varphi^{c_k}(\Lambda(\mathbb{Z}_p))$ lies in the centre.

Now by Remark 2.12 we have

$$\mathcal{O}_{\mathcal{E}}[H_1/H_k, \ell] \cong \varphi^{c_k}(\mathcal{O}_{\mathcal{E}}) \otimes_{\varphi^{c_k}(\Lambda(\mathbb{Z}_p)),\iota} \Lambda(H_0/H_k)$$

for any $k \geq 1$.

Since $\iota(\varphi^{p^{c_k}}(\Lambda(\mathbb{Z}_p)))$ lies in the centre of $\Lambda(H_0/H_k)$, the right hand side above is the localisation of $\Lambda(H_0/H_k)$ inverting the central element $\varphi^{p^{c_k}}(T)$ and taking the p-adic completion afterwards (ie. "microlocalisation" at $\varphi^{p^{c_k}}(T)$). However, in a p-adically complete ring T is invertible if and only if so is $\varphi^{p^{c_k}}(T)$. Indeed, we have

$$T \mid \varphi^{p^{c_k}}(T) = (T+1)^{p^{c_k}} - 1 = \sum_{i=1}^{p^{c_k}} \binom{p^{c_k}}{i} T^i \in T^{p^{c_k}}(1 + po[[T]][T^{-1}]) \ .$$

Hence we obtain

$$\varphi^{c_k}(\mathcal{O}_{\mathcal{E}}) \otimes_{\varphi^{c_k}(\Lambda(\mathbb{Z}_p)),\iota} \Lambda(H_0/H_k) \cong \Lambda_\ell(H_0/H_k)$$

as both sides are the microlocalisation of $\Lambda(H_0/H_k)$ at T. $\square$

3 Equivalence of categories

Let S be a φ-ring over any pro-p group H_0 satisfying (A), (B), and (C) (for now it would suffice to assume that S has an injective ring-endomorphism $\varphi\colon S\to S$). We define a φ-module over S to be a free S-module D of finite rank together with a semilinear action of φ such that the map

$$\begin{aligned} 1\otimes\varphi\colon S\otimes_{S,\varphi} D &\to D \\ r\otimes d &\mapsto r\varphi(d) \end{aligned} \tag{10}$$

is an isomorphism. Note that for rings S in which p is not invertible (such as $S=\mathcal{O}_{\mathcal{E}}$ and $\mathcal{O}_{\mathcal{E}}^{\dagger}$) this is the definition of an *étale* φ-module. However, for rings in which p is invertible (such as the Robba ring $\mathcal{R}$) this is the usual definition of a φ-module. We use this definition for both $S=R$ and $S=R[\![H_1,\ell]\!]$—the former being a φ-ring over $\mathbb{Z}_p$ and the latter being a φ-ring over H_0. We denote the category of φ-modules over R (resp. over $R[\![H_1,\ell]\!]$) by $\mathfrak{M}(R,\varphi)$ (resp. by $\mathfrak{M}(R[\![H_1,\ell]\!],\varphi)$). These are clearly additive categories. However, they are not abelian in general, as the kernel and cokernel might not be a free module over R, resp. over $R[\![H_1,\ell]\!]$.

Note that for modules M over $R[\![H_1,\ell]\!]$ (10) (with $D=M$) is equivalent to saying that each element $m\in M$ is uniquely decomposed as

$$m=\sum_{u\in J(H_0/\varphi^k(H_0))} u\varphi^k(m_{u,k})$$

for $k=1$, or equivalently, for all $k\geq 1$.

There is an obvious functor in both directions induced by ℓ_R and ι_R that we denote by

$$\begin{aligned} \mathbb{D}:=R\otimes_{R[\![H_1,\ell]\!],\ell}\cdot\colon\quad &\mathfrak{M}(R[\![H_1,\ell]\!],\varphi)\to\mathfrak{M}(R,\varphi) \\ \mathbb{M}:=R[\![H_1,\ell]\!]\otimes_{R,\iota}\cdot\colon\quad &\mathfrak{M}(R,\varphi)\to\mathfrak{M}(R[\![H_1,\ell]\!],\varphi)\ . \end{aligned}$$

The following is a generalization of Thm. 8.20 in [19]. The proof is also similar, but we include it here for the convenience of the reader.

Proposition 3.1. *The functors $\mathbb{D}$ and $\mathbb{M}$ are quasi-inverse equivalences of categories.*

Proof. We first note that since $\ell\circ\iota=\mathrm{id}_R$ we also have $\mathbb{D}\circ\mathbb{M}\cong\mathrm{id}_{\mathfrak{M}(R,\varphi)}$. So it remains to show that $\mathbb{D}$ is full and faithful.

For the faithfulness of $\mathbb{D}$ let $f\colon M_1\to M_2$ be a morphism in $\mathfrak{M}(R[\![H_1,\ell]\!],\varphi)$ such that $\mathbb{D}(f)=0$ which means that $f(M_1)\subseteq I_1M_2$. Let $m\in M_1$. For any $k\in\mathbb{N}$ we write $m=\sum_{u\in J(H_0/\varphi^k(H_0))}u\varphi^k(m_{u,k})$ and

$$f(m)=\sum_{u\in J(H_0/\varphi^k(H_0))}u\varphi^k f(m_{u,k})\in\varphi^k(I_1M_2)\subseteq I_{k+1}M_2$$

by Remark 2.11. Therefore $f(M_1)\subseteq I_{k+1}M_2$ for any $k\geq 0$ and therefore $f=0$ since M_2 is a finitely generated free module over $R[\![H_1,\ell]\!]$ and $\bigcap_{k\geq 0}I_{k+1}=0$ since $R[\![H_1,\ell]\!]\cong\varprojlim R[\![H_1,\ell]\!]/I_k$.

Now we prove that for any object M in $\mathfrak{M}(R[\![H_1,\ell]\!],\varphi)$ we have an isomorphism $\mathbb{M}\circ\mathbb{D}(M)\to M$. We start with an arbitrary finite $R[\![H_1,\ell]\!]$-basis $(\epsilon_i)_{1\leq i\leq d}$ of M (where d is the rank of M). As R-modules we have

$$M=\Big(\bigoplus_{1\leq i\leq d}\iota(R)\epsilon_i\Big)\oplus\Big(\bigoplus_{1\leq i\leq d}I_1\epsilon_i\Big)\ .$$

It is clear that the $R[\![H_1,\ell]\!]$-linear map from M to $\mathbb{M}(\mathbb{D}(M))$ sending ϵ_i to $1\otimes(1\otimes\epsilon_i)$ is bijective. It is φ-equivariant if and only if $\bigoplus_{1\le i\le d}\iota(R)\epsilon_i$ is φ-stable which is, of course, not true in general. We always have

$$\varphi(\epsilon_i)=\sum_{1\le j\le d}(a_{i,j}+b_{i,j})\epsilon_j \quad \text{where } a_{i,j}\in\iota(R)\ ,\ b_{i,j}\in I_1\ .$$

If the $b_{i,j}$ are not all 0, we will find elements $x_{i,j}\in I_1$ such that

$$\eta_i:=\epsilon_i+\sum_{1\le j\le d}x_{i,j}\epsilon_j$$

satisfies

$$\varphi(\eta_i)=\sum_{1\le j\le d}a_{i,j}\eta_j \quad \text{for } i\in I\ .$$

The conditions on the matrix $X:=(x_{i,j})_{1\le i,j\le d}$ are :

$$\varphi(\mathrm{id}+X)(A+B)=A(\mathrm{id}+X)$$

for the matrices $A:=(a_{i,j})_{1\le i,j\le d}$, $B:=(b_{i,j})_{1\le i,j\le d}$. The coefficients of A belong to the commutative ring $\iota(R)$. The matrix $A+B$ is invertible because the $R[\![H_1,\ell]\!]$-endomorphism f of M defined by

$$f(\epsilon_i)=\varphi(\epsilon_i) \quad \text{for } 1\le i\le d\ ,$$

is an automorphism of M as M lies in $\mathfrak{M}(R[\![H_1,\ell]\!],\varphi)$. Therefore the matrix $A=\ell(A+B)$ is also invertible. We are reduced to solve the equation

$$A^{-1}B+A^{-1}\varphi(X)(A+B)=X$$

in the indeterminate X. We are looking for the solution X in the form of an infinite sum

$$X=A^{-1}B+\ldots+(A^{-1}\varphi(A^{-1})\ldots\varphi^{k-1}(A^{-1})\varphi^k(A^{-1}B)\varphi^{k-1}(A+B)\ldots\varphi(A+B)(A+B))+\ldots\ .$$

The coefficients of $A^{-1}B$ belong to the two-sided ideal I_1 of $R[\![H_1,\ell]\!]$ and the coefficients of the k-th term of the series

$$(A^{-1}\varphi(A^{-1})\ldots\varphi^{k-1}(A^{-1})\varphi^k(A^{-1}B)\varphi^{k-1}(A+B)\ldots\varphi(A+B)(A+B))$$

belong to $\varphi^k(I_1)\subseteq I_{k+1}$. Hence the series converges since $R[\![H_1,\ell]\!]\cong\varprojlim_k R[\![H_1,\ell]\!]/I_k$. Its limit X is the unique solution of the equation. The coefficients of every term in the series belong to I_1 and I_1 is closed in $R[\![H_1,\ell]\!]$, hence $x_{i,j}\in I_1$ for $1\le i,j\le d$.

We still need to show that the set $(\eta_i)_{1\le i\le d}$ is an $R[\![H_1,\ell]\!]$-basis of M. Similarly to the above equation we may find a matrix Y with coefficients in I_1 such that we have

$$(A+B)(\mathrm{id}+Y)=\varphi(\mathrm{id}+Y)A.$$

Therefore we obtain

$$(A+B)(\mathrm{id}+Y)(\mathrm{id}+X)=\varphi\left((\mathrm{id}+Y)(\mathrm{id}+X)\right)(A+B)$$

which means that the map

$$\begin{array}{rcl}(\mathrm{id}+Y)(\mathrm{id}+X)\colon M & \to & M \\ \epsilon_i & \mapsto & (\mathrm{id}+Y)(\mathrm{id}+X)\epsilon_i\end{array}$$

is a φ-equivariant map such that $\mathbb{D}((\mathrm{id}+Y)(\mathrm{id}+X)) = \mathrm{id}$, hence $(\mathrm{id}+Y)(\mathrm{id}+X) = \mathrm{id}$ by the faithfulness of $\mathbb{D}$. By a similar computation we also obtain $A(\mathrm{id}+X)(\mathrm{id}+Y) = \varphi((\mathrm{id}+X)(\mathrm{id}+Y))A$ showing that $(\mathrm{id}+X)(\mathrm{id}+Y)$ is a φ-equivariant endomorphism of $\mathbb{M}\circ\mathbb{D}(M)$ reducing to the identity modulo I_1. Hence $(\mathrm{id}+Y)$ is a twosided inverse to the map $(\mathrm{id}+X)$, in particular $(\eta_i)_{1\leq i\leq d}$ is an $R[[H_1,\ell]]$-basis of M. So we obtain an isomorphism in $\mathfrak{M}(R[[H_1,\ell]],\varphi)$,

$$\Theta\ :\ M\to M(\mathbb{D}(M))\quad,\quad \Theta(\eta_i)=1\otimes(1\otimes\eta_i)\ \text{ for } 1\leq i\leq d\ ,$$

such that $\mathbb{D}(\Theta)$ is the identity morphism of $\mathbb{D}(M)$.

Now if $f\colon \mathbb{D}(M_1)\to\mathbb{D}(M_2)$ then for

$$\mathbb{M}(f)\colon M_1\cong\mathbb{M}\circ\mathbb{D}(M_1)\to\mathbb{M}\circ\mathbb{D}(M_2)\cong M_2$$

we have $\mathbb{D}\circ\mathbb{M}(f)=f$ therefore $\mathbb{D}$ is full. □

Remark 3.2. *There is a small mistake in Lemma 1 of [21]. The map ω is in fact not a p-valuation, since assertion (iii) stating that $\omega(g^p)=\omega(g)+1$ is false. It is only true in the weaker form $\omega(g^p)\geq\omega(g)+1$. However, this does not influence the validity of the rest of the paper as $N_{0,n}:=\{g\in N_0\mid\omega(g)\geq n\}$ is still a subgroup satisfying Lemma 2. Alternatively, it is possible to modify ω so that one truely obtains a p-valuation. I would like to take this opportunity to thank Torsten Schoeneberg for pointing this out to me.*

Remark 3.3. *Note that in the case of $R=\mathcal{O}_{\mathcal{E}}$ we may end the proof of Proposition 3.1 by saying that $\mathrm{id}+X$ is invertible since X lies in $I_1^{d\times d}$ and $\mathcal{O}_{\mathcal{E}}[[H_1,\ell]]\cong\Lambda_\ell(H_0)$ is I_1-adically complete. However, in the general situation $R[[H_1,\ell]]$ may not be complete I_1-adically. The reason for this is the fact that the ideals $(I_k)_{k\geq1}$ are only cofinal with the ideals I_1^k whenever R is killed by a power of p. Therefore if R is not p-adically complete, we do not have $R[[H_1,\ell]]\cong\varprojlim R[[H_1,\ell]]/I_1^k$ in general. Moreover, in case of $R=\mathcal{O}_{\mathcal{E}}$ Proposition 3.1 holds for not necessarily free modules, as well. See [19] for the proof of this.*

Remark 3.4. *The matrix Y in the proof of Prop. 3.1 is given by a convergent sum of the terms*

$$-(A+B)^{-1}\varphi((A+B)^{-1})\dots\varphi^{k-1}((A+B)^{-1})\varphi^k((A+B)^{-1}B)\varphi^{k-1}(A)\dots\varphi(A)A$$

for $k\geq0$ and a direct computation also shows that $(\mathrm{id}+Y)(\mathrm{id}+X)=\mathrm{id}=(\mathrm{id}+X)(\mathrm{id}+Y)$.

3.1 Reductive groups over $\mathbb{Q}_p$ and Whittaker functionals

Let p be a prime number let $\mathbb{Q}_p\subseteq K$ be a finite extension with ring of integer o_K, uniformizer p_K, and residue field $k=o_K/p_K$. This field will only play the role of coefficients, the reductive groups will all be defined over $\mathbb{Q}_p$. Following [18], let G be the $\mathbb{Q}_p$-rational points of a $\mathbb{Q}_p$-split connected reductive group over $\mathbb{Q}_p$. In particular, G is a locally $\mathbb{Q}_p$-analytic group. Moreover, we assume that the centre of G is connected. We fix a Borel subgroup $P=TN$ in

G with maximal split torus T and unipotent radical N. Let Φ^+ denote, as usual, the set of positive roots of T with respect to P and let $\Delta \subseteq \Phi^+$ be the subset of simple roots. For any $\alpha \in \Phi^+$ we have the root subgroup $N_\alpha \subseteq N$. We recall that $N = \prod_{\alpha\in\Phi^+} N_\alpha$ (set-theoretically) for any total ordering of Φ^+. Let $T_0 \subseteq T$ be the maximal compact subgroup. We fix a compact open subgroup $N_0 \subseteq N$ which is totally decomposed, in other words $N_0 = \prod_\alpha (N_0 \cap N_\alpha)$ for any total ordering of Φ^+. Hence $P_0 := T_0N_0$ is a group. We introduce the submonoid $T_+ \subseteq T$ of all $t \in T$ such that $tN_0t^{-1} \subseteq N_0$, or equivalently, such that $|\alpha(t)| \leq 1$ for any $\alpha \in \Delta$. Obviously, $P_+ := N_0T_+ = P_0T_+P_0$ is then a submonoid of P.

We fix once and for all isomorphisms of algebraic groups

$$\iota_\alpha \colon N_\alpha \xrightarrow{\cong} \mathbb{Q}_p$$

for $\alpha \in \Delta$, such that

$$\iota_\alpha(tnt^{-1}) = \alpha(t)\iota_\alpha(n)$$

for any $n \in N_\alpha$ and $t \in T$. We normalize these isomorphisms so that $\iota_\alpha(N_0 \cap N_\alpha) = \mathbb{Z}_p \subset \mathbb{Q}_p$. Since $\prod_{\alpha\in\Delta} N_\alpha$ is naturally a quotient of $N/[N,N]$ we may view any homomorphism

$$\ell \colon \prod_{\alpha\in\Delta} N_\alpha \to \mathbb{Q}_p$$

as a functional on N. We fix once and for all a homomorphism ℓ such that we have $\ell(N_0) = \mathbb{Z}_p$. Let $X^*(T) := \mathrm{Hom}_{alg}(T, \mathbb{G}_m)$ (resp. $X_*(T) := \mathrm{Hom}_{alg}(\mathbb{G}_m, T)$) be the group of algebraic characters (resp. cocharacters) of T. Since we assume that the centre of G is connected, the quotient $X^*(T)/\bigoplus_{\alpha\in\Delta} \mathbb{Z}\alpha$ is free. Hence we find a cocharacter ξ in $X_*(T)$ such that $\alpha \circ \xi = \mathrm{id}_{\mathbb{G}_m}$ for any α in Δ. It is injective and uniquely determined up to a central cocharacter. We fix once and for all such a ξ. It satisfies

$$\xi(\mathbb{Z}_p \setminus \{0\}) \subseteq T_+$$

and

$$\ell(\xi(a)n\xi(a^{-1})) = a\ell(n) \tag{11}$$

for any a in $\mathbb{Q}_p^\times$ and n in N since ℓ is a linear functional on the space $\prod_{\alpha\in\Delta} N_\alpha$ and therefore can be written as a linear combination of the isomorphisms $\iota_\alpha \colon N_\alpha \to \mathbb{Q}_p$.

For example, if $G = \mathrm{GL}_n(\mathbb{Q}_p)$, T is the group of diagonal matrices, and N is the group of unipotent upper triangular matrices, then we could choose $\xi \colon \mathbb{G}_m(\mathbb{Q}_p) = \mathbb{Q}_p^\times \to T = (\mathbb{Q}_p^\times)^n$,

$$\xi(x) := \begin{pmatrix} x^{n-1} & & & \\ & x^{n-2} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Put $\Gamma := \xi(\mathbb{Z}_p^\times)$ and $s := \xi(p)$. The element s acts by conjugation on the group N_0 such that $\bigcap_k s^kN_0s^{-k} = \{1\}$. We denote this action by $\varphi := \varphi_s$. This is compatible with the functional ℓ in the sense $\ell \circ \varphi = p\ell$ (see section 2) by (11). Therefore we may apply the theory of the preceding sections to any φ-ring R with the homomorphism $\ell \colon N_0 \to \mathbb{Z}_p$ and $N_1 := \mathrm{Ker}(\ell_{|N_0})$. We are going to apply the theory of section 2 in the setting $H_0 := N_0$ and $H_1 := N_1$.

Note that in [18] and [21] ℓ is assumed to be generic—we do not assume this here, though. We remark that for any $\alpha \in \Delta$ the restriction of ℓ to a fixed N_α is either zero or an isomorphism of N_α with $\mathbb{Q}_p$ and put $a_\alpha := \ell(\iota_\alpha^{-1}(1))$. By the assumption $\ell(N_0) = \mathbb{Z}_p$ we obtain $a_\alpha \in \mathbb{Z}_p$ for all $\alpha \in \Delta$ and $a_\alpha \in \mathbb{Z}_p^\times$ for at least one α in Δ. We put $T_{+,\ell} := \{t \in T_+ \mid tN_1t^{-1} \subseteq N_1\}$. The monoid $T_{+,\ell}$ acts on the group $\mathbb{Z}_p$ via $\ell\colon N_0 \to \mathbb{Z}_p$, too.

A (φ, Γ)-ring R is by definition a φ-ring (in the sense of section 2) together with an action of $\Gamma \cong \mathbb{Z}_p^\times$ commuting with φ and satisfying $\gamma(\chi(x)) = \chi(\xi^{-1}(\gamma)x)$. For example $\mathcal{O}_\mathcal{E}, \mathcal{O}_\mathcal{E}^\dagger, \mathcal{E}^\dagger, \mathcal{R}$ are (φ, Γ)-rings. Note that the endomorphism ring $\mathrm{End}(\mathbb{Z}_p)$ of the p-adic integers (as a topological abelian group) is isomorphic to $\mathbb{Z}_p$. On the other hand, the multiplicative monoid $\mathbb{Z}_p \setminus \{0\}$ is isomorphic to $\varphi^\mathbb{N}\Gamma$. Now having an action of φ and Γ on R we obtain an action of $T_{+,\ell}$ on R since the map $\ell\colon N_0 \to \mathbb{Z}_p$ induces a monoid homorphism $T_{+,\ell} \to \mathbb{Z}_p \setminus \{0\} \cong \varphi^\mathbb{N}\Gamma$. We denote the kernel of this monoid homomorphism by $T_{0,\ell}$. Similarly, we have a natural action of $T_{+,\ell}$ on the ring $R[[N_1, \ell]]$ by conjugation. Indeed, if $t \in T_{+,\ell}$ then since T is commutative we have

$$t\varphi^k(N_1)t^{-1} = ts^kN_1s^{-k}t^{-1} = s^ktN_1t^{-1}s^{-k} = \varphi^k(tN_1t^{-1}) \subseteq \varphi^k(N_1),$$

whence $tN_kt^{-1} \subseteq N_k$. Hence t acts naturally on the skew group ring $R[N_1/N_k, \ell]$ and by taking the limit we also obtain an action on $R[[N_1, \ell]]$. We denote the map on both R and $R[[N_1, \ell]]$ induced by the action of $t \in T_{+,\ell}$ by φ_t.

Now a $T_{+,\ell}$-module over R (resp. over $R[[N_1, \ell]]$) is a finitely generated free R-module D (resp. $R[[N_1, \ell]]$-module M) with a semilinear action of $T_{+,\ell}$ (denoted by $\varphi_t\colon D \to D$, resp. $\varphi_t\colon M \to M$ for any $t \in T_{+,\ell}$) such that the restriction of the $T_{+,\ell}$-action to $s \in T_{+,\ell}$ defines a φ-module over R (resp. over $R[[N_1, \ell]]$). We denote the category of $T_{+,\ell}$-modules over R (resp. over $R[[N_1, \ell]]$) by $\mathfrak{M}(R, T_{+,\ell})$ (resp. by $\mathfrak{M}(R[[N_1, \ell]], T_{+,\ell})$).

Lemma 3.5. *Let M be in $\mathfrak{M}(R[[N_1, \ell]], T_{+,\ell})$ and D be in $\mathfrak{M}(R, T_{+,\ell})$. Then the maps*

$$\begin{aligned} 1 \otimes \varphi_t\colon R[[N_1, \ell]] \otimes_{R[[N_1,\ell]],\varphi_t} M &\to M \\ r \otimes m &\mapsto r\varphi_t(m) \end{aligned}$$

and

$$\begin{aligned} 1 \otimes \varphi_t\colon R \otimes_{R,\varphi_t} D &\to D \\ r \otimes d &\mapsto r\varphi_t(d) \end{aligned}$$

are isomorphisms for any $t \in T_{+,\ell}$.

Proof. We only prove the statement for M (the statement for D is entirely analogous). First note that the subgroups $s^kN_0s^{-k}$ (resp. $s^kN_1s^{-k}$) form a system of neighbourhoods of 1 in N (resp. in $\mathrm{Ker}(\ell)$). On the other hand, if t is in $T_{+,\ell}$ then

$$t\mathrm{Ker}(\ell_{|N})t^{-1} = t\left(\bigcup_{k\in\mathbb{Z}} s^kN_1s^{-k}\right)t^{-1} = \bigcup_{k\in\mathbb{Z}} s^ktN_1t^{-1}s^{-k} = \mathrm{Ker}(\ell_{|N}).$$

since tN_1t^{-1} has finite index in N_1. Now since $t^{-1}N_0t$ and on $t^{-1}N_1t$ are compact, we find $k_0 > 0$ so that $t^{-1}N_0t \subseteq s^{-k_0}N_0s^{k_0}$ and $t^{-1}N_1t \subseteq s^{-k_0}N_1s^{k_0}$ whence $s^{k_0}t^{-1}$ lies in $T_{+,\ell}$. Since M is a φ-module over $R[[N_1, \ell]]$, the map

$$\begin{aligned} 1 \otimes \varphi_{s^{k_0}}\colon R[[N_1, \ell]] \otimes_{R[[N_1,\ell]],\varphi_{s^{k_0}}} M &\to M \\ r \otimes m &\mapsto r\varphi_{s^{k_0}}(m) \end{aligned}$$

is an isomorphism. Moreover, under the identifications

$$R[\![N_1,\ell]\!]\otimes_{R[\![N_1,\ell]\!],\varphi_t}(R[\![N_1,\ell]\!]\otimes_{R[\![N_1,\ell]\!],\varphi_{s^{k_0}t^{-1}}}M)\cong R[\![N_1,\ell]\!]\otimes_{R[\![N_1,\ell]\!],\varphi_{s^{k_0}}}M\cong$$
$$\cong R[\![N_1,\ell]\!]\otimes_{R[\![N_1,\ell]\!],\varphi_{s^{k_0}t^{-1}}}(R[\![N_1,\ell]\!]\otimes_{R[\![N_1,\ell]\!],\varphi_t}M)$$

we have

$$(1\otimes\varphi_t)\circ(1\otimes(1\otimes\varphi_{s^{k_0}t^{-1}}))=1\otimes\varphi_{s^{k_0}}=(1\otimes\varphi_{s^{k_0}t^{-1}})\circ(1\otimes(1\otimes\varphi_t)),$$

so $1\otimes\varphi_t$ is surjective by the equality on the left and injective by the equality on the right. □

Remark 3.6. *Note that the action of $T_{0,\ell}$ on a $T_{+,\ell}$-module D over R is linear since $T_{0,\ell}$ acts trivially on R. Therefore this action extends (uniquely) to the subgroup $T_\ell\leq T$ generated by the monoid $T_{0,\ell}$.*

Proof. By the étaleness of the action of φ_t for $t\in T_{0,\ell}$ we see immediately that φ_t is an automorphism of D since $\varphi_t\colon R\to R$ is the identity map. Therefore φ_t has a (left and right) inverse (as a linear transformation of the R-module D) which we denote by $\varphi_{t^{-1}}$. The remark follows noting that T_ℓ consists of the quotients of elements of $T_{0,\ell}$. □

In the case when $\ell=\ell_\alpha$ given by the projection of $\prod_{\beta\in\Delta}N_\beta$ to N_α for some fixed simple root $\alpha\in\Delta$ it is clear that $T_{+,\ell}=T_+$ as N_β is T_+-invariant for each $\beta\in\Phi^+$ and $\mathrm{Ker}(\ell)=\prod_{\alpha\neq\beta\in\Phi^+}N_\beta$. Therefore $T_\ell\cong(\mathbb{Q}_p^\times)^{n-1}$ where $n=\dim T$. This is the case in which a G-equivariant sheaf on G/P is constructed in [19] associated to any object D in $\mathfrak{M}(\mathcal{O}_\mathcal{E},T_{+,\ell})$. So an object in $\mathfrak{M}(\mathcal{O}_\mathcal{E},T_{+,\ell})$ is nothing else but a (φ,Γ)-module over $\mathcal{O}_\mathcal{E}$ with an additional linear action of the group T_ℓ (once we fixed the cocharacter ξ). In case of $G=\mathrm{GL}_2(\mathbb{Q}_p)$ this additional action is just an action of the centre $Z=T_\ell$ of G. In the work of Colmez [8, 9] on the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ the action of Z on an irreducible 2-dimensional étale (φ,Γ)-module D is given by the determinant (ie. the action of $\mathbb{Q}_p^\times\cong Z$ on $\bigwedge^2 D$). It is unclear at this point whether the action of T_ℓ can be chosen canonically (in a similar fashion) for a given n-dimensional irreducible étale (φ,Γ)-module D.

As a corollary of Prop. 3.1 we obtain

Proposition 3.7. *The functors $\mathbb{D}=R\otimes_{R[\![N_1,\ell]\!],\ell}\cdot$ and $\mathbb{M}=R[\![N_1,\ell]\!]\otimes_{R,\iota}\cdot$ are quasi-inverse equivalences of categories between $\mathfrak{M}(R[\![N_1,\ell]\!],T_{+,\ell})$ and $\mathfrak{M}(R,T_{+,\ell})$.*

Proof. Since we clearly have $\mathbb{D}\circ\mathbb{M}\cong\mathrm{id}_{\mathfrak{M}(R,T_{+,\ell})}$ and the faithfulness of $\mathbb{D}$ is a formal consequence of Prop. 3.1, it suffices to show that the isomorphism $\Theta\colon M\to\mathbb{M}\circ\mathbb{D}(M)$ is $T_{+,\ell}$-equivariant whenever M lies in $\mathfrak{M}(R[\![N_1,\ell]\!],T_{+,\ell})$. Let $t\in T_{+,\ell}$ be arbitrary and for an $m\in M$ write $m=\sum_{u\in J(N_0/\varphi_s^k(N_0))}u\varphi_s^k(m_{u,k})$. Since $\mathbb{D}(\Theta)=\mathrm{id}_{\mathbb{D}(M)}$, we have $(\Theta\circ\varphi_t-\varphi_t\circ\Theta)(M)\subseteq I_1\mathbb{M}\circ\mathbb{D}(M)$. We compute

$$(\Theta\circ\varphi_t-\varphi_t\circ\Theta)(m)=\sum_{u\in J(N_0/\varphi_s^k(N_0))}\varphi_t(u)\varphi_s^k\circ(\Theta\circ\varphi_t-\varphi_t\circ\Theta)(m_{u,k})\subseteq$$
$$\subseteq\varphi_s^k(I_1\mathbb{M}\circ\mathbb{D}(M))\subseteq I_{k+1}\mathbb{M}\circ\mathbb{D}(M)$$

for all $k\geq 0$ showing that Θ is φ_t-equivariant. □

4 The case of overconvergent and Robba rings

4.1 The locally analytic distribution algebra

Let p be a prime and put $\epsilon_p = 1$ if p is odd and $\epsilon_p = 2$ if $p = 2$. If H is a compact locally $\mathbb{Q}_p$-analytic group then we denote by $D(H,K)$ the algebra of K-valued locally analytic distributions on H. Recall that $D(H,K)$ is equal to the strong dual of the locally convex vector space $C^{an}(H,K)$ of K-valued locally $\mathbb{Q}_p$-analytic functions on H with the convolution product.

Recall that a topologically finitely generated pro-p group H is uniform, if it is powerful (ie. $H/\overline{H^{p^{\epsilon_p}}}$ is abelian) and for all $i \geq 1$ we have $|P_i(H) : P_{i+1}(H)| = |H : P_2(H)|$ where $P_1(H) = H$ and $P_{i+1}(H) = \overline{P_i(H)^p[P_i(H),H]}$ (see [10] for more details). Now if H is uniform, then it has a bijective global chart

$$\begin{array}{rcl} \mathbb{Z}_p^d & \to & H \\ (x_1,\dots,x_d) & \mapsto & h_1^{x_1}\cdots h_d^{x_d} \end{array}$$

where $h_1,\dots,h_d$ is a fixed (ordered) minimal set of topological generators of H. Putting $b_i := h_i - 1 \in \mathbb{Z}[G]$, $\mathbf{b}^{\mathbf{k}} := b_1^{k_1}\cdots b_d^{k_d}$ for $\mathbf{k} = (k_i) \in \mathbb{N}^d$ we can identify $D(H,K)$ with the ring of all formal series

$$\lambda = \sum_{\mathbf{k}\in\mathbb{N}^d} d_{\mathbf{k}}\mathbf{b}^{\mathbf{k}}$$

with $d_{\mathbf{k}}$ in K such that the set $\{|d_{\mathbf{k}}|\rho^{\epsilon_p|\mathbf{k}|}\}_{\mathbf{k}}$ is bounded for all $0 < \rho < 1$. Here the first $|\cdot|$ is the normalized absolute value on K and the second one denotes the degree of $\mathbf{k}$, that is $\sum_i k_i$. For any ρ in $p^{\mathbb{Q}}$ with $p^{-1} < \rho < 1$, we have a multiplicative norm $\|\cdot\|_\rho$ on $D(H,K)$ [16] given by

$$\|\lambda\|_\rho := \sup_{\mathbf{k}} |d_{\mathbf{k}}|\rho^{\epsilon_p|\mathbf{k}|}\ .$$

The family of norms $\|\cdot\|_\rho$ defines the Fréchet topology on $D(H,K)$. The completion with respect to the norm $\|\cdot\|_\rho$ is denoted by $D_{[0,\rho]}(H,K)$.

4.2 Microlocalization

Let G be the group of $\mathbb{Q}_p$-points of a $\mathbb{Q}_p$-split connected reductive group with a fixed Borel subgroup $P = TN$. We also choose a simple root α for the Borel subgroup P and let $\ell = \ell_\alpha$ be the functional given by the projection

$$\ell_\alpha \colon N \to N/[N,N] \to \prod_{\beta\in\Delta} N_\beta \to N_\alpha \overset{\iota_\alpha}{\to} \mathbb{Q}_p\ .$$

Therefore we have $T_{+,\ell} = T_+$ as N_β is T_+-invariant for each $\beta \in \Phi^+$. We assume further that N_0 is *uniform*.

Let us begin by recalling the definition of the classical Robba ring for the group $\mathbb{Z}_p$. The distribution algebra $D(\mathbb{Z}_p,K)$ of $\mathbb{Z}_p$ can clearly be identified with the ring of power series (in variable T) with coefficients in K that are convergent in the p-adic open unit disc. Now put

$$\mathcal{A}_{[\rho,1)} := \text{the ring of all Laurent series } f(T) = \sum_{n\in\mathbb{Z}} a_nT^n \text{ that converge for } \rho \leq |T| < 1.$$

For $\rho \leq \rho'$ we have a natural inclusion $\mathcal{A}_{[\rho,1)} \hookrightarrow \mathcal{A}_{[\rho',1)}$ so we can form the inductive limit

$$\mathcal{R} := \varinjlim_{\rho \to 1} \mathcal{A}_{[\rho,1)}$$

defining the Robba ring. $\mathcal{R}$ is a (φ, Γ)-ring over $\mathbb{Z}_p$ with the maps $\chi\colon \mathbb{Z}_p \to \mathcal{R}^\times$ and $\varphi\colon \mathcal{R} \to \mathcal{R}$ such that $\chi(1) = 1 + T$, $\varphi(T) = (T+1)^p - 1$, and $\gamma(T) = (1+T)^{\xi^{-1}(\gamma)} - 1$ for $\gamma \in \Gamma$.

Recall that the ring

$$\mathcal{O}_{\mathcal{E}}^\dagger := \{\sum_{n \in \mathbb{Z}} a_n T^n \mid a_n \in o_K \text{ and there exists a } \rho < 1 \text{ s.t. } |a_n|\rho^n \to 0 \text{ as } n \to -\infty\}$$

is called the ring of overconvergent power series. It is a subring of both $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{R}$. We put $\mathcal{E}^\dagger := K \otimes_{o_K} \mathcal{O}_{\mathcal{E}}^\dagger$ which is also a subring of the Robba ring. These rings are also (φ, Γ)-rings.

The rings $\mathcal{O}_{\mathcal{E}}^\dagger[[N_1, \ell]]$ and $\mathcal{R}[[N_1, \ell]]$ constructed in the previous sections are only overconvergent (resp. Robba) in the variable b_α for the fixed simple root α. In all the other variables b_β they behave like the Iwasawa algebra $\Lambda(N_1)$ since we took the completion with respect to the ideals generated by $(N_k - 1)$. Moreover, in the projective limit $\mathcal{O}_{\mathcal{E}}^\dagger[[N_1, \ell]] \cong \varprojlim_k \mathcal{O}_{\mathcal{E}}^\dagger[N_1/N_k, \ell]$ the terms are not forced to share a common region of convergence. In this section we construct the rings $\mathcal{R}^{int}(N_1, \ell)$ and $\mathcal{R}(N_1, \ell)$ with better analytic properties.

We start with constructing a ring $\mathfrak{R}_0 = \mathfrak{R}_0(N_0, K, \alpha)$ as a certain microlocalization of the distribution algebra $D(N_0, K)$. We fix the topological generator n_α of $N_0 \cap N_\alpha$ such that $\ell_\alpha(n_\alpha) = 1$. This is possible since we normalized $\iota_\alpha\colon N_\alpha \xrightarrow{\sim} \mathbb{Q}_p$ so that $\iota_\alpha(N_0 \cap N_\alpha) = \mathbb{Z}_p$. Further, we fix topological generators n_β of $N_0 \cap N_\beta$ for each $\alpha \neq \beta \in \Phi^+$. Since N_0 is uniform of dimension $|\Phi^+|$, the set $A := \{n_\beta \mid \beta \in \Phi^+\}$ is a minimal set of topological generators of the group N_0. Moreover, $A \setminus \{n_\alpha\}$ is a minimal set of generators of the group $N_1 = \mathrm{Ker}(\ell) \cap N_0$. Further, we put $b_\beta := n_\beta - 1$. For any real number $p^{-1} < \rho < 1$ in $p^{\mathbb{Q}}$ the formula $\|b_\beta\|_\rho := \rho$ (for all $\beta \in \Phi^+$) defines a multiplicative norm on $D(N_0, K)$. The completion of $D(N_0, K)$ with respect to this norm is a Banach algebra which we denote by $D_{[0,\rho]}(N_0, K)$. Let now $p^{-1} < \rho_1 < \rho_2 < 1$ be real numbers in $p^{\mathbb{Q}}$. We take the generalized microlocalization (cf. the Appendix of [20]) of the Banach algebra $D_{[0,\rho_2]}(N_0, K)$ at the multiplicatively closed set $\{(n_\alpha - 1)^i\}_{i \geq 1}$ with respect to the pair of norms (ρ_1, ρ_2). This provides us with the Banach algebra $D_{[\rho_1,\rho_2]}(N_0, K, \alpha)$. Recall that the elements of this Banach algebra are equivalence classes of Cauchy sequences $((n_\alpha - 1)^{-k_n} x_n)_n$ (with $x_n \in D_{[0,\rho_2]}(N_0, K)$) with respect to the norm $\|\cdot\|_{\rho_1,\rho_2} := \max(\|\cdot\|_{\rho_1}, \|\cdot\|_{\rho_2})$.

Letting ρ_2 tend to 1 we define $D_{[\rho_1,1)}(N_0, K, \alpha) := \varprojlim_{\rho_2 \to 1} D_{[\rho_1,\rho_2]}(N_0, K, \alpha)$. This is a Fréchet-Stein algebra (the proof is completely analogous to that of Theorem 5.5 in [20], but it is not a formal consequence of that). However, we will not need this fact in the sequel so we omit the proof. Now the partial Robba ring $\mathfrak{R}_0 := \mathfrak{R}_0(N_0, K, \alpha) := \varinjlim_{\rho_1 \to 1} D_{[\rho_1,1)}(N_0, K, \alpha)$ is defined as the injective limit of these Fréchet-Stein algebras. We equip $\mathfrak{R}_0$ with the inductive limit topology of the Fréchet topologies of $D_{[\rho_1,1)}(N_0, K, \alpha)$. By the following parametrization the partial Robba ring can be thought of as a skew Laurent series ring on the variables b_β ($\beta \in \Phi^+$) with certain convergence conditions such that only the variable b_α is invertible. Note that in [20] a "full" Robba ring is constructed such that all the variables b_β are invertible. We denote the corresponding "fully" microlocalized Banach algebras by $D_{[\rho_1,\rho_2]}(N_0, K)$. In all these rings we will often omit K from the notation if it is clear from the context.

Remark 4.1. *The microlocalization of quasi-abelian normed algebras (Appendix of [20]) is somewhat different from the microlocalisation constructing $\Lambda_\ell(N_0)$ where first a localization* *(with respect to an Ore set) is constructed and then the completion is taken. The set we are inverting here does not satisfy the Ore property, so the localization in the usual sense does not exist. However, we may complete and localize at the same time in order to obtain a microlocalized ring directly.*

In order to be able to work with these rings we will show that their elements can be viewed as Laurent series. The discussion below is completely analogous to the discussion before Prop. A.24 in [20]. However, for the convenience of the reader, we explain the method specialized to our case here. We introduce the affinoid domain

$$A_\alpha[\rho_1, \rho_2] := \{(z_\beta)_{\beta\in\Phi^+} \in \mathbb{C}_p^{\Phi^+} : \rho_1 \leq |z_\alpha| \leq \rho_2, 0 \leq |z_\beta/z_\alpha| \leq 1 \text{ for } \alpha \neq \beta \in \Phi^+\} \ .$$

This has the affinoid subdomain

$$X^{\Phi^+}_{[\rho_1,\rho_2]} := \{(z_\beta)_{\beta\in\Phi^+} \in \mathbb{C}_p^{\Phi^+} : \rho_1 \leq |z_{\beta_1}| = \ldots = |z_{\beta_{|\Phi^+|}}| \leq \rho_2\}$$

(where $\{\beta_1, \ldots, \beta_{|\Phi^+|}\} = \Phi^+$) as defined in [20] (Prop. A.24).

Lemma 4.2. *The ring $\mathcal{O}_K(A_\alpha[\rho_1, \rho_2])$ of K-analytic functions on $A_\alpha[\rho_1, \rho_2]$ is the ring of all Laurent series*

$$f(\mathbf{Z}) = \sum_{\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}} d_{\mathbf{k}}\mathbf{Z}^{\mathbf{k}}$$

with $d_{\mathbf{k}} \in K$ and such that $\lim_{\mathbf{k}\to\infty} |d_{\mathbf{k}}|\rho^{\mathbf{k}} = 0$ for any $\rho_1 \leq \rho \leq \rho_2$. Here

$$\mathbf{Z}^{\mathbf{k}} := \prod_{\beta\in\Phi^+} Z_\beta^{k_\beta} \qquad \text{and} \qquad \rho^{\mathbf{k}} := \rho^{\sum_{\beta\in\Phi^+} k_\beta}$$

and $\mathbf{k} \to \infty$ means that $\sum_{\beta\in\Phi^+} |k_\beta| \to \infty$. This is the subring of $\mathcal{O}_K(X^{\Phi^+}_{[\rho_1,\rho_2]})$ consisting of elements in which the variables Z_β appear only with nonnegative exponent for all $\alpha \neq \beta \in \Phi^+$.

Proof. Since $X^{\Phi^+}_{[\rho_1,\rho_2]} \subseteq A_\alpha[\rho_1, \rho_2]$, we clearly have $\mathcal{O}_K(A_\alpha[\rho_1, \rho_2]) \subseteq \mathcal{O}_K(X^{\Phi^+}_{[\rho_1,\rho_2]})$. Moreover, the power series in $\mathcal{O}_K(A_\alpha[\rho_1, \rho_2])$ converge for $z_\beta = 0$ ($\beta \neq \alpha$), hence these variables appear with nonnegative exponent. On the other hand, if we have a power series $f(\mathbf{Z}) \in \mathcal{O}_K(X^{\Phi^+}_{[\rho_1,\rho_2]})$ such that the variables Z_β have nonnegative exponent for all $\alpha \neq \beta \in \Phi^+$ then it also converges in the region $A_\alpha[\rho_1, \rho_2]$ as we have the trivial estimate $|\prod_{\beta\in\Phi^+} z_\beta^{k_\beta}| \leq |z_\alpha|^{\sum_{\beta\in\Phi^+} k_\beta}$ in this case. □

Since $\rho^{\mathbf{k}} \leq \max(\rho_1^{\mathbf{k}}, \rho_2^{\mathbf{k}})$ for any $\rho_1 \leq \rho \leq \rho_2$ and any $\mathbf{k} \in \mathbb{Z}^{(\alpha)} \times \mathbb{N}^{\Phi^+\setminus\{\alpha\}}$ the convergence condition on f is equivalent to

$$\lim_{\mathbf{k}\to\infty} |d_{\mathbf{k}}|\rho_1^{\mathbf{k}} = \lim_{\mathbf{k}\to\infty} |d_{\mathbf{k}}|\rho_2^{\mathbf{k}} = 0 \ .$$

The spectral norm on the affinoid algebra $\mathcal{O}_K(A_\alpha[\rho_1, \rho_2])$ (for the definition of these notions see [12]) is given by

$$\begin{aligned}\|f\|_{A_\alpha[\rho_1,\rho_2]} &= \sup_{\rho_1\leq\rho\leq\rho_2} \max_{\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}} |d_{\mathbf{k}}|\rho^{\mathbf{k}} \\ &= \max\left(\max_{\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}} |d_{\mathbf{k}}|\rho_1^{\mathbf{k}}, \max_{\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}} |d_{\mathbf{k}}|\rho_2^{\mathbf{k}}\right) \ .\end{aligned}$$

Setting $\mathbf{b}^{\mathbf{k}} := \prod_{\beta\in\Phi^+} b_\beta^{k_\beta}$ for some fixed ordering of Φ^+ and for any $\mathbf{k} = (k_\beta)_{\beta\in\Phi^+} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+\setminus\{\alpha\}}$ we claim that $f(\mathbf{b}) := \sum_{\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}} d_{\mathbf{k}}\mathbf{b}^{\mathbf{k}}$ converges in $D_{[\rho_1,\rho_2]}(N_0,K,\alpha)$ for $f \in \mathcal{O}_K(A_\alpha[\rho_1,\rho_2])$. As a consequence of Prop. A.21 and Lemma A.7.iii in [20] we have

$$\|\mathbf{b}^{\mathbf{k}}\|_{\rho_1,\rho_2} = \max(\rho_1^{\mathbf{k}}, \rho_2^{\mathbf{k}})$$

for any $\mathbf{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+\setminus\{\alpha\}}$. Hence

$$\lim_{\mathbf{k}\to\infty} \|d_{\mathbf{k}}\mathbf{b}^{\mathbf{k}}\|_{\rho_1,\rho_2} = \lim_{\mathbf{k}\to\infty} \max(|d_{\mathbf{k}}|\rho_1^{\mathbf{k}}, |d_{\mathbf{k}}|\rho_2^{\mathbf{k}}) = \max(\lim_{\mathbf{k}\to\infty} |d_{\mathbf{k}}|\rho_1^{\mathbf{k}}, \lim_{\mathbf{k}\to\infty} |d_{\mathbf{k}}|\rho_2^{\mathbf{k}}) = 0\ .$$

Therefore

$$\begin{aligned} \mathcal{O}_K(A_\alpha[\rho_1,\rho_2]) &\longrightarrow D_{[\rho_1,\rho_2]}(N_0,K,\alpha) \\ f &\longmapsto f(\mathbf{b}) \end{aligned}$$

is a well defined K-linear map. In order to investigate this map we introduce the filtration

$$F^i D_{[\rho_1,\rho_2]}(N_0,K,\alpha) := \{e \in D_{[\rho_1,\rho_2]}(N_0,K,\alpha) : \|e\|_{\rho_1,\rho_2} \le |p|^i\} \qquad \text{for } i \in \mathbb{R}$$

on $D_{[\rho_1,\rho_2]}(N_0,K,\alpha)$. Since K is discretely valued and $\rho_1,\rho_2 \in p^{\mathbb{Q}}$ this filtration is quasi-integral in the sense of [16] §1. The corresponding graded ring $gr^\cdot D_{[\rho_1,\rho_2]}(N_0,K,\alpha)$, by Prop. A.21 in [20], is commutative. We let $\sigma(e) \in gr^\cdot D_{[\rho_1,\rho_2]}(N_0,K,\alpha)$ denote the principal symbol of any element $e \in D_{[\rho_1,\rho_2]}(N_0,K,\alpha)$.

Proposition 4.3. *i. $gr^\cdot D_{[\rho_1,\rho_2]}(N_0,K,\alpha)$ is a free $gr^\cdot K$-module with basis $\{\sigma(\mathbf{b}^{\mathbf{k}}) : \mathbf{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+\setminus\{\alpha\}}\}$.*

ii. The map

$$\begin{aligned} \mathcal{O}_K(A_\alpha[\rho_1,\rho_2]) &\xrightarrow{\cong} D_{[\rho_1,\rho_2]}(N_0,K,\alpha) \\ f &\longmapsto f(\mathbf{b}) \end{aligned}$$

is a K-linear isometric bijection.

Proof. Since $\{b_\alpha^{-l}\mu : l \ge 0, \mu \in D_{[0,\rho_1]}(N_0,K)\}$ is dense in $D_{[\rho_1,\rho_2]}(N_0,K,\alpha)$ every element in the graded ring $gr^\cdot D_{[\rho_1,\rho_2]}(N_0,K,\alpha)$ is of the form $\sigma(b_\alpha^{-l}\mu)$. Suppose that $\mu = \sum_{\mathbf{k}\in\mathbb{N}_0^d} d_{\mathbf{k}}\mathbf{b}^{\mathbf{k}}$. Then $b_\alpha^{-l}\mu = \sum_{\mathbf{k}\in\mathbb{N}_0^d} d_{\mathbf{k}} b_\alpha^{-l}\mathbf{b}^{\mathbf{k}}$ and, using Lemma A.7.iii [20] we compute

$$\begin{aligned} \|b_\alpha^{-l}\mu\|_{\rho_1,\rho_2} &= \max(\|b_\alpha\|_{\rho_1}^{-l}\|\mu\|_{\rho_1}, \|b_\alpha\|_{\rho_2}^{-l}\|\mu\|_{\rho_2}) = \max(\max_{\mathbf{k}\in\mathbb{N}_0^d} |d_{\mathbf{k}}|\rho_1^{\mathbf{k}-l}, \max_{\mathbf{k}\in\mathbb{N}_0^d} |d_{\mathbf{k}}|\rho_2^{\mathbf{k}-l}) \\ &= \max_{\mathbf{k}\in\mathbb{N}_0^d} |d_{\mathbf{k}}| \max(\rho_1^{\mathbf{k}-l}, \rho_2^{\mathbf{k}-l}) = \max_{\mathbf{k}\in\mathbb{N}_0^d} |d_{\mathbf{k}}| |b_\alpha^{-l}\mathbf{b}^{\mathbf{k}}|_{\rho_1,\rho_2}\ . \end{aligned}$$

It follows that $gr^\cdot D_{[\rho_1,\rho_2]}(N_0,K,\alpha)$ as a $gr^\cdot K$-module is generated by the principal symbols $\sigma(b_\alpha^{-l}\mathbf{b}^{\mathbf{k}})$ with $\mathbf{k} \in \mathbb{N}_0^d$, $l \ge 0$. But it also follows that, for a fixed $l \ge 0$, the principal symbols $\sigma(b_\alpha^{-l}\mathbf{b}^{\mathbf{k}})$ with $\mathbf{k}$ running over $\mathbb{N}_0^d$ are linearly independent over $gr^\cdot K$. By Prop. A.21 in [20] we may permute the factors in $\sigma(b_\alpha^{-l}\mathbf{b}^{\mathbf{k}})$ arbitrarily. Hence $gr^\cdot D_{[\rho_1,\rho_2]}(N_0,K,\alpha)$ is a free $gr^\cdot K$-module with basis $\{\sigma(\mathbf{b}^{\mathbf{k}}) : \mathbf{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+\setminus\{\alpha\}}\}$.

On the other hand, we of course have

$$\begin{aligned}\|f(\mathbf{b})\|_{\rho_1,\rho_2} &\leq \max_{\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}} |d_{\mathbf{k}}||\mathbf{b}^{\mathbf{k}}|_{\rho_1,\rho_2}\\ &= \max_{\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}} |d_{\mathbf{k}}|\max(\rho_1^{\mathbf{k}},\rho_2^{\mathbf{k}})\\ &= \max(\max_{\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}} |d_{\mathbf{k}}|\rho_1^{\mathbf{k}}, \max_{\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}} |d_{\mathbf{k}}|\rho_2^{\mathbf{k}})\\ &= |f|_{A_\alpha[\rho_1,\rho_2]}\ .\end{aligned}$$

This means that if we introduce on $\mathcal{O}_K(A_\alpha[\rho_1,\rho_2])$ the filtration defined by the spectral norm then the asserted map respects the filtrations, and by the above reasoning it induces an isomorphism between the associated graded rings. Hence, by completeness of these filtrations, it is an isometric bijection. □

Now we turn to the construction of $\mathcal{R}(N_1,\ell)$. The problem with (naïve) microlocalization is that the ring $\mathfrak{R}_0$ is not finitely generated over $\varphi(\mathfrak{R}_0)$. The reason for this is that φ improves the order of convergence for a power series in $\mathfrak{R}_0$. In the case $G\neq \mathrm{GL}_2(\mathbb{Q}_p)$ the operator $\varphi=\varphi_s$ acts by conjugation on N_β by raising to the $\beta(s)$-th power. Whenever $\beta\in\Phi^+\setminus\Delta$ is not a simple root then $\beta(s)=p^{m_\beta}>\alpha(s)=p$ where m_β is the degree of the map $\beta\circ\xi\colon\mathbb{G}_m\to\mathbb{G}_m$.

Lemma 4.4. *We have $\|b_\beta\|_\rho=\|b_\alpha\|_\rho=\rho$ and*

$$\|\varphi(b_\beta)\|_\rho=\max_{0\leq j\leq m_\beta}(\rho^{p^j}p^{j-m_\beta})<\max(\rho^p,p^{-1}\rho)=\|\varphi(b_\alpha)\|_\rho$$

for any $p^{-1}<\rho<1$. In general, we have $\|\varphi_t(b_\beta)\|_\rho=\max_{0\leq j\leq \mathrm{val}_p(\beta(t))}(\rho^{p^j}p^{j-\mathrm{val}_p(\beta(t))})$.

Proof. We compute

$$\|\varphi_t(b_\beta)\|_\rho=\|(1+b_\beta)^{\beta(t)}-1\|_\rho=\|\sum_{i=1}^{\infty}\binom{\beta(t)}{i}b_\beta^i\|_\rho=\max_{0\leq j\leq \mathrm{val}_p(\beta(t))}(\rho^{p^j}p^{j-\mathrm{val}_p(\beta(t))})\ .$$

Here we use the trivial estimate $\mathrm{val}_p\binom{n}{k}=\mathrm{val}_p(\frac{n}{k}\binom{n-1}{k-1})\geq\mathrm{val}_p(n)-\mathrm{val}_p(k)$ for $n:=\beta(t)\in\mathbb{Z}_p$ and $k\in\mathbb{N}$. We see immediately that whenever $m_\beta>1$ then $\rho^{p^j}p^{j-m_\beta}<\rho^p$ for $1\leq j\leq m_\beta$ and $p^{-m_\beta}\rho<p^{-1}\rho$. □

Now choose an ordering $<$ on Φ^+ such that (i) $m_{\beta_1}<m_{\beta_2}$ implies $\beta_1>\beta_2$ and (ii) $\alpha>\beta$ for any $\alpha\neq\beta,\beta_1,\beta_2\in\Phi^+$. Then by Prop. 4.3 any element in $\mathfrak{R}_0$ has a skew Laurent-series expansion

$$f(\mathbf{b})=\sum_{\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}}c_{\mathbf{k}}\mathbf{b}^{\mathbf{k}}$$

such that there exists $p^{-1}<\rho<1$ such that for all $\rho<\rho_1<1$ we have $|c_{\mathbf{k}}|_p\rho_1^{\sum k_\beta}\to 0$ as $\sum|k_\beta|\to\infty$. By Lemma 4.4 and the discussion above we clearly have the following

Example 4.5. *Let $\beta\in\Phi^+\setminus\Delta$ be a non-simple root. Then the series $\sum_{n=1}^\infty b_\beta^n b_\alpha^{-n}$ does not belong to $\mathfrak{R}_0(N_0)$. However, the series $\sum_{n=1}^\infty\varphi(b_\beta^n b_\alpha^{-n})$ converges in each $D_{[\rho_1,\rho_2]}(N_0,\alpha)$ (for arbitrary $p^{-1}<\rho_1<\rho_2<1$) hence defines an element in $\mathfrak{R}_0(N_0)$. Therefore we cannot have a continuous left inverse ψ to φ on $\mathfrak{R}_0(N_0)$ as otherwise $\psi(\sum_{n=1}^\infty\varphi(b_\beta^n b_\alpha^{-n}))=$*

$\sum_{n=1}^{\infty} b_\beta^n b_\alpha^{-n}$ *would converge. In particular, we cannot write* $\mathfrak{R}_0(N_0)$ *as the topological direct sum* $\bigoplus_{u\in N_0/\varphi(N_0)} u\varphi(\mathfrak{R}_0(N_0))$ *of closed subspaces in* $\mathfrak{R}_0(N_0)$ *as otherwise the operator*

$$\begin{aligned} \psi\colon \mathfrak{R}_0(N_0) &\to \mathfrak{R}_0(N_0) \\ \sum_{u\in J(N_0/\varphi(N_0))} u\varphi(f_u) &\mapsto \varphi^{-1}(u_0)f_{u_0} \end{aligned}$$

for the unique $u_0 \in J(N_0/\varphi(N_0)) \cap \varphi(N_0)$ *would be a continuous left inverse to* φ*. In fact, we even have* $\mathfrak{R}_0(N_0) \neq \bigoplus_{u\in N_0/\varphi(N_0)} u\varphi(\mathfrak{R}_0(N_0))$ *algebraically, however, the proof of this requires the forthcoming machinery (see Remark 4.10).*

In order to overcome the above counter-example we are going to consider the ring $\mathcal{R}(N_1,\ell)$ of all the skew power series of the form $f(\mathbf{b})$ such that $f(\varphi_t(\mathbf{b}))$ is convergent in $\mathfrak{R}_0$ for some $t \in T_+$. A priori it is not clear that these series form a ring, so we are going to give a more conceptual construction.

Take an arbitrary element $t \in T_+$. The conjugation by t on N_0 gives an isomorphism $\varphi_t\colon N_0 \to \varphi_t(N_0)$ of pro-p groups (since it is injective). Hence $\varphi_t(N_0)$ is also a uniform pro-p group with minimal set of generators $\{\varphi_t(n_\beta)\}_{\beta\in\Phi^+}$. So we may define the distribution algebra $D(\varphi_t(N_0)) := D(\varphi_t(N_0), K)$. The inclusion $\varphi_t(N_0) \hookrightarrow N_0$ induces an injective homomorphism of Fréchet-algebras $\iota_{1,t}\colon D(\varphi_t(N_0)) \hookrightarrow D(N_0)$. It is well-known [16] that we have

$$D(N_0) = \bigoplus_{n\in J(N_0/\varphi_t(N_0))} n\iota_{1,t}(D(\varphi_t(N_0)))$$

as right $D(\varphi_t(N_0))$-modules. Moreover, the direct summands are closed in $D(N_0)$. For each real number $p^{-1} < \rho < 1$ the ρ-norm on $D(N_0)$ defines a norm $r_t(\rho)$ on $D(\varphi_t(N_0))$ by restriction. Note that this is different from the ρ-norm on $D(\varphi_t(N_0))$ (using the uniform structure on $\varphi_t(N_0)$). However, the family $(r_t(\rho))_\rho$ of norms defines the Fréchet topology on $D(\varphi_t(N_0))$. On the other hand, whenever r is a norm on $D(\varphi_t(N_0))$ then we may extend r to a norm $q_t(r)$ on $D(N_0)$ by putting

$$\Big\| \sum_{n\in J(N_0/\varphi_t(N_0))} n\iota_{1,t}(x_n)\Big\|_{q_t(r)} := \max(\|x_n\|_r).$$

These norms define the Fréchet topology on $D(N_0)$. More precisely, if $\beta(t) = p^{m(\beta,t)}u(\beta,t)$ with $m(\beta,t) := \mathrm{val}_p(\beta(t)) \geq 0$ integer and $u(\beta,t) \in \mathbb{Z}_p^\times$, then we have

Lemma 4.6.

$$\|x\|_\rho \leq \|x\|_{q_t(r_t(\rho))} \leq \rho^{-\sum_{\beta\in\Phi^+}(p^{m(\beta,t)}-1)}\|x\|_\rho$$

for any $p^{-\frac{1}{\max_{\beta\in\Phi^+} p^{m(\beta,t)}}} < \rho < 1$ *and* $x \in D(N_0)$*. In particular, the norms* ρ *and* $q_t(r_t(\rho))$ *define the same topology.*

Proof. The inequality on the left is clear from the triangle inequality. For the other inequality note that our assumption on ρ implies in particular that

$$\rho^{p^{m(\beta,t)}} = \rho^{p^j}\rho^{p^{m(\beta,t)}-p^j} > \rho^{p^j}p^{-\frac{p^{m(\beta,t)}-p^j}{p^{m(\beta,t)}}} > \rho^{p^j}p^{j-m(\beta,t)}$$

for all $0 \leq j < m(\beta,t)$. Hence by Lemma 4.4, we have $\rho^{p^{m(\beta,t)}} = \|\binom{\beta(t)}{p^{m(\beta,t)}} b_\beta^{p^{m(\beta,t)}}\|_\rho = \|\varphi_t(b_\beta)\|_\rho$. Moreover, there exists an *invertible* element y in the Iwasawa algebra $\Lambda(N_{0,\beta})$ such that $y\varphi_t(b_\beta) \equiv \binom{\beta(t)}{p^{m(\beta,t)}} b_\beta^{p^{m(\beta,t)}} \pmod p$ (as both sides have the same principal term). However, by the choice of ρ, $|p| = 1/p < \rho^{p^{m(\beta,t)}} = \|\varphi_t(b_\beta)\|_\rho = \|\varphi_t(b_\beta)\|_{q_t(r_t(\rho))}$. Therefore we also have

$$\rho^{p^{m(\beta,t)}} = \|b_\beta^{p^{m(\beta,t)}}\|_\rho = \|\binom{\beta(t)}{p^{m(\beta,t)}} b_\beta^{p^{m(\beta,t)}}\|_\rho = \|\varphi_t(b_\beta)\|_\rho = \|\varphi_t(b_\beta)\|_{q_t(r_t(\rho))} =$$
$$= \|y\varphi_t(b_\beta)\|_{q_t(r_t(\rho))} = \|\binom{\beta(t)}{p^{m(\beta,t)}} b_\beta^{p^{m(\beta,t)}}\|_{q_t(r_t(\rho))} = \|b_\beta^{p^{m(\beta,t)}}\|_{q_t(r_t(\rho))}$$

whence

$$\|b_\beta^{k_\beta}\|_{q_t(r_t(\rho))} = \|b_\beta^{p^{m(\beta,t)}}\|_{q_t(r_t(\rho))}^{k_{1,\beta}} \|b_\beta^{k_{2,\beta}}\|_{q_t(r_t(\rho))} \leq \rho^{k_{1,\beta}p^{m(\beta,t)}} \leq \rho^{-p^{m(\beta,t)}+1}\|b_\beta^{k_\beta}\|_\rho \tag{12}$$

where $k_\beta = p^{m(\beta,t)}k_{1,\beta} + k_{2,\beta}$ with $0 \leq k_{2,\beta} \leq p^{m(\beta,t)} - 1$ and $k_{1,\beta}$ nonnegative integers.

Now consider an element of $D(N_0)$ of the form

$$x = \sum_{\mathbf{k}=(k_\beta)\in\mathbb{N}^{\Phi^+}} c_{\mathbf{k}} \prod_{\beta\in\Phi^+} b_\beta^{k_\beta} \ .$$

We may assume without loss of generality that $J(N_0/\varphi_t(N_0)) = \{\prod_{\beta\in\Phi^+} n_\beta^{j_\beta} \mid 0 \leq j_\beta \leq p^{m(\beta,t)} - 1\}$ where the product is taken in the reversed order. Let $\eta \in \Phi^+$ be the largest root (with respect to the ordering $<$ defined after Lemma 4.4) such that there exists a $\mathbf{k} \in \mathbb{N}^{\Phi^+}$ with $c_{\mathbf{k}} \neq 0$ and $k_\eta \neq 0$. We are going to show the estimate

$$\|x\|_{q_t(r_t(\rho))} \leq \rho^{-\sum_{\beta\leq\eta}(p^{m(\beta,t)}-1)}\|x\|_\rho$$

by induction on η. This induction has in fact finitely many steps since $|\Phi^+| < \infty$. At first we write $b_\eta^{k_\eta} = \sum_{j_\eta=0}^{p^{m(\eta,t)}-1} n_\eta^{j_\eta} f_{\mathbf{k},j_\eta}(\varphi_t(b_\eta))$ for each $\mathbf{k} \in \mathbb{N}^{\Phi^+}$. Note that—by the choice of the ordering on Φ^+—for any fixed η the set $\prod_{\beta<\eta} N_{0,\beta}$ is a normal subgroup of N_0. Moreover, the conjugation by any element of N_0 preserves the ρ-norm on $D(N_0)$. Therefore we may write

$$\prod_{\beta\leq\eta} b_\beta^{k_\beta} = \sum_{j_\eta=0}^{p^{m(\eta,t)}-1} n_\eta^{j_\eta} x_{\mathbf{k},j_\eta} f_{\mathbf{k},j_\eta}(\varphi_t(b_\eta))$$

such that

$$x_{\mathbf{k},j_\eta} := n_\eta^{-j_\eta}\left(\prod_{\beta<\eta} b_\beta^{k_\beta}\right) n_\eta^{j_\eta} \in D(\prod_{\beta<\eta} N_{0,\beta}) \ .$$

By (12) we have

$$\|f_{\mathbf{k},j_\eta}(\varphi_t(b_\eta))\|_{q_t(r_t(\rho))} = \|f_{\mathbf{k},j_\eta}(\varphi_t(b_\eta))\|_\rho \leq \|b_\eta^{k_\eta}\|_{q_t(r_t(\rho))} \leq \rho^{-p^{m(\eta,t)}+1}\|b_\eta^{k_\eta}\|_\rho \ .$$

Since the $r_t(\rho)$-norm is multiplicative on $D(\varphi_t(N_0))$, for any $a \in D(N_0)$ and $b \in D(\varphi_t(N_0))$ we also have $\|a\iota_{1,t}(b)\|_{q_t(r_t(\rho))} = \|a\|_{q_t(r_t(\rho))}\|b\|_{r_t(\rho)}$. Indeed, if we decompose a as $a =$

$\sum_{n\in J(N_0/\varphi_t(N_0))} n\iota_{1,t}(a_n)$ then we have $a\iota_{1,t}(b) = \sum_{n\in J(N_0/\varphi_t(N_0))} n\iota_{1,t}(a_n b)$. Now $f_{\mathbf{k},j_\eta}(\varphi_t(b_\eta))$ lies in $\iota_{1,t}(D(\varphi_t(N_0)))$, so we see that

$$\|x_{\mathbf{k},j_\eta} f_{\mathbf{k},j_\eta}(\varphi_t(b_\eta))\|_{q_t(r_t(\rho))} = \|x_{\mathbf{k},j_\eta}\|_{q_t(r_t(\rho))} \|f_{\mathbf{k},j_\eta}(\varphi_t(b_\eta))\|_{q_t(r_t(\rho))} \ .$$

On the other hand, the inductional hypothesis tells us that

$$\|x_{\mathbf{k},j_\eta}\|_{q_t(r_t(\rho))} \leq \rho^{-\sum_{\beta<\eta}(p^{m(\beta,t)}-1)} \|x_{\mathbf{k},j_\eta}\|_\rho = \rho^{-\sum_{\beta<\eta}(p^{m(\beta,t)}-1)} \|\prod_{\beta<\eta} b_\beta^{k_\beta}\|_\rho \ .$$

Hence we compute

$$\|x\|_{q_t(r_t(\rho))} = \|\sum_{\mathbf{k}} c_{\mathbf{k}} \sum_{j_\eta=0}^{p^{m(\eta,t)}-1} n_\eta^{j_\eta} x_{\mathbf{k},j_\eta} f_{\mathbf{k},j_\eta}(\varphi_t(b_\eta))\|_{q_t(r_t(\rho))} \leq$$

$$\leq \max_{\mathbf{k},j_\eta} \left(|c_{\mathbf{k}}| \|x_{\mathbf{k},j_\eta} f_{\mathbf{k},j_\eta}(\varphi_t(b_\eta))\|_{q_t(r_t(\rho))}\right) \leq$$

$$\leq \max_{\mathbf{k}} \left(|c_{\mathbf{k}}| \rho^{-\sum_{\beta<\eta}(p^{m(\beta,t)}-1)} \|\prod_{\beta<\eta} b_\beta^{k_\beta}\|_\rho \rho^{-p^{m(\eta,t)}+1} \|b_\eta^{k_\eta}\|_\rho \right) = \rho^{-\sum_{\beta\leq\eta}(p^{m(\beta,t)}-1)} \|x\|_\rho \ .$$

□

In particular, for each $p^{-\frac{1}{\max_\beta p^{m(\beta,t)}}} < \rho < 1$ the completion of $D(N_0)$ with respect to the topology defined by $\|\cdot\|_\rho$ and by $\|\cdot\|_{q_t(r_t(\rho))}$ are the same, ie.

$$D_{[0,\rho]}(N_0) = \bigoplus_{n\in J(N_0/\varphi_t(N_0))} n\iota_{1,t}(D_{r_t([0,\rho])}(\varphi_t(N_0))) \tag{13}$$

where $D_{r_t([0,\rho])}(\varphi_t(N_0))$ denotes the completion of $D(\varphi_t(N_0))$ with respect to the norm $r_t(\rho)$.

Now we turn to the microlocalization and first of all note that $\varphi_t(b_\alpha) = (b_\alpha+1)^{\alpha(t)} - 1$ is divisible by b_α. So if $\varphi_t(b_\alpha)$ is invertible in a ring then so is b_α. On the other hand, if $p^{-\frac{1}{p^{m(\alpha,t)}}} < \rho < 1$ then by Lemma 4.4 we have

$$\|\varphi_t(b_\alpha) - \binom{\alpha(t)}{p^{m(\alpha,t)}} b_\alpha^{p^{m(\alpha,t)}}\|_\rho < \|\binom{\alpha(t)}{p^{m(\alpha,t)}} b_\alpha^{p^{m(\alpha,t)}}\|_\rho \ .$$

Hence $\varphi_t(b_\alpha)$ is invertible in the Banach algebra $D_{[\rho_1,\rho_2]}(N_0,\alpha)$ for any $p^{-\frac{1}{p^{m(\alpha,t)}}} < \rho_1 < \rho_2 < 1$ since it is close to the invertible element $\binom{\alpha(t)}{p^{m(\alpha,t)}} b_\alpha^{p^{m(\alpha,t)}}$ (as the binomial coefficient $\binom{\alpha(t)}{p^{m(\alpha,t)}}$ is not divisible by p). This shows that the microlocalisation of $D_{[0,\rho_2]}(N_0)$ with respect to the multiplicative set $\varphi_t(b_\alpha)^{\mathbb{N}}$ and norm $\max(\rho_1,\rho_2)$ equals $D_{[\rho_1,\rho_2]}(N_0,\alpha)$. Therefore for each $p^{-\frac{1}{p^{m(\alpha,t)}}} < \rho_1 < \rho_2 < 1$ we obtain

$$D_{[\rho_1,\rho_2]}(N_0,\alpha) = \bigoplus_{n\in J(N_0/\varphi_t(N_0))} n\iota_{0,1}(D_{r_t([\rho_1,\rho_2])}(\varphi_t(N_0),\alpha))$$

by microlocalizing both sides of (13). Now letting ρ_2 tend to 1 and then also $\rho_1 \to 1$ we get

$$\mathfrak{R}_0(N_0,\alpha) = \bigoplus_{n\in J(N_0/\varphi_t(N_0))} n\iota_{1,t}(\mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)) \tag{14}$$

for all $t \in T_+$. Here we define

$$\mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha) := \varinjlim_{\rho_1\to 1}\varprojlim_{\rho_2\to 1} D_{r_t([\rho_1,r(\rho_2)])}(\varphi_t(N_0),\alpha)$$

which is in general different from $\mathfrak{R}_0(\varphi_t(N_0),\alpha)$ (in which by definition we use norms ρ such that $\|\varphi_t(b_\beta)\|_\rho = \|\varphi_t(b_\alpha)\|_\rho$) by Example 4.5. Indeed, for $t = s$ the sum $\sum_{n=1}^\infty \varphi(b_\beta^n b_\alpha^{-n})$ converges in $\mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)$, but not in $\mathfrak{R}_0(\varphi_t(N_0),\alpha)$.

By the entirely same proof we also obtain

$$\mathfrak{R}_{0,r_{t_1}(\cdot)}(\varphi_{t_1}(N_0),\alpha) = \bigoplus_{n\in J(\varphi_{t_1}(N_0)/\varphi_{t_1t_2}(N_0))} n\iota_{t_1,t_1t_2}(\mathfrak{R}_{0,r_{t_1t_2}(\cdot)}(\varphi_{t_1t_2}(N_0),\alpha)) \tag{15}$$

for each pair $t_1,t_2 \in T_+$ where ι_{t_1,t_1t_2} is the inclusion of the rings above induced by the natural inclusion $\varphi_{t_1t_2}(N_0) \hookrightarrow \varphi_{t_1}(N_0)$.

Now we would like to define continuous homomorphisms

$$\begin{aligned}\varphi_{t_2t_1,t_1}\colon \mathfrak{R}_{0,r_{t_1}(\cdot)}(\varphi_{t_1}(N_0),\alpha) &\to \mathfrak{R}_{0,r_{t_1t_2}(\cdot)}(\varphi_{t_1t_2}(N_0),\alpha)\\ \varphi_{t_1}(b_\beta) &\mapsto \varphi_{t_1t_2}(b_\beta)\end{aligned}$$

induced by the group isomorphism $\varphi_{t_2}\colon \varphi_{t_1}(N_0) \to \varphi_{t_1t_2}(N_0)$ so that we can take the injective limit

$$\mathfrak{R}(N_1,\ell) := \varinjlim_t \mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)$$

with respect to the maps $\varphi_{t_2t_1,t_1}$. This is not possible for all t_2 since the map φ_{t_2} will not always be norm-decreasing on monomials $\mathbf{b}^{\mathbf{k}}$ for $\mathbf{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+\setminus\{\alpha\}}$. To overcome this we define the pre-ordering $\leq_\alpha$ (depending on the choice of the simple root α) on T_+ the following way: $t_1 \leq_\alpha t_2$ if and only if $|\beta(t_2t_1^{-1})| \leq |\alpha(t_2t_1^{-1})| \leq 1$ for all $\beta \in \Phi^+$. (In other words if and only if we have $m(\beta, t_2t_1^{-1}) \geq m(\alpha, t_2t_1^{-1}) \geq 0$.) In particular, $t_1 \leq_\alpha t_2$ implies $t_2t_1^{-1} \in T_+$ and it is equivalent to $1 \leq_\alpha t_2t_1^{-1}$. We also have $1 \leq_\alpha s$ for any $\alpha \in \Delta$. It is clear that $\leq_\alpha$ is transitive and reflexive. Moreover, if $t_2 \leq_\alpha t_1 \leq_\alpha t_2$ then $|\beta(t_2t_1^{-1})| = 1$ for all $\beta \in \Phi^+$ whence $t_2t_1^{-1}$ lies in T_0. Therefore $\leq_\alpha$ defines a partial ordering on the quotient monoid T_+/T_0.

Lemma 4.7. *The partial ordering $\leq_\alpha$ on T_+/T_0 is right filtered, ie. any finite subset of T_+/T_0 has a common upper bound with respect to $\leq_\alpha$.*

Proof. Let $t_1,t_2 \in T_+$ be arbitrary with $|\alpha(t_1)| \leq |\alpha(t_2)|$. Since the simple roots $\beta \in \Delta$ are linearly independent in $X^*(T) = \mathrm{Hom}_{alg}(T,\mathbb{G}_m)$, and the pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$ is perfect, we may choose $s_{\overline{\alpha}} \in T$ so that $|\beta(s_{\overline{\alpha}})| < |\alpha(s_{\overline{\alpha}})| = 1$ for all $\alpha \neq \beta \in \Delta$. Since all the positive roots are positive linear combinations of the simple roots, we see immediately that $s_{\overline{\alpha}} \in T_+$. Moreover, if $\alpha \neq \gamma \in \Phi^+$ then γ is not a scalar multiple of α hence writing $\gamma = \sum_{\beta\in\Delta} m_{\beta,\gamma}\beta$ there is a $\alpha \neq \beta \in \Delta$ with $m_{\beta,\gamma} > 0$ whence $|\gamma(s_{\overline{\alpha}})| < 1$. So we have $t_1 \leq_\alpha t_1 s_{\overline{\alpha}}^k$ for any $k \geq 0$ and $t_2 \leq_\alpha t_1 s_{\overline{\alpha}}^k$ for k big enough. □

Fix an element $1 \leq_\alpha t \in T_+$ and let $p^{-\frac{1}{\max_{\beta\in\Phi^+} p^{m(\beta,t)+m(\alpha,t)}}} < \rho_1 < \rho_2 < 1$ be a real numbers in $p^{\mathbb{Q}}$. Note that $\varphi_t\colon N_0 \to \varphi_t(N_0)$ is an isomorphism of pro-p groups. Hence it induces an isometric isomorphism

$$\begin{aligned}\varphi_t\colon D_{[0,\rho_2^{p^{m(\alpha,t)}}]}(N_0) &\to D_{[0,\rho_2^{p^{m(\alpha,t)}}]}(\varphi_t(N_0))\\ \sum_{\mathbf{k}} c_{\mathbf{k}} \prod_\beta b_\beta^{k_\beta} &\mapsto \sum_{\mathbf{k}} c_{\mathbf{k}} \prod_\beta \varphi_t(b_\beta)^{k_\beta}\end{aligned}$$

of Banach algebras where $D_{[0,\rho_2^{p^{m(\alpha,t)}}]}(\varphi_t(N_0))$ denotes the completion of $D(\varphi_t(N_0))$ with respect to the $\rho_2^{p^{m(\alpha,t)}}$-norm defined by the set of generators $\{\varphi_t(n_\beta)\}_{\beta\in\Phi^+}$ of $\varphi_t(N_0)$. To avoid confusion, from now on we denote by the subscript ρ, N_0 the ρ-norm (as before) on $D(N_0)$ and by the subscript $\rho,\varphi_t(N_0)$ the ρ-norm on $D(\varphi_t(N_0))$. By Lemma 4.4 we have

$$\|\varphi_t(b_\beta)\|_{\rho,N_0} = \rho^{p^{m(\beta,t)}} \leq \rho^{p^{m(\alpha,t)}} = \|\varphi_t(b_\alpha)\|_{\rho,N_0}$$

for any $\beta\in\Phi^+$ and $\rho=\rho_1$ or $\rho=\rho_2$ because of our assumption $1\leq_\alpha t$. This shows that for any monomial $\prod_{\beta\in\Phi^+}\varphi_t(b_\beta)^{k_\beta}$ (with $k_\beta\geq 0$ for all $\beta\in\Phi^+$) we have

$$\|\prod_{\beta\in\Phi^+}\varphi_t(b_\beta)^{k_\beta}\|_{r_t(\rho)} = \|\prod_{\beta\in\Phi^+}\varphi_t(b_\beta)^{k_\beta}\|_{\rho,N_0} \leq \rho^{p^{m(\alpha,t)}\mathbf{k}} = \|\prod_{\beta\in\Phi^+}\varphi_t(b_\beta)^{k_\beta}\|_{\rho^{p^{m(\alpha,t)}},\varphi_t(N_0)}$$

since both norms are multiplicative on $D(\varphi_t(N_0))$. Hence we obtain a norm decreasing homomorphism

$$D_{[0,\rho_2^{p^{m(\alpha,t)}}]}(N_0) \overset{\sim}{\to} D_{[0,\rho_2^{p^{m(\alpha,t)}}]}(\varphi_t(N_0)) \to D_{r_t([0,\rho_2])}(\varphi(N_0)) \hookrightarrow D_{r_t([\rho_1,\rho_2])}(\varphi(N_0),\alpha)\ .$$

Moreover, the element $\varphi_t(b_\alpha)$ is invertible in $D_{r_t([\rho_1,\rho_2])}(\varphi_t(N_0),\alpha)$ and for each $\rho_1\leq\rho\leq\rho_2$ and $x\in D_{[0,\rho_2]}(N_0)$ we have

$$\|\varphi_t(x)\varphi_t(b_\alpha)^{-k}\|_{r_t(\rho)} \leq \|x\|_{\rho^{p^{m(\alpha,t)}},N_0}\|b_\alpha^{-k}\|_{\rho^{p^{m(\alpha,t)}},N_0}\ .$$

Therefore by the universal property of microlocalisation (Prop. A.18 in [20]) we obtain a norm decreasing homomorphism

$$\begin{aligned}\varphi_{t,1}\colon D_{[\rho_1^{p^{m(\alpha,t)}},\rho_2^{p^{m(\alpha,t)}}]}(N_0,\alpha) &\to D_{r_t([\rho_1,\rho_2])}(\varphi_t(N_0),\alpha)\\ b_\beta &\mapsto \varphi_t(b_\beta)\ .\end{aligned} \tag{16}$$

Note that this above map is not surjective in general by Example 4.5.

Lemma 4.8. *The map* (16) *is injective.*

Proof. Take an element $f(\mathbf{b})=\sum_{\mathbf{k}} d_{\mathbf{k}}\mathbf{b}^{\mathbf{k}}\in D_{[\rho_1^{p^{m(\alpha,t)}},\rho_2^{p^{m(\alpha,t)}}]}(N_0,\alpha)$ and pairwise distinct $\mathbf{k}_1,\dots,\mathbf{k}_r\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}$. Note that $\|\varphi_t(b_\beta)\|_{\rho,N_0} > \|\varphi_t(b_\beta)-\binom{\beta(t)}{p^{m(\beta,t)}}b_\beta^{p^{m(\beta,t)}}\|_{\rho,N_0}$ hence we obtain

$$\|\sum_{j=1}^r d_{\mathbf{k}_j}\varphi_t(\mathbf{b})^{\mathbf{k}_j}\|_{r_t(\rho_1),r_t(\rho_2)} = \|\sum_{j=1}^r d_{\mathbf{k}_j}\prod_{\beta\in\Phi^+}\left(\binom{\beta(t)}{p^{m(\beta,t)}}b_\beta^{p^{m(\beta,t)}}\right)^{k_{j,\beta}}\|_{\rho_1,\rho_2} =$$

$$\max_j\|d_{\mathbf{k}_j}\prod_{\beta\in\Phi^+}\left(\binom{\beta(t)}{p^{m(\beta,t)}}b_\beta^{p^{m(\beta,t)}}\right)^{k_{j,\beta}}\|_{\rho_1,\rho_2} = \max_j\|d_{\mathbf{k}_j}\varphi_t(\mathbf{b})^{\mathbf{k}_j}\|_{r_t(\rho_1),r_t(\rho_2)}$$

using Prop. 4.3 as we have $\prod_{\beta\in\Phi^+}b_\beta^{p^{m(\beta,t)}k_{j_1,\beta}}\neq\prod_{\beta\in\Phi^+}b_\beta^{p^{m(\beta,t)}k_{j_2,\beta}}$ for $1\leq j_1\neq j_2\leq r$.

Since the map $\varphi_{t,1}$ is norm decreasing, we have $\|d_{\mathbf{k}}\varphi_t(\mathbf{b})^{\mathbf{k}}\|_{r_t(\rho_1),r_t(\rho_2)}\to 0$ as $\mathbf{k}\to\infty$. Therefore we also have $\|\sum_{\mathbf{k}} d_{\mathbf{k}}\varphi_t(\mathbf{b})^{\mathbf{k}}\|_{r_t(\rho_1),r_t(\rho_2)} = \max_{\mathbf{k}}\|d_{\mathbf{k}}\varphi_t(\mathbf{b})^{\mathbf{k}}\|_{r_t(\rho_1),r_t(\rho_2)}$ which is nonzero if there exists a $\mathbf{k}$ with $d_{\mathbf{k}}\neq 0$. Therefore the injectivity. ∎

Taking projective and injective limits we obtain an injective ring homomorphism

$$\varphi_{t,1}\colon \mathfrak{R}_0(N_0,\alpha)\hookrightarrow \mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)$$

for any $1\leq_\alpha t\in T_+$.

Remark 4.9. *Note that $\mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)$ is a subring of $\mathfrak{R}_0(N_0,\alpha)$ via the map $\iota_{1,t}$ (for all $t\in T_+$). Hence for $1\leq_\alpha t$ we obtain a ring homomorphism $\varphi_t=\iota_{1,t}\circ\varphi_{t,1}\colon \mathfrak{R}_0(N_0,\alpha)\to \mathfrak{R}_0(N_0,\alpha)$. However, if $1\not\leq_\alpha t$ for some $t\in T_+$ then we in fact do not have a continuous ring homomorphism $\varphi_t\colon \mathfrak{R}_0(N_0,\alpha)\to\mathfrak{R}_0(N_0,\alpha)$. Indeed, in this case there exists a $\beta\in\Phi^+$ such that $|\beta(t)|>|\alpha(t)|$ so there exist integers $k_\beta>k_\alpha$ such that $\|\varphi_t(b_\beta^{k_\beta}b_\alpha^{-k_\alpha})\|_\rho=\rho^{k_\beta/|\beta(t)|-k_\alpha/|\alpha(t)|}>1$ for any $p^{-|\alpha(t)|}<\rho<1$ therefore $\sum_{n=1}^\infty\varphi_t(b_\beta^{nk_\beta}b_\alpha^{-nk_\alpha})$ does not converge in $\mathfrak{R}_0(N_0,\alpha)$ even though $\sum_{n=1}^\infty b_\beta^{nk_\beta}b_\alpha^{-nk_\alpha}$ does.*

Remark 4.10. *If $\Phi^+\neq\Delta$ (e.g. if $G=\mathrm{GL}_n(\mathbb{Q}_p)$, $n>2$) then we have*

$$\mathfrak{R}_0(N_0,\alpha)=\bigoplus_{u\in J(N_0/\varphi_t(N_0))}u\iota_{1,s}(\mathfrak{R}_{0,r_s(\cdot)}(\varphi(N_0),\alpha))\supsetneq\bigoplus_{n\in J(N_0/\varphi(N_0))}u\varphi(\mathfrak{R}_0(N_0,\alpha))$$

by (14) *(with the choice $t=s$) and Example 4.5 (which shows that $\varphi_{s,1}$ is not surjective).*

In a similar fashion we get for $t_1\in T_+$ (and $1\leq_\alpha t\in T_+$) an injective homomorphism

$$\varphi_{tt_1,t_1}\colon \mathfrak{R}_{0,r_{t_1}(\cdot)}(\varphi_{t_1}(N_0),\alpha)\to\mathfrak{R}_{0,r_{tt_1}(\cdot)}(\varphi_{tt_1}(N_0),\alpha)\ .$$

In view of Lemma 4.7 we define

$$\mathcal{R}(N_1,\ell):=\varinjlim_{t\in T_+}\mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)$$

with respect to the maps φ_{t_1,t_2} for $t_2\leq_\alpha t_1$.

Now take any $t\in T_+$ (not necessarily satisfying $1\leq_\alpha t$). The map

$$\varphi_t:=\varinjlim_{t_1}\iota_{t_1,tt_1}\colon\mathcal{R}(N_1,\ell)\to\mathcal{R}(N_1,\ell)$$

is defined as the direct limit of the inclusion maps

$$\iota_{t_1,tt_1}\colon\mathfrak{R}_{0,r_{tt_1}(\cdot)}(\varphi_{tt_1}(N_0),\alpha)\hookrightarrow\mathfrak{R}_{0,r_{t_1}(\cdot)}(\varphi_{t_1}(N_0),\alpha)$$

induced by $\varphi_{tt_1}(N_0)\subseteq\varphi_{t_1}(N_0)$. By definition, for any $t\in T_+$ the ring $\mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)$ consists of formal power series $\sum_{\mathbf{k}}c_{\mathbf{k}}\varphi_t(\mathbf{b})^{\mathbf{k}}$ that converge in $\mathfrak{R}_0(N_0,\alpha)$. Therefore the map

$$\bigcup_{t\in T_+}\left\{\sum_{\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}}c_{\mathbf{k}}\mathbf{b}^{\mathbf{k}}\ \Big|\ \sum_{\mathbf{k}}c_{\mathbf{k}}\varphi_t(\mathbf{b})^{\mathbf{k}}\text{ convergent in }\mathfrak{R}_0(N_0,\alpha)\right\}\to\mathcal{R}(N_1,\ell)$$

$$\sum_{\mathbf{k}}c_{\mathbf{k}}\mathbf{b}^{\mathbf{k}}\mapsto\sum_{\mathbf{k}}c_{\mathbf{k}}\varphi_t(\mathbf{b})^{\mathbf{k}}\in\mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)\hookrightarrow\mathcal{R}(N_1,\ell)$$

is well-defined and bijective since $\sum_{\mathbf{k}}c_{\mathbf{k}}\varphi_t(\mathbf{b})^{\mathbf{k}}$ converges for some $t\in T_+$ and the connecting homomorphisms in the injective limit defining $\mathcal{R}(N_1,\ell)$ are injective and given by φ_{t_1,t_2} for $t_2\leq_\alpha t_1$.

Hence we may identify

$$\mathcal{R}(N_1,\ell) = \bigcup_{t\in T_+} \left\{ \sum_{\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}} c_{\mathbf{k}}\mathbf{b}^{\mathbf{k}} \;\middle|\; \sum_{\mathbf{k}} c_{\mathbf{k}}\varphi_t(\mathbf{b})^{\mathbf{k}} \text{ convergent in } \mathfrak{R}_0(N_0,\alpha) \right\} \tag{17}$$

and obtain

Proposition 4.11. *The natural map $\varphi_t\colon \mathcal{R}(N_1,\ell)\to\mathcal{R}(N_1,\ell)$ is injective for all $t\in T_+$ and we have the decomposition*

$$\mathcal{R}(N_1,\ell) = \bigoplus_{n\in J(N_0/\varphi_t(N_0))} n\varphi_t(\mathcal{R}(N_1,\ell)) \ .$$

In particular, $\mathcal{R}(N_1,\ell)$ is a free (right) module over itself via φ_t and it is a φ-ring over N_0 with $\varphi=\varphi_s$ in the sense of Definition 2.9.

Proof. By (15) we have

$$\mathfrak{R}_{0,r_{t_1}(\cdot)}(\varphi_{t_1}(N_0),\alpha) = \bigoplus_{n\in J(N_0/\varphi_t(N_0))} \varphi_{t_1}(n)\iota_{t_1,tt_1}(\mathfrak{R}_{0,r_{tt_1}(\cdot)}(\varphi_{tt_1}(N_0),\alpha))$$

for any $t_1\in T_+$. The statement follows by taking the injective limit of both sides (with respect to t_1) and noting that $\varphi_{t_1,1}(n)=\varphi_{t_1}(n)\in\varphi_{t_1}(N_0)\subseteq\mathfrak{R}_{0,r_{t_1}(\cdot)}(\varphi_{t_1}(N_0),\alpha)$ for $n\in N_0\subseteq\mathfrak{R}_0(N_0)$ and $1\leq_\alpha t_1$ therefore n corresponds to $\varinjlim_{1\leq_\alpha t_1}(\varphi_{t_1}(n))_{t_1}$ via the identification (17). $\square$

Remark 4.12. *The ring $\mathcal{R}(N_1,\ell)$ via the description* (17) *consists of exactly those Laurent-series*

$$x = \sum_{\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}} c_{\mathbf{k}}\mathbf{b}^{\mathbf{k}}$$

that converge on the open annulus of the form

$$\left\{\rho_2<|z_\alpha|<1,\ |z_\beta|\leq|z_\alpha|^r \text{ for } \beta\in\Phi^+\setminus\{\alpha\}\right\} \ . \tag{18}$$

for some $p^{-1}<\rho_2<1$ and $1\leq r\in\mathbb{Z}$.

Proof. If $x\in\mathcal{R}(N_1,\ell)$ then there exists a $t\in T_+$ such that $\varphi_t(x)$ converges in $\mathfrak{R}_0(N_0)$, ie. it converges in the norm $\|b_\beta\|_\rho=\rho$ for all $\beta\in\Phi^+$ for some fixed $p^{-1}<\rho_0<1$ and all $\rho\in(\rho_0,1)$. By Lemma 4.7 we may assume that $|\alpha(t)|=1$ whence $\|\varphi_t(b_\alpha)\|_\rho=\rho$ for all $\rho<1$ as we may take $t=s_{\overline{\alpha}}^k$ for k large enough. Now let $\rho_2:=\rho_0$ and $r:=\max_{\beta\in\Phi^+}([|1/\beta(t)|]+1)\in\mathbb{Z}$. Then x converges on the annulus (18) as we have $\rho^r\leq\rho^{1/|\beta(t)|}\leq\|\varphi_t(b_\beta)\|_\rho$ for all $\beta\in\Phi^+\setminus\{\alpha\}$ by Lemma 4.4.

Conversely for any fixed $p^{-1}<\rho_2<1$ and integer $r\geq1$ we need to find a $t\in T_+$ and a $\rho_0\in(p^{-1},1)$ such that for all $\rho\in(\rho_0,1)$ we have $\rho_2<\|\varphi_t(b_\alpha)\|_\rho<1$ and $\|\varphi_t(b_\beta)\|_\rho\leq\|\varphi_t(b_\alpha)\|_\rho^r$. We take $t:=s_{\overline{\alpha}}^k$ and $\rho_0:=\max(\rho_2,p^{-|\beta(t)|}\mid\beta\in\Phi^+\setminus\{\alpha\})$ where $k:=\max_{\beta\in\Phi^+\setminus\{\alpha\}}([-\frac{\log r}{\log|\beta(s_{\overline{\alpha}})|}]+1)$ (for the definition of $s_{\overline{\alpha}}$ see the proof of Lemma 4.7). Indeed, since $|\alpha(s_{\overline{\alpha}}^k)|$ equals 1, we have $\rho_2<\rho=\|\varphi_{s_{\overline{\alpha}}^k}(b_\alpha)\|_\rho<1$ (for any k). On the other hand, we have $|\beta(s_{\overline{\alpha}})| < 1$ for all $\alpha \neq \beta \in \Phi^+$ (whence, in particular, the definition of k makes sense), so we obtain

$$\|\varphi_t(b_\beta)\|_\rho = \max_{0\leq j\leq \mathrm{val}_p(\beta(t))}(\rho^{p^j}p^{j-\mathrm{val}_p(\beta(t))}) = \rho^{p^{\mathrm{val}_p(\beta(t))}} = \rho^{1/|\beta(s_{\overline{\alpha}})|^k} \leq \rho^r$$

for all $\beta \in \Phi^+ \setminus \{\alpha\}$ by Lemma 4.4, the choice of k so that we have $r \leq 1/|\beta(s_{\overline{\alpha}})|^k$, and the choice of $p^{-|\beta(t)|} < \rho$ so that we have $\max_{0\leq j\leq \mathrm{val}_p(\beta(t))}(\rho^{p^j}p^{j-\mathrm{val}_p(\beta(t))}) = \rho^{p^{\mathrm{val}_p(\beta(t))}}$. $\square$

4.3 Bounded rings

Let us denote by $\mathfrak{R}_0^b$ (resp. by $\mathfrak{R}_0^{int}$) the set of elements $x \in \mathfrak{R}_0$ such that $\lim_{\rho\to 1}\|x\|_{\rho,\rho}$ exists (resp. exists and is at most 1). These are subrings of $\mathfrak{R}_0$ by Prop. A.28 in [20]. Moreover, since φ_t is norm-decreasing for any $1 \leq_\alpha t$ (see (16)), these subrings are stable under the action of φ_t ($1 \leq_\alpha t \in T_+$). We put

$$\begin{aligned}\mathfrak{R}^b_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha) &:= \mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)\cap\mathfrak{R}_0^b(N_0,\alpha)\ , \quad \text{and}\\ \mathfrak{R}^{int}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha) &:= \mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)\cap\mathfrak{R}_0^{int}(N_0,\alpha)\end{aligned}$$

where the intersection is taken inside $\mathfrak{R}_0$ under the inclusion $\iota_{1,t}\colon \mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha) \hookrightarrow \mathfrak{R}_0$. Hence

$$\mathcal{R}^b(N_1,\ell) := \varinjlim_t \mathfrak{R}^b_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha) \text{ and } \mathcal{R}^{int}(N_1,\ell) := \varinjlim_t \mathfrak{R}^{int}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)$$

are T_+-stable subrings of $\mathcal{R}(N_1,\ell)$ (the injective limit is taken with repsect to the maps φ_{t_1,t_2} for $t_1 \leq_\alpha t_2 \in T_+$ as in the construction of $\mathcal{R}(N_1,\ell)$). Further, Lemma 4.6 shows that for any $t \in T_+$ and $x \in \mathfrak{R}_0$ we have

$$\lim_{\rho\to 1}\|x\|_\rho = \lim_{\rho\to 1}\|x\|_{q_t(r_t(\rho))}\ . \tag{19}$$

Indeed, we may use Lemma 4.6 in the context of $\mathfrak{R}_0$ the following way. The elements of $D_{[\rho_1,\rho_2]}(N_0,\alpha)$ are Cauchy sequences $(a_n\varphi_t(b_\alpha)^{-k_n})_{n\in\mathbb{N}}$ (in the norm $\max(\|\cdot\|_{\rho_1},\|\cdot\|_{\rho_2})$) with $a_n \in D_{[0,\rho_2]}(N_0)$ and $k_n \geq 0$. Since $\|\cdot\|_\rho$ is multiplicative for any $\rho_1 \leq \rho \leq \rho_2$ in $p^{\mathbb{Q}}$ and so is its restriction to $D(\varphi_t(N_0))$ we compute

$$\|a_n\varphi_t(b_\alpha)^{-k_n}\|_\rho\rho^{\sum_{\beta\in\Phi^+}(p^{m(\beta,t)}-1)} = \frac{\|a_n\|_\rho}{\|\varphi_t(b_\alpha)^{k_n}\|_\rho\rho^{-\sum_{\beta\in\Phi^+}(p^{m(\beta,t)}-1)}} \leq \frac{\|a_n\|_{q_t(r_t(\rho))}}{\|\varphi_t(b_\alpha)^{k_n}\|_{q_t(r_t(\rho))}} =$$

$$= \|a_n\varphi_t(b_\alpha)^{-k_n}\|_{q_t(r_t(\rho))} \leq \frac{\|a_n\|_\rho\rho^{-\sum_{\beta\in\Phi^+}(p^{m(\beta,t)}-1)}}{\|\varphi_t(b_\alpha)^{k_n}\|_\rho} \leq \|a_n\varphi_t(b_\alpha)^{-k_n}\|_\rho\rho^{-\sum_{\beta\in\Phi^+}(p^{m(\beta,t)}-1)}\ .$$

If $\rho \to 1$ and $n \to \infty$ we obtain (19). Combining this observation with (14) we obtain

$$\mathfrak{R}_0^b(N_0,\alpha) = \bigoplus_{n\in J(N_0/\varphi_t(N_0))} n\iota_{1,t}(\mathfrak{R}^b_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha))\ ;$$

$$\mathfrak{R}_0^{int}(N_0,\alpha) = \bigoplus_{n\in J(N_0/\varphi_t(N_0))} n\iota_{1,t}(\mathfrak{R}^{int}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha))\ .$$

So by a similar argument as for $\mathcal{R}(N_1,\ell)$ we also obtain

$$\mathcal{R}^b(N_1,\ell) = \bigoplus_{n\in J(N_0/\varphi_t(N_0))} n\varphi_t(\mathcal{R}^b(N_1,\ell)) \ ;$$
$$\mathcal{R}^{int}(N_1,\ell) = \bigoplus_{n\in J(N_0/\varphi_t(N_0))} n\varphi_t(\mathcal{R}^{int}(N_1,\ell)),$$

in other words these are φ-rings over N_0 in the sense of Definition 2.9.

Remark 4.13. *Note that by Lemma A.27 in [20] an element* $\sum_{\mathbf{k}\in\mathbb{N}^{\Phi^+\setminus\{\alpha\}}\times\mathbb{Z}} c_{\mathbf{k}}\mathbf{b}^{\mathbf{k}} \in \mathcal{R}(N_1,\ell)$ *(under the parametrization* (17)*) lies in* $\mathcal{R}^b(N_1,\ell)$ *(resp. in* $\mathcal{R}^{int}(N_1,\ell)$*) if and only if* $|c_{\mathbf{k}}|$ *is bounded (resp.* ≤ 1*) for* $\mathbf{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}$.

4.4 Relation with the completed Robba ring and overconvergent ring

Lemma 4.14. *There exists a continuous (in the weak topology of* $\Lambda_\ell(N_0)$*) injective ring homomorphism* $j_{int}\colon \mathcal{R}^{int}(N_1,\ell)\to\Lambda_\ell(N_0)$ *respecting Laurent series expansions. The image of* j_{int} *is contained in* $\mathcal{O}_{\mathcal{E}}^{\dagger}[\![N_1,\ell]\!]\subset\Lambda_\ell(N_0)$.

Proof. We proceed in 3 steps. In Step 1 we construct a map $j_{int,0} = j_{int|\mathfrak{R}_0^{int}}\colon \mathfrak{R}_0^{int}\to\Lambda_\ell(N_0)$ which is a priori continuous and o_K-linear. In Step 2 we show that $j_{int,0}$ is multiplicative hence a ring homomorphism. In Step 3 we extend it to $\mathcal{R}^{int}(N_1,\ell)$ and show that the image lies in $\mathcal{O}_{\mathcal{E}}^{\dagger}[\![N_1,\ell]\!]\subset\mathcal{O}_{\mathcal{E}}[\![N_1,\ell]\!]=\Lambda_\ell(N_0)$.

Step 1. By Lemma 4.3 and Remark 4.13 we may write any element in $\mathfrak{R}_0^{int}$ in a Laurent series expansion $\sum_{\mathbf{k}\in\mathbb{N}^{\Phi^+\setminus\{\alpha\}}\times\mathbb{Z}} c_{\mathbf{k}}\mathbf{b}^{\mathbf{k}}$ with coefficients $c_{\mathbf{k}}$ in o_K. So we may collect all the terms containing $b_\alpha^{k_\alpha}$ for some fixed k_α into an element of the Iwasawa algebra $\Lambda(N_1)$ to obtain an expansion $\sum_{n\in\mathbb{Z}} b_\alpha^n f_n$ with $f_n\in\Lambda(N_1)$. These power series satisfy the convergence property that there exists a real number $p^{-1}<\rho_1<1$ such that $\rho^n\|f_n\|_\rho\to 0$ as $|n|\to\infty$ for all $\rho_1<\rho<1$. In particular, if $n\to-\infty$ then $f_n\to 0$ in the compact topology of $\Lambda(N_1)$. Hence the sum $\sum_n b_\alpha^n f_n$ also converges in $\Lambda_\ell(N_0)$. This way we obtained a right $\Lambda(N_0)$-linear injective map $j_{int,0}\colon\mathfrak{R}_0^{int}\to\Lambda_\ell(N_0)$.

Recall that the weak topology (see [17], [18], [19] for instance) on $\Lambda_\ell(N_0)$ is defined by the open neighbourhoods of 0 of the form $\mathcal{M}(r)=\mathcal{M}_\ell(N_0)^r+\mathcal{M}(N_0)^r$ where $\mathcal{M}_\ell(N_0)=\Lambda_\ell(N_0)\mathcal{M}(N_1)$ denotes the maximal ideal of $\Lambda_\ell(N_0)$ and $\mathcal{M}(N_i)$ denotes the maximal ideal of $\Lambda(N_i)\subseteq\Lambda_\ell(N_0)$ $(i=0,1)$. For any fixed $p^{-1}<\rho_1<\rho<1$ the preimage of $\mathcal{M}(r)$ in $\mathfrak{R}_0^{int}\cap D_{[\rho_1,1)}(N_1,\alpha)$ contains the open ball $\{x\mid \|x\|_\rho<p^{-r}\}$. Indeed, if $x=\sum_{n\in\mathbb{Z}} b_\alpha^n f_n$ then for any $n<0$ we have $\|f_n\|_\rho<p^{-r}$ hence $f_n\in\mathcal{M}(N_1)^r$ and $b_\alpha^n f_n\in\mathcal{M}_\ell(N_0)$. On the other hand, the positive part $\sum_{n\geq 0} b_\alpha^n f_n$ lies in $\Lambda(N_0)$ and has ρ-norm smaller than p^{-r} therefore lies in $\mathcal{M}(N_0)^r$. Hence the continuity.

Step 2. Now by the continuity and linearity of $j_{int,0}$ it suffices to show that it is multiplicative on monomials $\mathbf{b}^{\mathbf{k}}$. Moreover, each monomial is a linear combination of elements of the form $b_\alpha^n g$ with $g\in N_0$. In order to expand the product $(b_\alpha^{n_1}g_1)(b_\alpha^{n_2}g_2)$ into a skew Laurent series it suffices to expand $g_1 b_\alpha^{n_2}$ with $n_2<0$. However, if $g_1 b_\alpha^{n_2}=\sum_n b_\alpha^n h_n$ is the expansion in $\mathcal{R}^{int}(N_1,\ell)$ then $\sum_{|n|<n_0} b_\alpha^n h_n b_\alpha^{-n_2}$ tends to g_1 (as $n_0\to+\infty$) in the topology of $\mathfrak{R}_0^{int}$ (induced by the norms) hence also in the weak topology. Therefore the expansion in $\Lambda_\ell(N_0)$ is also $g_1 b_\alpha^{n_2}=\sum_n b_\alpha^n h_n$. So the above constructed map $j_{int,0}$ is indeed a ring homomorphism as claimed.

Step 3. Finally, take an element $x \in \mathcal{R}^{int}(N_1, \ell)$. There exists an element $1 \leq_\alpha t \in T_+$ such that $\varphi_t(x)$ lies in the image of the composite map

$$\mathfrak{R}^{int}_{0,r_t(\cdot)}(\varphi_t(N_0), \alpha) \hookrightarrow \mathfrak{R}^{int}_0(N_0, \alpha) \hookrightarrow \mathcal{R}^{int}(N_1, \ell)$$

where the first arrow is induced by the inclusion $\varphi_t(N_0) \subseteq N_0$. Now if we reduce $j_{int,0}(\varphi_t(x)) \in \Lambda_\ell(N_0)$ modulo the ideal generated by $N_l - 1$ for some integer $l \geq 1$ then we obtain an element in $\varphi_t(\mathcal{O}^\dagger_\mathcal{E}[N_1/N_l, \ell])$. Indeed, $\varphi_t(\mathcal{O}_\mathcal{E}[N_1/N_l, \ell])$ is a closed subspace in $\mathcal{O}_\mathcal{E}[N_1/N_l, \ell]$ and all the monomials $j_{int,0}(\varphi_t(\mathbf{b}^\mathbf{k}))$ map into this subspace under the reduction modulo $(N_l - 1)$. Hence the image lies in $\varphi_t(\mathcal{O}_\mathcal{E}[N_1/N_l, \ell]$. Moreover, by the convergence property of elements in $\mathfrak{R}^{int}_0$, we may expand

$$\varphi_t(x) = \sum_{n \in \mathbb{Z}} b^n_\alpha f_n$$

with $f_n \in \Lambda(N_1)$ and $\rho^n \|f_n\|_\rho \to 0$ as $n \to \infty$ for all $\rho_1 < \rho < 1$ and a fixed $p^{-1} < \rho_1 < 1$ depending on x. Since the reduction map $\Lambda(N_1) \to o[N_1/N_l]$ is continuous in the ρ-norm, we obtain that the reduction of $j_{int,0}(\varphi_t(x))$ modulo $(N_l - 1)$ also lies in $\mathcal{O}^\dagger_\mathcal{E}[N_1/N_l, \ell]$. Hence we have $j_{int,0}(\varphi_t(x)) \pmod{N_l - 1} \in \varphi_t(\mathcal{O}^\dagger_\mathcal{E}[N_1/N_l, \ell]) = \varphi_t(\mathcal{O}_\mathcal{E}[N_1/N_l, \ell]) \cap \mathcal{O}^\dagger_\mathcal{E}[N_1/N_l, \ell]$. Taking the limit we see (using (7)) that $j_{int,0}(\varphi_t(x))$ lies in

$$\varprojlim_l \varphi_t(\mathcal{O}^\dagger_\mathcal{E}[N_1/N_l, \ell]) = \varphi_t(\mathcal{O}^\dagger_\mathcal{E}[\![N_1, \ell]\!]) \ .$$

So we put $j_{int}(x) := \varphi_t^{-1}(j_{int,0}(\varphi_t(x)))$. This extends the ring homomorphism $j_{int,0}$ to a continuous ring homomorphism $j_{int}\colon \mathcal{R}^{int}(N_1, \ell) \hookrightarrow \mathcal{O}^\dagger_\mathcal{E}[\![N_1, \ell]\!] \subset \Lambda_\ell(N_0)$ by Lemma 4.7. Moreover, j_{int} is T_+-equivariant as it respects power series expansions. □

Now the following proposition compares $\mathcal{R}(N_1, \ell)$ with the previous construction $\mathcal{R}[\![N_1, \ell]\!]$.

Proposition 4.15. *There exists a natural T_+-equivariant ring homomorphism*

$$j\colon \mathcal{R}(N_1, \ell) \to \mathcal{R}[\![N_1, \ell]\!]$$

with dense image.

Proof. At first we construct the map $j_0 = j_{|\mathfrak{R}_0}$ on $\mathfrak{R}_0 \subset \mathcal{R}(N_1, \ell)$ with dense image. We are going to show that for any open characteristic subgroup $H \leq N_1$ we have an isomorphism $\mathfrak{R}_0/\mathfrak{R}_0(H-1) \cong \mathcal{R}[N_1/H, \ell]$. Note that N_1 being a compact p-adic Lie group, N_1 has a system of neighbourhoods of 1 consisting of open uniform characteristic subgroups (in fact N_1 is uniform—since so is N_0 by assumption—and one can take repeatedly the Frattini subgroups of N_1 which are characteristic subgroups, ie. stable under all the continuous automorphisms of N_1). So we may assume without loss of generality that H is uniform with topological generators $h_1, h_2, \dots, h_d$ with $d = \dim N_1$ as a p-adic Lie group.

Under the parametrization in Prop. 4.3 the elements of $\mathfrak{R}_0$ can be written as power series $\sum_{n \in \mathbb{Z}} b^n_\alpha f_n$ with $f_n \in D(N_1, K)$ and the convergence property that there exists a real number $\rho_1 < 1$ such that $\rho^n \|f_n\|_\rho \to 0$ (as $|n| \to \infty$) for all $\rho_1 \leq \rho < 1$. Now note that we have $D(N_1, K) = \bigoplus_{u \in J(N_1/H)} uD(H, K)$. Hence the right ideal $D(N_1, K)(H-1)$ in $D(N_1, K)$ is generated by the elements $h_i - 1$ for $1 \leq i \leq d$ and it is the kernel of the natural projection $\pi_H\colon D(N_1, K) \to D(N_1/H) = K[N_1/H]$. Moreover, this quotient map factors through the inclusion $D(N_1,K) \hookrightarrow D_{[0,\rho]}(N_1,K)$ for any $p^{-1} < \rho < 1$. Hence $\rho^n \|\pi_H(f_n)\| \to 0$ where $\|x\| := \max_u |x_u|$ with $x = \sum_{u\in N_1/H} x_u u$, $x_u \in K$. Therefore we obtain a map

$$\begin{array}{rcll} \pi_H & : & \mathfrak{R}_0 & \to \mathcal{R}[N_1/H,\ell] = \bigoplus_{u\in N_1/H} \mathcal{R}u \\ & & \sum_{n\in\mathbb{Z}} b_\alpha^n f_n & \mapsto \sum_{u\in N_1/H}\sum_{n\in\mathbb{Z}} \pi_H(f_n)_u T^n u \ . \end{array}$$

A priori this map is only known to be K-linear, continuous, and surjective between topological K-vectorspaces. So for the multiplicativity it suffices to show that $\pi_H(\mathbf{b}^{\mathbf{k}_1}\mathbf{b}^{\mathbf{k}_2}) = \pi_H(\mathbf{b}^{\mathbf{k}_1})\pi_H(\mathbf{b}^{\mathbf{k}_2})$ for monomials $\mathbf{b}^{\mathbf{k}_i}$ with $\mathbf{k}_i \in \mathbb{N}\times\mathbb{Z}^d$ $(i=1,2)$. On the other hand, these monomials are contained in the subring $\mathfrak{R}_0^{int}$. By Lemma 4.14 we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{R}_0^{int} & \longrightarrow & \mathfrak{R}_0 \\ {\scriptstyle \pi_{H,\mathcal{O}_\mathcal{E}^\dagger[\![N_1,\ell]\!]}}\downarrow & & \downarrow{\scriptstyle \pi_H} \\ \mathcal{O}_\mathcal{E}^\dagger[N_1/H,\ell] & \longrightarrow & \mathcal{R}[N_1/H,\ell] \end{array}$$

of o-modules such that all the maps are ring homomorphisms except possibly for π_H. However, from the commutativity of the diagram it follows that also π_H is multiplicative on monomials therefore a ring homomorphism. By taking the projective limit of maps π_H we obtain a ring homomorphism $j_0\colon \mathfrak{R}_0 \to \mathcal{R}[\![N_1,\ell]\!]$ with dense image and extending $j_{int,0}\colon \mathfrak{R}_0^{int} \hookrightarrow \mathcal{O}_\mathcal{E}^\dagger$.

Finally, the homomorphism j_0 is extended to $\mathcal{R}(N_1,\ell)$ as in the proof of Lemma 4.14. The T_+-equivariance is clear on monomials by Lemma 4.14 and follows in general from the continuity and linearity. □

Remark 4.16. *The above constructed map* $j\colon \mathcal{R}(N_1,\ell) \to \mathcal{R}[\![N_1,\ell]\!]$ *is not injective in general. Indeed, for any root* $\beta \neq \alpha$ *in* Φ^+ *the element* $\log(n_\beta) = \log(1+b_\beta)$ *lies in* $D(N_1) \subset \mathcal{R}(N_1,\ell)$. *It is easy to see that* $\log(1+b_\beta)$ *is divisible by* $\varphi^r(b_\beta)$ *for any nonnegative integer* r. *Indeed, we clearly have* $b_\beta \mid \log(1+b_\beta)$. *Applying* φ^r *on the both sides of the divisibility we obtain*

$$\varphi^r(b_\beta) \mid \varphi^r(\log(1+b_\beta)) = \log(1+b_\beta)^{p^{rm_\beta}} = p^{rm_\beta}\log(1+b_\beta) \mid \log(1+b_\beta)$$

as p^{rm_β} *is invertible in* $\mathcal{R}$. *Therefore* $\log(1+b_\beta)$ *lies in the kernel of* π_H *for all* $H = N_r$ *hence also in the kernel of* j.

Remark 4.17. *Note that via the inclusion* $\mathcal{O}_\mathcal{E}^\dagger \subseteq \mathcal{R}$ *we also have* $\mathcal{O}_\mathcal{E}^\dagger[\![N_1,\ell]\!] \subseteq \mathcal{R}[\![N_1,\ell]\!]$. *However, if* $N_1 \neq 1$ *then we have* $j_{int}(\mathcal{R}^{int}(N_1,\ell)) \neq j(\mathcal{R}(N_1,\ell)) \cap \mathcal{O}_\mathcal{E}^\dagger[\![N_1,\ell]\!] \subset \mathcal{R}[\![N_1,\ell]\!]$.

Proof. Assume $N_1 \neq 1$, so we have a positive root $\beta \neq \alpha \in \Phi^+$. We proceed in 3 steps. In Step 1 we are going to construct an element $x \in \mathcal{R}(N_1,\ell)$ with several properties. In Step 2 we are going to show that $j(x)$ lies in $\mathcal{O}_\mathcal{E}^\dagger[\![N_1,\ell]\!] \subset \mathcal{R}[\![N_1,\ell]\!]$. In Step 3 we prove that $j(x)$ does not lie in $j_{int}(\mathcal{R}^{int}(N_1,\ell))$. Note that the other inclusion $j_{int}(\mathcal{R}^{int}(N_1,\ell)) \subset j(\mathcal{R}(N_1,\ell)) \cap \mathcal{O}_\mathcal{E}^\dagger[\![N_1,\ell]\!]$ is obvious.

Step 1. We denote by $s_n := \sum_{i=1}^n (-1)^{i+1}\frac{b_\beta^i}{i}$ the nth estimating sum of $\log(1+b_\beta) \in \mathcal{R}(N_1,\ell)$. Note that $k_n := [\log_p n]$ is the smallest positive integer such that

$$p^{k_n} s_n \in \mathbb{Z}_p[N_{\beta,0}] \subseteq \mathcal{R}(N_1,\ell) \tag{20}$$

where $[\cdot]$ denotes the integer part of a real number. We further choose a sequence of real numbers $p^{-1} < \rho_1 < \cdots < \rho_n < \cdots < 1$ in $p^{\mathbb{Q}}$ such that $\lim_{n\to\infty} \rho_n = 1$. Now for any fixed positive integer n let i_n be the smallest positive integer satisfying the following properties

$$\begin{aligned} \log_{\rho_{n-1}}(\|p^{k_{i_{n-1}}} \log(1+b_\beta)\|_{\rho_{n-1}}) + 1 < \log_{\rho_n}(\|p^{k_{i_n}} \log(1+b_\beta)\|_{\rho_n}) \ ; \\ \frac{\|\log(1+b_\beta)\|_{\rho_n}}{p^n} > \|\log(1+b_\beta) - s_{i_n}\|_{\rho_n} \ ; \\ p^{k_{i_n}/2} > \|\log(1+b_\beta)\|_{\rho_n} \ ; \\ \|\varphi^i(\log(1+b_\beta))\|_{\rho_j} > \|\varphi^i(\log(1+b_\beta) - s_{i_n})\|_{\rho_j} \end{aligned} \tag{21}$$

for all $1 \leq i, j \leq n$. Such an i_n exists as for any fixed $1 \leq i, j \leq n$ we have $\lim_{k\to\infty} \|\varphi^i(\log(1+b_\beta) - s_k)\|_{\rho_j} = 0$. The first condition in (21) makes the definition of i_n inductive. As a consequence, we have $\|\log(1+b_\beta)\|_{\rho_n} = \|s_{i_n}\|_{\rho_n}$ by the ultrametric inequality. Now define $j_n \in \mathbb{Z}$ so that

$$\rho_n^{j_n+1} < \frac{\|s_{i_n}\|_{\rho_n}}{p^{k_{i_n}}} = \|p^{k_{i_n}} s_{i_n}\|_{\rho_n} = \|p^{k_{i_n}} \log(1+b_\beta)\|_{\rho_n} \leq \rho_n^{j_n} \ . \tag{22}$$

(In other words $j_n = [\log_{\rho_n}(\|p^{k_{i_n}} \log(1+b_\beta)\|_{\rho_n})]$.) By (20) we have $j_n \geq 0$. Moreover, by the first condition in (21) the sequence $(j_n)_n$ is strictly increasing: $j_{n-1} < j_n$ for all $n > 1$. On the other hand, $(-1)^{p^{k_{i_n}}} b_\beta^{p^{k_{i_n}}}$ is a summand in $p^{k_{i_n}} s_{i_n}$, therefore we have $\rho_n^{p^{k_{i_n}}} \leq \|p^{k_{i_n}} s_{i_n}\|_{\rho_n} \leq \rho_n^{j_n}$ whence

$$j_n \leq p^{k_{i_n}} \leq i_n \ . \tag{23}$$

Put $x := \sum_{n=1}^{\infty} p^{k_{i_n}}(\log(1+b_\beta) - s_{i_n}) b_\alpha^{-j_n}$. Our goal in this step is to show that the sum x converges in $\mathfrak{R}_0(N_0, \alpha) \subset \mathcal{R}(N_1, \ell)$. For this it suffices to verify that for any fixed $k \geq 1$ we have $\|p^{k_{i_n}}(\log(1+b_\beta) - s_{i_n}) b_\alpha^{-j_n}\|_{\rho_k} \to 0$ as $n \to \infty$. Note that in the power series expansion of $\log(1+b_\beta) - s_{i_n}$ all the terms have degree $> i_n \geq j_n$ by (23). Therefore in the power series expansion of x all the terms have positive degree. In particular, for $k < n$ we have $\|y\|_{\rho_k} \leq \|y\|_{\rho_n}$ whenever y is a monomial in the expansion of x. By (21) and (22) we obtain

$$\begin{aligned} \|p^{k_{i_n}}(\log(1+b_\beta) - s_{i_n}) b_\alpha^{-j_n}\|_{\rho_k} \leq \|p^{k_{i_n}}(\log(1+b_\beta) - s_{i_n}) b_\alpha^{-j_n}\|_{\rho_n} < \\ < \frac{\|p^{k_{i_n}} \log(1+b_\beta)\|_{\rho_n}}{p^n} \rho_n^{-j_n} \leq \frac{1}{p^n} \end{aligned}$$

for $k < n$. Hence we have $x \in \mathfrak{R}_0(N_0, \alpha) \subset \mathcal{R}(N_1, \ell)$.

Step 2. Note that by Remark 4.16 $\log(1+b_\beta)$ lies in the kernel of π_H for all open normal subgroup $H \leq N_1$. Hence by the continuity of π_H we obtain $\pi_H(x) = \sum_{n=1}^{\infty} \pi_H(-p^{k_{i_n}} s_{i_n} b_\alpha^{-j_n}) \in \mathcal{O}_{\mathcal{E}}^{\dagger}[N_1/H, \ell] \subseteq \mathcal{R}[N_1/H, \ell]$ as we have $-p^{k_{i_n}} s_{i_n} \in \mathbb{Z}_p[N_1]$ and $\mathcal{O}_{\mathcal{E}}^{\dagger}$ is closed in $\mathcal{R}$.

Step 3. Assume finally that $j_{int}(z) = j(x)$ for some $z \in \mathcal{R}^{int}(N_1, \ell)$. Note that both z and $j(x) \in \mathcal{O}_{\mathcal{E}}^{\dagger}[\![N_1, \ell]\!] \subset \mathcal{O}_{\mathcal{E}}[\![N_1, \ell]\!]$ have a power series expansion. By the injectivity of j_{int} these expansions are equal. Hence put $z = \sum_{\mathbf{k} \in \mathbb{Z} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}} d_{\mathbf{k}} \mathbf{b}^{\mathbf{k}}$ with $d_{\mathbf{k}} \in \mathbb{Z}_p$. By the definition of $\mathcal{R}^{int}(N_1, \ell)$ there exists an element $t \in T_+$ such that $\varphi_t(z)$ lies in $\mathfrak{R}_0^{int}$. This means that there exists a positive integer K_0 such that for all fixed $k \geq K_0$ and $\varepsilon > 0$ we have $\|\varphi_t(d_{\mathbf{k}} \mathbf{b}^{\mathbf{k}})\|_{\rho_k} < \varepsilon$ for all but finitely many $\mathbf{k} \in \mathbb{Z} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}$. In particular, for any fixed $k \geq K_0$ we have

$$\|\varphi_t(-p^{k_{i_n}} s_{i_n} b_\alpha^{-j_n})\|_{\rho_k} < \varepsilon$$

for all but finitely many positive integers n since the sequence j_n is strictly increasing by construction therefore the terms in $x = \sum_{n=1}^{\infty} p^{k_{i_n}}(\log(1+b_\beta) - s_{i_n}) b_\alpha^{-j_n}$ cannot cancel each other. Now we clearly have $\|\varphi_t(b_\alpha)\|_{\rho_k} \leq \rho_k$. On the other hand, we compute (for $n > \max(k, m(\beta,t))$ large enough)

$$\|\varphi_t(-p^{k_{i_n}} s_{i_n})\|_{\rho_k} = \frac{\|\varphi^{m(\beta,t)}(s_{i_n})\|_{\rho_k}}{p^{k_{i_n}}} = \frac{\|\varphi^{m(\beta,t)}(\log(1+b_\beta))\|_{\rho_k}}{p^{k_{i_n}}} = \frac{\|\log(1+b_\beta)\|_{\rho_k}}{p^{m(\beta,t)+k_{i_n}}} \ .$$

Hence we obtain

$$\begin{aligned} \varepsilon > \|\varphi_t(-p^{k_{i_n}} s_{i_n} b_\alpha^{-j_n})\|_{\rho_k} &\geq \frac{\|\log(1+b_\beta)\|_{\rho_k}}{p^{m(\beta,t)+k_{i_n}} \rho_k^{j_n}} > \frac{\rho_k \|\log(1+b_\beta)\|_{\rho_k}}{p^{m(\beta,t)+k_{i_n}} \|p^{k_{i_n}} \log(1+b_\beta)\|_{\rho_n}^{\log_{\rho_n} \rho_k}} = \\ &= \frac{\rho_k \|\log(1+b_\beta)\|_{\rho_k}}{p^{m(\beta,t)}} \frac{p^{k_{i_n}(\log_{\rho_n}\rho_k - 1)}}{\|\log(1+b_\beta)\|_{\rho_n}^{\log_{\rho_n}\rho_k}} > \frac{\rho_k \|\log(1+b_\beta)\|_{\rho_k}}{p^{m(\beta,t)}} p^{k_{i_n}(1/2\cdot\log_{\rho_n}\rho_k - 1)} \end{aligned}$$

using (21) and (22). This is a contradiction as the right hand side above tends to ∞ as $n \to \infty$. Therefore $j(x)$ is not in the image of j_{int} as claimed. □

Remark 4.18. *The elements of $\mathcal{R}[\![N_1,\ell]\!]$ cannot be expanded as a skew Laurent series of the form $\sum_{\mathbf{k}\in\mathbb{Z}^{\Phi^+}} d_{\mathbf{k}} \mathbf{b}^{\mathbf{k}}$ in general. Indeed, the sum $\sum_{n=1}^{\infty} \varphi^n(b_\beta)/p^{2^n} = \sum_{n=1}^{\infty}((b_\beta+1)^{p^n} - 1)/p^{2^n}$ converges in $\mathcal{R}[\![N_1,\ell]\!]$ for any simple root $\beta \neq \alpha$ but does not have a skew Laurent-series expansion as the coefficient of b_β in its expansion would be the non-convergent sum $\sum_{n=1}^{\infty} p^{n-2^n}$.*

We end this section by a diagram showing all the rings constructed.

$$\begin{array}{ccccccc}
\mathcal{O}_{\mathcal{E}} & \hookrightarrow & & & & & \mathcal{O}_{\mathcal{E}}[\![N_1,\ell]\!] = \Lambda_\ell(N_0) \\
\uparrow & & & & & & \uparrow \\
\mathcal{O}_{\mathcal{E}}^{\dagger} & \hookrightarrow & \mathfrak{R}_0^{int}(N_0,\alpha) & \hookrightarrow & \mathcal{R}^{int}(N_1,\ell) & \xrightarrow{j_{int}} & \mathcal{O}_{\mathcal{E}}^{\dagger}[\![N_1,\ell]\!] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{E}^{\dagger} & \hookrightarrow & \mathfrak{R}_0^{bd}(N_0,\alpha) & \hookrightarrow & \mathcal{R}^{bd}(N_1,\ell) & \xrightarrow{j_{int}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p} & \mathcal{E}^{\dagger}[\![N_1,\ell]\!] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{R} & \hookrightarrow & \mathfrak{R}_0(N_0,\alpha) & \hookrightarrow & \mathcal{R}(N_1,\ell) & \xrightarrow{j} & \mathcal{R}[\![N_1,\ell]\!]
\end{array}$$

Here $\mathcal{R}(N_1,\ell)$ consists of Laurent series $\sum_{\mathbf{k}} c_{\mathbf{k}} \mathbf{b}^{\mathbf{k}}$ with $c_{\mathbf{k}} \in K$ that converge on the open annulus of the form

$$\left\{\rho_2 < |z_\alpha| < 1,\ |z_\beta| \leq |z_\alpha|^r \text{ for } \beta \in \Phi^+ \setminus \{\alpha\}\right\}$$

for some $0 < \rho_2 < 1$ and $1 \leq r \in \mathbb{Z}$. The elements of $\mathfrak{R}_0(N_0,\alpha)$ are exactly those for which we can take $r = 1$. Their analogous integral (resp. bounded) versions consist of those Laurent series having the same convergence condition for which $c_{\mathbf{k}} \in o_K$ for all $\mathbf{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+\setminus\{\alpha\}}$ (resp. for which $\{c_{\mathbf{k}} \mid \mathbf{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+\setminus\{\alpha\}}\} \subset K$ bounded).

4.5 Towards an equivalence of categories for overconvergent and Robba rings

Note that Propositions 3.1 and 3.7 apply in both the cases $R = \mathcal{O}_{\mathcal{E}}$ and $R = \mathcal{O}_{\mathcal{E}}^{\dagger}$. In both cases the category $\mathfrak{M}(R, \varphi)$ is the category of *étale* φ-modules over R. Moreover, by the main result of [6] (see also [13]), we also have an equivalence of categories between finite free étale (φ, Γ)-modules over $\mathcal{O}_{\mathcal{E}}^{\dagger}$ and finite free étale (φ, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$ given by the base change $\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}^{\dagger}} \cdot$. On the other hand, T_ℓ acts by automorphisms on an object D in $\mathfrak{M}(\mathcal{O}_{\mathcal{E}}, T_+)$ and also on an object $D^{\dagger}$ in $\mathfrak{M}(\mathcal{O}_{\mathcal{E}}^{\dagger}, T_+)$. Since automorphisms correspond to automorphism in an equivalence of categories, we obtain

Proposition 4.19. *The functors*

$$\begin{aligned} \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}^{\dagger}} \cdot : \mathfrak{M}(\mathcal{O}_{\mathcal{E}}^{\dagger}, T_+) &\to \mathfrak{M}(\mathcal{O}_{\mathcal{E}}, T_+) \\ \cdot^{\dagger} : \mathfrak{M}(\mathcal{O}_{\mathcal{E}}, T_+) &\to \mathfrak{M}(\mathcal{O}_{\mathcal{E}}^{\dagger}, T_+) \end{aligned}$$

are quasi-inverse equivalences of categories.

Note that for the Robba ring $\mathcal{R}$ étaleness is stronger than what we assumed for a module $D_{rig}^{\dagger}$ to belong to $\mathfrak{M}(\mathcal{R}, \varphi)$. The category $\mathfrak{M}(\mathcal{R}, \varphi)$ is just the category of φ-modules over the Robba ring. Recall that an object $D_{rig}^{\dagger}$ in $\mathfrak{M}(\mathcal{R}, \varphi)$ is *étale* (or unit-root, or pure of slope zero) whenever it comes from an overconvergent étale φ-module $D^{\dagger}$ over the ring of "overconvergent" power series $\mathcal{O}_{\mathcal{E}}^{\dagger}$ by base extension. We denote by $\mathfrak{M}^0(\mathcal{R}, \varphi)$ the category of étale φ-modules over the Robba ring $\mathcal{R}$. We consequently define the categories $\mathfrak{M}^0(\mathcal{R}, T_+)$, $\mathfrak{M}^0(\mathcal{R}[[N_1, \ell]], \varphi)$, and $\mathfrak{M}^0(\mathcal{R}[[N_1, \ell]], T_+)$ as the full subcategory of étale objects in the corresponding categories without superscript 0. Note that via the equivalence of categories 3.7, étale objects correspond to each other. Combining this observation with the main result of [1] leads to

Corollary 4.20. *We have a commutative diagram of equivalences of categories*

$$\begin{array}{ccccc} \mathfrak{M}^0(\mathcal{R}, T_+) & \leftarrow & \mathfrak{M}(\mathcal{E}^{\dagger}, T_+) & \rightarrow & \mathfrak{M}(\mathcal{E}, T_+) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{M}^0(\mathcal{R}[[N_1, \ell]], T_+) & \leftarrow & \mathfrak{M}(\mathcal{E}^{\dagger}[[N_1, \ell]], T_+) & \rightarrow & \mathfrak{M}(\mathcal{E}[[N_1, \ell]], T_+). \end{array}$$

Proof. The left horizontal arrows are also equivalences of categories by [1] noting that T_ℓ acts via automorphisms on both types of objects in the upper row. □

Remark 4.21. *The category $\mathfrak{M}^0(\mathcal{R}[[N_1, \ell]], T_+)$ of étale T_+-modules is embedded into the bigger category $\mathfrak{M}(\mathcal{R}[[N_1, \ell]], T_+)$. So we may speak of* trianguline *objects in $\mathfrak{M}^0(\mathcal{R}[[N_1, \ell]], T_+)$ as in the classical case (see for instance [2]). Indeed, we call an object $M_{rig}^{\dagger}$ in $\mathfrak{M}^0(\mathcal{R}[[N_1, \ell]], T_+)$ trianguline if it becomes a successive extension of objects in $\mathfrak{M}(\mathcal{R}[[N_1, \ell]], T_+)$ of rank 1 after a finite base extension $L \otimes_K \cdot$. It is clear that trianguline objects correspond to trianguline objects via the first vertical arrow in Corollary 4.20.*

Remark 4.22. *It would be interesting to construct a noncommutative version of the "big" rings $\tilde{\mathbf{A}}_{\mathbb{Q}_p}$ and $\tilde{\mathbf{A}}_{\mathbb{Q}_p}^{\dagger}$ in [13] and generalize (the proofs of) Theorems 2.3.5, 2.4.5, and 2.6.2 to this noncommutative setting. For this, one would need a generalization for results in the present paper to base fields other than $\mathbb{Q}_p$.*

Remark 4.23. *Since we have the natural inclusions* $\mathcal{O}_{\mathcal{E}}^{\dagger} \hookrightarrow \mathcal{R}^{int}(N_1,\ell) \hookrightarrow \mathcal{O}_{\mathcal{E}}^{\dagger}[\![N_1,\ell]\!]$*, we have a fully faithful functor*

$$\Theta := \left(\mathcal{R}^{int}(N_1,\ell)\otimes_{\mathcal{O}_{\mathcal{E}}^{\dagger}}\cdot\right)\circ\left(\mathcal{O}_{\mathcal{E}}^{\dagger}\otimes_{\ell,\mathcal{O}_{\mathcal{E}}^{\dagger}[[N_1,\ell]]}\cdot\right)\circ\left(\mathcal{O}_{\mathcal{E}}^{\dagger}[\![N_1,\ell]\!]\otimes_{\mathcal{R}^{int}(N_1,\ell)}\cdot\right)$$

from the category $\mathfrak{M}(\mathcal{R}^{int}(N_1,\ell),T_+)$ *to itself. Whether or not it is essentially surjective (or equivalently that it is naturally isomorphic to the identity functor) is not clear. However, we have* $\Theta\cong\Theta\circ\Theta$ *naturally.*

Proof. The faithfulness is clear since the objects in the category $\mathfrak{M}(\mathcal{R}^{int}(N_1,\ell),T_+)$ are free modules, the maps $\mathcal{O}_{\mathcal{E}}^{\dagger} \hookrightarrow \mathcal{R}^{int}(N_1,\ell) \hookrightarrow \mathcal{O}_{\mathcal{E}}^{\dagger}[\![N_1,\ell]\!]$ are injective, and the functor $\mathcal{O}_{\mathcal{E}}^{\dagger}\otimes_{\ell,\mathcal{O}_{\mathcal{E}}^{\dagger}[[N_1,\ell]]}\cdot$ in the middle is an equivalence of categories by Prop. 3.7. The assertion $\Theta\cong\Theta\circ\Theta$ is also clear by Prop. 3.7. For the fullyness let $f\colon\Theta(\mathcal{M}_1)\to\Theta(\mathcal{M}_2)$ be a morphism in $\mathfrak{M}(\mathcal{R}^{int}(N_1,\ell))$. Then we have $\Theta(f-\Theta(f))=0$ and by the faithfulness of Θ obtain $f=\Theta(f)$. □

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